Group C*-algebra

G = finite group R ring (commut) $R[G]$ group ring

$f : G \to R$ functions   pointwise addition
and convolution product

$$(f_1 * f_2)(g) = \sum_h f_1(h) f_2(h^{-1}g) = \sum_{g=h_1h_2} f_1(h_1) f_2(h_2)$$

if R unital then have → δ_g = delta function supported at $g \in G$ $= \begin{cases} 1_R & h=g \\ 0 & h \neq g \end{cases}$

(Notation: * of convolution product not to be confused w/ * of adjoint)

$$\delta_{g_1} * \delta_{g_2} = \delta_{g_1 g_2} \qquad (*)$$

any $f = \sum_{g \in G} a_g \delta_g$

⇒ can also describe $R[G]$ as formal sums

$$\sum_{g \in G} a_g \delta_g \qquad a_g \in R$$

with multiplication (*)
(any a_g commuting with δ_g)

Involution $f^*(g) = \overline{f(g^{-1})}$ when $R = \mathbb{C}$

$\delta_g^* = \delta_{g^{-1}}$ unitaries implementing the group elements

Norm on $\mathbb{C}[G]$

$\mathbb{C}[G]$ Hilbert space (still G finite here)   So w/ basis δ_g (same as $\ell^2(G)$ when finite G)

$$\langle f_1, f_2 \rangle = \sum_{g \in G} \overline{f_1(g)} f_2(g)$$

Left regular representation of algebra $\mathbb{C}[G]$ on Hilbert space $\mathbb{C}[G]$ $(= \ell^2(G))$

$$\lambda(f)h = f*h$$

(left multipl. in convolution *-product)

G infinite discrete group

$f : G \to \mathbb{R}$ ~~/~~ $\mathbb{C}$ functions with finite support

then same def. of convolution $f_1 * f_2$

of involution $f^*(g) = \overline{f(g^{-1})}$

and representation on $\ell^2(G)$

$$\lambda(f)\ \Sigma c_g \delta_g = \Sigma a_g \delta_g * \Sigma c_g \delta_g \qquad c_g \in \ell^2(G)$$

when $f = \Sigma a_g \delta_g$

left regular representation

$$\lambda : \mathbb{C}[G] \longrightarrow B(\ell^2(G))$$

group ring

note: compatible w/ involution

$$\lambda(f^*) = \lambda(f)^* \quad \text{adjoint operator}$$

$$\langle h_1, h_2 \rangle = \sum_g \overline{h_1(g)}\, h_2(g) \qquad h(g) \in \ell^2(G) = \sum_{g \in G} c_g \delta_g$$

$$\langle h_1, \lambda(f) h_2 \rangle = \sum_g \overline{h_1(g)} \sum_{g = g_1 g_2} f(g_1) h_2(g_2)$$

$$\sum_{g = g_1 g_2} \overline{h_1(g_1 g_2)}\ f(g g_2^{-1})\ h_2(g_2)$$

$$\sum_{g} \sum_{g = g_1 g_2} \overline{h_1(g)\ \underbrace{\overline{f((g_2 g^{-1})^{-1})}}}\ h_2(g_2)$$

$$\sum_{g_2} \sum_g \overline{h_1(g)\, f^*(g_2 g^{-1})}\ h_2(g_2)$$

then norm $\|\cdot\|$ in $B(\ell^2(G))$

<u>reduced</u> group C^*-algebra

$f : G \to \mathbb{C}$

$f^*(g) = \overline{f(g^{-1})}$

$\langle h_1, h_2 \rangle = \sum_g \overline{h_1(g)}\, h_2(g)$ $\qquad \Rightarrow \quad \lambda(f^*) = \lambda(f)^*$

$\langle h_1, \lambda(f) h_2 \rangle = \sum_g \overline{h_1(g)} \sum_{g = g_1 g_2} f(g_1)\, h_2(g_2)$

$= \sum_{g = g_1 g_2} \overline{h_1(g)}\, \underbrace{f(g g_2^{-1})}\, h_2(g_2)$

$\sum \overline{h_1(g) f^*(g_2 g^{-1})}\, h_2(g_2)$

$\langle \lambda(f)^* h_1, h_2 \rangle$

$\sum_{g_2 = g_2 g^{-1} \cdot g} f^*(\tilde{g}_1)\, h_1(\tilde{g}_2)\, h_2(g_2)$

$\tilde{g}_1 \quad \tilde{g}_2$

Other norm <u>max</u> $\quad C^*_{max}(G) \neq C^*_{red}(G)$ (in general different for certain classes of groups "slow growth" same)

through considering all unitary representations

$\pi : G \to B(\mathcal{H})$ in fact ($\pi : G \to \mathcal{U}(\mathcal{H})$) group homom. to unitary operators on a Hilbert space

