Tuesday Jan 5 2010

1

Geometry and Noncommutative geometry

- <u>Geometry adapted to quantum world</u>
  ↳ mathematical setting as in Quantum Mechanics
  (Hilbert spaces, operator algebras)
  Heisenberg uncertainty principle: x, p not commuting

  Further developments in quantum & high energy physics
  more mathematical tools (category theory, motives, ...)
  Similar role in NCG

- Methods to continue to do geometry on objects that are <u>not</u> smooth manifolds <u>as if</u> they were smooth manifolds

  (Tools of geometry: topology; homology; differential forms; vector bundles; connections; curvatures; integration; measures; gravity + matter)

  extend to "spaces" that are not manifolds
  (bad quotients; fractals; quantum groups; deformation quantization; ~~almost~~ almost-commutative geometries; triangulated categories, dG-categories ...)

  Extend various type of geometry (topology; measure theory; smooth; Riemannian)

- NCG and physics. quantum Hall effect (challange from integral to fractional); quantum field theory (on NC spacetimes; limits of string theories); models for particle physics and cosmology; open string theory

NC spaces as categories / algebras
like wave/particle complementarity in quantum physics

Connes "Noncommutative Geometry" 1994

A preliminary excursus into the language and tools of operator algebras

Idea: describe completely the geometry in terms not of points in space but of the algebra of functions on the space (ring of coordinates)

$\hookrightarrow$ developed in algebraic geometry theory of schemes
in NC geometry through operator algebras

* Differential NCG: X nc space $\equiv$ "$C(X)$" algebra C^*algebra of continuous functions
Gelfand-Naimark correspondence

* Algebraic NCG: X nc space $\equiv$ $\mathcal{T}_X$ category of sheaves (triangulated and/or dg-category)

- Recent interesting interplay: NC tori (Connes, Rieffel, Manin, Polishchuk ...)
Motives (Keller, Kontsevich, Kaledin)
Motivic Donaldson-Thomas invariants (Kontsevich-Soibelman) Hall algebras

Separable Hilbert space $\mathcal{H}$ vector space over $\mathbb{C}$

$\langle , \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ hermitian (linear in one var. antilin in other)

pos. def. $\langle x,x \rangle \geq 0$ & $\langle x,x \rangle = 0$ iff $x=0$

$\|x\| = \langle x,x \rangle^{1/2}$ norm: $\mathcal{H}$ complete in this norm

(countable max orthonormal set)

Ref for functional analysis Zimmer "Essential results of functional analysis" UChicago Press

C^*-algebra

$\mathbb{R}$-algebra A = ab. group which is both ring & $\mathbb{R}$-module so that ring multipl. is $\mathbb{R}$-bilinear ($\mathbb{R}$ commut. ring)
$r(xy) = (rx)y = x(ry)$

associative always assumed

A algebra over $\mathbb{C}$

$\mathbb{C}$-vector space w/ associative multipl.

normed: $\|xy\| \le \|x\| \cdot \|y\|$ (Banach alg. complete in $\|\cdot\|$)

- involution $* : A \to A$ antilinear $*(\lambda_1 a + \lambda_2 b) = \bar{\lambda}_1 a^* + \bar{\lambda}_2 b^*$
  (i.e. it extends complex conjugation of scalars)

  s.t. $(a^*)^* = a$

  $(ab)^* = b^* a^* \quad \forall a, b \in A$

  e.g. $A = C(X)$
  $f(x) \longmapsto \overline{f(x)}$

  $\|ab\| \le \|a\| \cdot \|b\|$

- C^*-norm on A $\quad \|\cdot\|$ s.t. A is complete in this norm (Banach space)

  $\|a^* a\| = \|a\|^2 \quad \forall a \in A$

A is unital if $\exists\, 1 \in A$ s.t. $a1 = 1a = a \quad \forall a \in A$

$\Rightarrow 1^* = 1 \; ; \; \|1\| = 1$

Morphisms: $\varphi : A \to B$ $\quad *$-morphism

continuous homomorphism, algebra homom. respecting involution

(note it follows they are contractions $\|\varphi(a)\| \le \|a\|$)