$g \mapsto U_g$

any unitary rep λ extends to rep of group ring $\mathbb{C}[G] \to B(\mathcal{H})$

group ring $\mathbb{C}[G]$ still <u>fin</u> lin comb. $\sum a_g \delta_g$

$$\pi\left(\sum_g a_g \delta_g\right) = \sum_g a_g U_g$$

$\| f \| = \sup_\pi \| \pi(f) \|_{B(\mathcal{H})} \qquad f \in \mathbb{C}[G]$

over different unitary reps.

completion in this norm $C^*_{max}(G)$

$\pi : G \to \mathcal{U}(\mathcal{H})$

$\Downarrow$

$\pi : C^*(G) \to B(\mathcal{H})$

Not same in general

e.g. F_2 = free group on two generators

$C^*_r(F_2)$ simple (R. Powers 1975)

but then cannot have fin dim repres. (there would be Ker not simple)

but F_2 has fin dim unitary reps

$\Rightarrow C^*_{max}(F_2)$ not simple

Suppose G abelian $\Rightarrow C^*(G)$ is commutative

($\mathbb{C}[G]$ commut iff G abelian)

then $\exists\, X$ loc. comp. top. space

s.t. $C^*(G) = C_0(X)$

X is set of multiplicative lin functionals on $C^*(G)$

$\Updownarrow$

group homomorphisms $G \longrightarrow \mathbb{C}^*$

(note: $\delta_g * \delta_{g^{-1}} = \delta_e = 1$ in $C^*(G)$)

(unitary))

i.e. X is group of characters of G

$X = \hat{G}$ Pontrjagin dual

$$C^*(G) = C_0(\hat{G})$$

Usual Pontrjagin duality for loc. comp. abelian groups

if G not abelian topol. group loc. comp

~~...~~ $C^*(G)$ as a Noncommutative space "is"

the replacement for Pontrjagin dual $\hat{G}$

$\mathrm{Aut}(A)$ $\quad A = C^*\text{-alg}$

$\alpha: A \to A$ $\quad$ $*$-isomorphisms

inner automorphisms $\quad u \in \mathcal{U}(A)$ unitary

$$\alpha_u(a) = u\, a\, u^*$$

C^*-dynamical system (A, G, σ)

$A = C^*$ alg. $\quad G$ loc. comp. group

$\sigma: G \longrightarrow \mathrm{Aut}(A)$ group homom. $\quad g \mapsto \alpha_g(a)$ continuous $\forall a \in A$

i.e. have an action of G by automorphisms of A

Commutative case $A = C_0(X)$

and $G \circlearrowright X$ acting on X by homeomorphisms

$$\alpha_g(f)(x) = f(g^{-1}x)$$

induced group action on $C_0(X)$

Describing quotients in noncommutative geometry

$G \circlearrowright X$ group action on a loc. comp. Hausdorff space

X/G = quotient = space of orbits

if action of G is ________

then X/G is still Hausdorff

otherwise in general not

Usual description of functions on quotient

$$C(X/G) = C(X)^G$$

those functions that are constant along orbits

$f(g^{-1}x) = f(x)$

(assume X, X/G compact for simplicity)

But: Usually not enough such functions

example: G acting on X with dense orbits

X/G only continuous functions are the constant functions

"Better" way to describe functions on the quotient

$X/\sim$ $\sim$ = equivalence relation

$R \subset X \times X$ $R = \{(x,y) : x \sim y\}$

$C(X/\sim)$ instead of this: invariant functions

take vector space $C(R)$ with product different from commutative pointwise product

convolution

$$(f_1 * f_2)(x,y) = \sum_{\substack{z: \\ (x,z),(z,y) \in R}} f_1(x,z) f_2(z,y) \qquad (x,y) \in R$$

formal definition (need finite sum and then norm completion)