$\leadsto$ if unital algebras also require $\varphi(1) = 1$

[Note: occasionally will consider morphisms when $\varphi(1)$ is an idempotent $\varphi(1)^2 = \varphi(1) = \varphi(1)^*$ (projection) but not necessarily $= 1$]

Basic example: X locally compact Hausdorff space

$C_0(X)$ = algebra of continuous functions vanishing at ∞
(completion in sup norm of compactly supp. functions)

$\|f\| = \sup_{x \in X} |f(x)|$ (max achieved) $\qquad f^*(x) = \overline{f(x)}$

commutative C^*-algebra

$\{x \in X : \|f(x)\| \ge \varepsilon\}$ compact for all $\varepsilon > 0$

Simple example of a noncommutative C^*-algebra
$M_n(\mathbb{C})$ matrices (simple)
finite dimensional

Variant: $C_0(X, M_n(\mathbb{C}))$ X loc. comp. Hausdorff

alg. of continuous functions $f: X \to M_n(\mathbb{C})$
vanishing at infinity

$f^*(x) = (f(x))^*$ adjoint matrix at each pt of x

$\|f\| = \max_{x \in X} \|f(x)\|$ ← norm in $M_n(\mathbb{C})$

Similarly can construct $C_0(X, A)$ for any C^*-alg. A
also $M_n(A)$ for A a C^* algebra

<u>Note</u>: $C_0(X)$ commutative $C_0(X, M_n(\mathbb{C}))$ non-commutative

In many ways would like to think of these as two models of the same space X ⇝ not isomorphic algebras but "Morita equivalent"

In several applications of NCG (e.g. particle physics models) use freedom to move between different models of same underlying "space"; <u>not</u> pass to quotient by Morita equivalence; in other settings want NC spaces <u>up to</u> Morita equivalence

<u>Note</u>: $M_n(C_0(X, A)) \underset{\text{isom.}}{\cong} C_0(X, M_n(A))$

Case of commutative C^* algebras: suppose X, Y loc comp. Hausdorff spaces
$C_0(X)$, $C_0(Y)$ corresp. C^* algebras

suppose <u>X compact</u> (so $C(X)$ alg.) and $\varphi: C(X) \to C_0(Y)$ morphism then φ comes from a map of underlying topological spaces :

$\exists\, U \subset Y$ compact open subset of Y (component of Y compact)

$\alpha : U \to X$ continuous map

$$\varphi(f)(y) = \begin{cases} f(\alpha(y)) & y \in U \\ 0 & y \notin U \end{cases}$$

U = support of idempotent $\varphi(1)$

(more generally if X loc. comp. same if α proper map)

Example when can describe fully morphisms in NC case

$$A = M_{k_1}(\mathbb{C}) \oplus \cdots \oplus M_{k_r}(\mathbb{C}) \qquad B = M_{n_1}(\mathbb{C}) \oplus \cdots \oplus M_{n_s}(\mathbb{C})$$

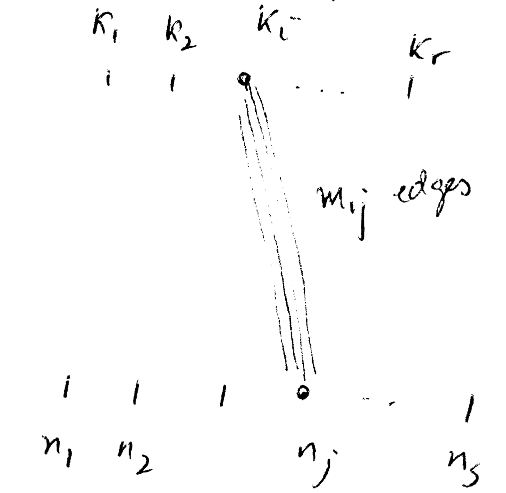
morphisms given by Bratteli diagrams

$$M_{k_i}(\mathbb{C}) \hookrightarrow A \xrightarrow{\varphi} B \longrightarrow M_{n_j}(\mathbb{C})$$

$$\sum_{i=1}^{r} k_i m_{ij} \leq n_j \qquad 1 \leq j \leq s$$

assign multiplicities

represented by graph



$a = (a_1, \ldots, a_r) \longmapsto b = \varphi(a) = (b_1, \ldots, b_s)$

$b_j \in M_{n_j}(\mathbb{C})$

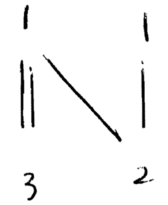
$\shortparallel$

block diagonal matrix with m_{1j} copies of a_1 then m_{2j} copies of a_2 ....