See more precisely in the case of a group action:

start with case of finite group G

formal sums $\sum_{g \in G} a_g u_g$ where $a_g \in A$ u_g unitaries

and with relation

$$u_g a = \alpha_g(a) u_g \qquad \text{where } \alpha_g : G \to \mathrm{Aut}(A) \text{ action}$$

then $(a u_g)^* = u_g^* a^* = u_g^* a^* u_g u_g^* = \alpha_{g^{-1}}(a^*) u_g^*$

Also view $\sum_{g \in G} a_g u_g$ as functions $f : G \to A$

So usual convolution formula for product

$$(f_1 * f_2)(g) = \sum_{g = g_1 g_2} f_1(g_1) \alpha_{g_1}(f_2(g_2))$$

Call this involutive ring (not yet normed) $A[G]$

$$f^*(g) = \alpha_g(f(g^{-1})^*)$$

Representations:

suppose $\pi : A \to B(\mathcal{H})$ representation of A on a Hilbert space $\mathcal{H}$

$\ell^2(G, \mathcal{H})$ functions $\xi : G \to \mathcal{H}$

with $\langle \xi_1, \xi_2 \rangle = \sum_{g \in G} \langle \xi_1(g), \xi_2(g) \rangle_{\mathcal{H}}$

then extend representation π to $A[G]$

$$\rho : AG \to B(\ell^2(G, \mathcal{H}))$$

$$(\rho(f)\xi)(g) = \sum_{h \in G} \pi\left(\alpha_g^{-1}(f(h))\right)\xi(h^{-1}g)$$

$$\sum_{g = g_1 g_2} \pi\left(\alpha_g^{-1}(f(g_1))\right) \xi(g_2)$$

Note: product if write

$f_1 = \sum_{g \in G} a_g u_g \qquad f_2 = \sum_{h \in G} b_h u_h$

$$f_1 * f_2 = \sum a_g u_g b_h u_h$$
$$= \sum a_g \alpha_g(b_h) u_{gh}$$
$$= \sum_{g = g_1 g_2} a_{g_1} \alpha_{g_1}(b_{g_2}) u_g$$

in terms of $f(g) = \sum a_g u_g$

with $u_g a u_g^* = \alpha_g(a)$

$\rho(u_g) = U_g$ unitary operator on $\mathcal{H}$

with $(U_g \xi)(h) = \xi(g^{-1}h)$

$$(\rho(f)\xi)(g) = \sum_{h \in G} \pi(a_h)(U_h \xi)(g) = \sum_{h \in G} \pi(a_h) \xi(h^{-1}g)$$

$$\rho\left(\left(\sum a_g u_g\right)\left(\sum b_h u_h\right)\right) \xi = \rho\left(\sum a_{g_1}\alpha_{g_1}(b_{g_2}) u_g\right) \xi$$

$$= \sum_{g = g_1 g_2} \pi\left(a_{g_1} \alpha_{g_1}(b_{g_2})\right) \xi(g^{-1}h)$$

$f(h) = \sum_{g \in G} a_g \underset{\delta_g(h)}{\underset{\parallel}{a_g(h)}} = a_h \qquad \alpha_g^{-1}(a_h)$

$$\sum \pi\left(a_{g_1} \alpha_{g_1}(b_{g_2})\right) U_{g_1 g_2} \xi = \rho\left(\sum a_g u_g\right) \sum \pi(b_{g_2})\, U_{g_2} \xi$$

need to have

$$\rho(f_1 * f_2) = \rho(f_1)\,\rho(f_2)$$

conv. prod — prod as operators on $\ell^2(G, \mathcal{H})$

$$(f_1 * f_2)(g) = \sum_{g = g_1 g_2} f_1(g_1)\, \alpha_{g_1}(f_2(g_2))$$

$$(\rho(f_1 * f_2)\xi)(g) = \sum_{g = g_1 g_2 g_3} \pi\big(\alpha_g^{-1}(f_1(g_1)\, \alpha_{g_1}(f_2(g_2)))\big)\, \xi(g_3)$$

$$\alpha_g^{-1}(f_1(g_1)) \cdot \alpha_{g_2 g_3}^{-1}(f_2(g_2))$$

$$(\rho(f_1)\rho(f_2)\xi)(g) = \rho(f_1) \sum_{g = g_1 g_2} \pi(\alpha_g^{-1}(f_2(g_1)))\, \xi(g_2)$$

$$= \sum_{g = g_1 g_2} \pi(\alpha_g^{-1}(f_1(g_1)))\, \rho(f_2)(\xi)(g_2)$$

$$= \sum_{g = g_1 g_2 g_3} \pi(\alpha_g^{-1}(f_1(g_1)))\, \pi(\alpha_{g_2 g_3}^{-1}(f_2(g_2)))\, \xi(g_3)$$

To get right twisting by α_g action need

$$(\rho(f)\xi)(g) = \sum_{g = g_1 g_2} \pi(\alpha_g^{-1}(f(g_1)))\, \xi(g_2)$$

then see that product matches

Then induced norm on $A[G]$ from
ρ representation (assume ρ injective)