up to conjugation by an element of B

(inner autom. of B)

example:



$$\mathbb{C}\oplus\mathbb{C} \xrightarrow{\varphi} M_3(\mathbb{C})\oplus M_2(\mathbb{C})$$

$$(\lambda,\mu) \longmapsto \left(\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \right)$$

One point compactification:

A C^*-algebra (not unital)

$A^+ = A \oplus \mathbb{C}$ as vector space

with product

$$(a,\lambda)(b,\mu) = (ab + \lambda b + \mu a,\ \lambda\mu)$$

$\forall\, a,b \in A \quad \lambda,\mu \in \mathbb{C}$

unit element $(0,1)$

embedding $A \hookrightarrow A'$ $\quad a \mapsto (a,0)$ ~~dense~~

$$(a,\lambda)^* = (a^*, \bar\lambda)$$

$$\|(a,\lambda)\| = \sup_{\|b\|\le 1} \{ \|ab + \lambda b\| \}$$

$a \mapsto (a,0)$ isometry

$\varphi: A \to B \Rightarrow$
$\varphi^+: A^+ \to B^+$
$\varphi^+(a,\lambda) = (\varphi(a),\lambda)$

check: $\|(a,\lambda)^*(a,\lambda)\| = \|(a,\lambda)\|^2$

(note suffices to check $\geq$)

if A unital $(a,\lambda) \mapsto (a+\lambda 1, \lambda)$ isom $\Rightarrow$ $A^+ \cong A \oplus \mathbb{C}$ as algebras

X loc. comp. Hausdorff space $\quad X^+ = X \cup \{\infty\}$ 1-pt. compactif.

$$C_0(X)^+ \xrightarrow{\cong} C(X^+) \qquad (f,\lambda) \mapsto f+\lambda \ \text{ w/ } (f+\lambda)(\infty) = \lambda$$

$0 \to C_0(X) \to C(X^+) \to \mathbb{C} \to 0$ extension

Spectrum A unital C^*-algebra $a \in A$

$$\sigma_A(a) = \{ \lambda \in \mathbb{C} : a - \lambda \text{ not invertible in } A \}$$

where $\lambda = \lambda \cdot 1 \in A$ $\quad R_\lambda(a) = (\lambda - a)^{-1}$ resolvent function

spectral radius $r(a) = \sup \{ |\lambda| : \lambda \in \sigma_A(a) \}$

Lemma $\sigma_A(a)$ is non-empty compact set
$R_\lambda(a)$ analytic on $\mathbb{C} \setminus \sigma_A(a)$

Pf: Suppose $|\lambda| > \|a\|$: then $|\lambda|^{-n} \|a\|^n \geq \|\lambda^{-n} a^n\|$
decreases geometrically so series converges in norm

$$\sum_{n \geq 0} \lambda^{-n-1} a^n$$

$$(\lambda - a) \sum_{n=0}^{k} \lambda^{-n-1} a^n = 1 - \lambda^{-n-2} a^{n+1}$$

so lim of series is $(\lambda - a)^{-1} = R_\lambda(a)$

$\rightsquigarrow$ $R_\lambda(a)$ analytic and Laurent expansion at $\lambda = \infty$

$$\lim_{|\lambda| \to \infty} \|R_\lambda(a)\| \leq \lim_{|\lambda| \to \infty} |\lambda|^{-1} (1 - |\lambda|^{-1} \|a\|)^{-1} = 0$$

$$(\lambda - a)^{-1} = \sum_{n \geq 0} (\lambda - \lambda_0)^n (\lambda_0 - a)^{-n-1}$$ Taylor exp around λ_0 when $\lambda_0 - a$ invertible

$\Rightarrow$ analytic on complement of spectrum (resolvent).
in particular $|\lambda| > \|a\|$ in resolvent $\Rightarrow$ $\sigma(a) \subset \{ \lambda : |\lambda| \leq \|a\| \}$
resolvent open compact