$\Downarrow$

left-regular repres. $A \rtimes_\alpha G$ resulting C^*-algebra (reduced norm)

Otherwise: $A \rtimes_\alpha G$ universal way to implement α-action via Unitary operators

All (π, U) repres on $\mathcal{H}$ $\pi : A \to B(\mathcal{H})$
$U : G \to \mathcal{U}(\mathcal{H})$

satisfying condition

$$\pi(\alpha_g(a)) = U_g \pi(a) U_g^*$$

Then norm on $A[G]$ from sup over these representations and resulting maximal $A \rtimes_\alpha G$
has univ. property any such (π, U) extends to

$$\rho : A \rtimes_\alpha G \to B(\mathcal{H})$$

Case of infinite discrete group similar

using finite support functions $f : G \to A$ and all as above, then completing in norm

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More general than equivalence relations defined by group actions

<u>Groupoids</u>: A small category where all the morphisms are invertible

in particular a group is a groupoid w/ only one object

More explicit description

Set $G^{(0)}$ (objects, or units of the groupoid)

Set $G^{(1)}$ (arrows; morphisms)

maps $s, t : G^{(1)} \longrightarrow G^{(0)}$ source and target
(sometime s, r
or d, r)
domain, range

__Note__: think of $G^{(0)} \subset G^{(1)}$ by identifying each object
$x \in G^{(0)}$ with the morphism $1_x \in G^{(1)}$
in this case $s(1_x) = t(1_x) = x$

(~~sometime~~

if $g_1, g_2 \in G^{(1)}$ with $s(g_1) = t(g_2)$
then $\exists\ g_1 g_2 \in G^{(1)}$

can compose only
arrows ~~that~~ are
"consecutive

$s(g_1 g_2) = s(g_2)$
$t(g_1 g_2) = t(g_1)$

$\bullet \xrightarrow{g_2} \bullet \xrightarrow{g_1} \bullet$

$h = 1_x \in G^{(0)}$ then $hg = g$ for all $g \in G^{(1)}$ with
$t(g) = x$
and $gh = g$ for all $g \in G^{(1)}$ w $s(g) = x$

$\forall g \in G^{(1)}\ \exists\ g^{-1} \in G^{(1)}$ s.t.

$t(g^{-1}) = s(g)$ $\quad s(g^{-1}) = t(g)$ and

$g g^{-1} = t(g)$ $\quad g^{-1} g = s(g)$

Groupoid $\mathcal{G} = (\mathcal{G}^{(0)}, \mathcal{G}^{(1)}, s, t)$ is

principal if (s,t) injective

$$(s,t) : \mathcal{G}^{(1)} \to \mathcal{G}^{(0)} \times \mathcal{G}^{(0)}$$

transitive if (s,t) surjective

for $x \in \mathcal{G}^{(0)}$ set $\mathcal{G}^{(1)}(u) = \{ g \in \mathcal{G}^{(1)} : s(g) = t(g) = u \}$

isotropy group of $u \in \mathcal{G}^{(0)}$

principal = isotropy groups all trivial

transitive = isotropy groups all conjugate

Group actions $G \overset{\alpha}{\circlearrowright} X$ on sets $\Rightarrow$ groupoids

$\mathcal{G}^{(1)} = X \times G$ $\quad$ $\mathcal{G}^{(0)} = X$ $\quad$ (identified w/ $X \times \{e\}$ in $\mathcal{G}^{(1)}$)

$s(x,g) = g^{-1}(x) = \alpha_g(x)$ $\quad$ $t(x,s) = x$

can compose $(x,g)\cdot(y,h)$ iff $t(y,h) = s(x,g)$

then $= (x, gh)$ $\quad$ $y = g^{-1}(x) = \alpha_g(x)$

$$\begin{tikzcd} x & g^{-1}(x) \arrow[l, "g"'] & h^{-1}g^{-1}(x) = h^{-1}(y) \arrow[l, "h"'] \end{tikzcd}$$

$\alpha_g(x)$ $\quad$ $\alpha_{gh}(x) = \alpha_h(y)$

$$(x,g)^{-1} = (g^{-1}(x), g^{-1}) = (\alpha_g(x), g^{-1})$$

$$\alpha_g(x) = g^{-1}(x) \xrightarrow{g^{-1}} \alpha_{g^{-1}}\alpha_g(x) = g g^{-1}(x) = x$$

Convention

$x \xrightarrow{(x,g)} \alpha_g(x)$ or

$x \xleftarrow{(x,g)} \alpha_g(x)$

give opposite algebra

the groupoid $\mathcal{G} = (X \times G, X, (\alpha, [illegible]))$ is principal

iff G acts freely on X

Groupoid algebra: Topological groupoid (loc. comp. operations and s,t continuous)

$f: \mathcal{G}^{(1)} \to \mathbb{C}$ with finite support (if discrete) or else

$C_c(\mathcal{G}^{(1)})$ compactly supported continuous functions

Only look at discrete case now

then product convolution as before

$$(f_1 * f_2)(g) = \sum_{g = g_1 g_2} f_1(g_1) f_2(g_2)$$

where $g_1 \cdot g_2 = g$ implies $t(g_2) = s(g_1)$

[In case of $C^*(G)$ and of $C_0(X) \rtimes_\alpha G$ this gives groupoid algebra w/ corresp. groupoids as above]

involution & norm

$$f^*(g) = \overline{f(g^{-1})}$$

and norm through representations

Again have sets $x \in \mathcal{G}^{(0)}$ $\quad \mathcal{G}_x^{(1)} = \{ g \in \mathcal{G}^{(1)} : s(g) = x \}$

$\ell^2(\mathcal{G}_x^{(1)})$ Hilbert space (still assuming discrete)

$$\langle \xi_1, \xi_2 \rangle = \sum_{g \in \mathcal{G}_x^{(1)}} \overline{\xi_1(g)}\, \xi_2(g)$$

$\rho_x : C_c(\mathcal{G}^{(1)}) \longrightarrow B(\ell^2(\mathcal{G}_x^{(1)}))$

w/ convol. prod & involution as above

by setting

$$(\rho_x(f)\,\xi)(g) = \sum_{g=g_1g_2} f(g_1)\,\xi(g_2) = (f*\xi)(g)$$

as product of functions on groupoid restricted to $\mathcal{G}_x^{(1)} \subset \mathcal{G}^{(1)}$

$s(g)=x \Rightarrow s(g_2)=x$

Again need check that

$\rho_x(f_1 * f_2) = \rho_x(f_1)\,\rho_x(f_2)$ prod in $\mathcal{B}(\mathcal{H})$

$\rho_x(f^*) = \rho_x(f)^*$ adjoint in $\mathcal{B}(\mathcal{H})$

$$\rho_x(f_1)\,(\rho_x(f_2)\,\xi) = \rho_x(f_1)\,(f_2 * \xi) = f_1 * f_2 * \xi \quad ✓$$

$$\langle \xi_1, \rho_x(f)\,\xi_2 \rangle = \langle \xi_1, f*\xi_2 \rangle = \sum_g \overline{\xi_1(g)} \sum_{g=g_1g_2} f(g_1)\,\xi_2(g_2)$$

$$= \sum_{g_2=g_1^{-1}g} \overline{\xi_1(g)}\, f(g_1)\, \xi_2(g_2) = \sum_{g_2} \underbrace{\sum_{g_2=g_1^{-1}g} \overline{f((g_1^{-1})^{-1})\, \xi_1(g)}\, \xi_2(g_2)}_{\sum_{g_2} \left(\sum_{g_2=hg} f^*(h)\,\xi_1(g)\right) \xi_2(g_2)}$$

$$= \langle \rho_x(f^*)\,\xi_1, \xi_2 \rangle$$

More generally: $\mathcal{G}^{(1)}$ need not be discrete but if $s^{-1}(x)$ & $t^{-1}(x)$ are discrete for all $x \in \mathcal{G}^{(0)}$ can do same thing

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<u>Norm</u> $\|f\| = \sup_{x \in \mathcal{G}^{(0)}} \|\rho_x(f)\|_{\mathcal{B}(\ell^2(\mathcal{G}_x^{(1)}))}$

assume $\mathcal{G}^{(0)}$ compact

Example: Equivalence relations on finite sets

Groupoid C^*-algebras are direct sums of matrix algebras

$$\mathcal{G}^{(1)} = \coprod_{E \in \text{equivalence classes}} E \times E$$

$$C^*(\mathcal{G}) = \bigoplus C^*(E \times E)$$

if $\# E = n$ this is $M_n(\mathbb{C})$

e.g. $E = \{1,2\}$ $\delta_{(1,1)}$, $\delta_{(1,2)}$, $\delta_{(2,1)}$, $\delta_{(2,2)}$ 2×2 matrix units

Convolution product is matrix product