Lemma: Spectral radius $r(a) = \lim_{n \to \infty} \|a^n\|^{1/n}$

Pf: $$R_\lambda(a) = (\lambda - a)^{-1} = \sum_{n \geq 0} \lambda^{-n-1} a^n$$

$R_\lambda(a)$ analytic for $|\lambda| > r(a)$ so for $|\lambda| \geq r > r(a)$
series conv. absolutely & uniformly

$r^{-n-1} \|a^n\|$ converge to 0

Taylor coefficients

$$\Rightarrow \limsup_{n\to\infty} \|a^n\|^{1/n} \leq r(a)$$

for $\alpha \in \sigma_A(a)$ s.t. $|\alpha| = r(a)$

$|\alpha| \leq \|a\|$ so $|\alpha^n|^{1/n} \leq \|a^n\|^{1/n}$

$$\underset{r(a)}{\overset{\|}{}} \leq \inf_n \|a^n\|^{1/n}$$

<u>Unitary elements</u> $u \in A$ s.t. $uu^* = u^*u = 1$

$\Rightarrow \|u\| = 1 \quad (\|u\|^2 = \|uu^*\| = \|1\| = 1)$

$\Rightarrow \sigma(u) \subset \{\lambda : |\lambda| \leq 1\}$

$\sigma(u^{-1}) \subset \{\lambda : |\lambda| \geq 1\}$ but $u^{-1} = u^*$ also unitary

so $\sigma(u) \subset \{\lambda : |\lambda| = 1\} = S^1$ spectrum on the circle

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<u>Spectrum and morphisms</u> : $\varphi : A \to B$ unital

$\sigma_B(\varphi(a)) \subset \sigma_A(a)$

(or else more generally $\sigma_B(\varphi(a)) \subset \sigma_A(a) \cup \{0\}$)

[<u>Note</u> : in fact one shows that if $B \subset A$ unital
then $\sigma_A(a) = \sigma_B(a)$]

Self adjoint elements

(As in quantum mechanics: unitary operators implement symmetry and dynamics; self-adjoint operators observables)

$h^* = h$

$a \in A$ uniquely $a = h + ik$ $\quad h, k$ self-adjoint

$$h = \frac{1}{2}(a + a^*) \qquad k = \frac{1}{2}(a - a^*)$$

Spectral radius: $r_A(h) = \|h\|$ for $h^* = h$

Pf: $\|h\|^2 = \|h h^*\| = \|h^2\| \qquad \|h^{2^n}\| = \|h\|^{2^n}$

$$r(h) = \lim_{n \to \infty} \|h^{2^n}\|^{1/2^n} = \|h\|$$

Spectrum $\sigma_A(h) \subset \mathbb{R}$ for $h^* = h$

operator $e^{ih} := 1 + ih + \frac{1}{2}(ih)^2 + \dots + \frac{1}{n!}(ih)^n + \dots$
convergent series

$$(e^{ih})^* = e^{-ih}$$

Notice that if $a, b \in A$ with $[a, b] = 0$ commuting elements
then $e^{a+b} = e^a e^b$

$\Rightarrow e^{ih}$ unitary $\quad \sigma_A(e^{ih}) \subset S^1$

Moreover if $\lambda \in \sigma(h)$ i.e. $h - \lambda$ not invertible
then also $e^{ih} - e^{i\lambda}$ not invertible
(if it were: $\exists a$ $(e^{ih} - e^{i\lambda}) a = 1$ but
$[a, h] = 0$ so $e^{i\lambda}(e^{i(h-\lambda)} - 1) a = 1$
$= (h - \lambda) b = 1$ would be invertible)

$\Rightarrow e^{i\lambda} \in \sigma_A(e^{ih}) \subset S^1 \quad \Rightarrow \lambda \in \mathbb{R}$

Gelfand - Naimark theorem

Gelfand transform

$\varphi : \underset{\text{unital}}{A} \to \mathbb{C}$ multiplicative linear functional

$\Rightarrow$ continuous of norm $\|\varphi\| = 1$

Pf: suppose $\exists\, a \in A$ w/ $\|a\| < 1$ with $\varphi(a) = 1$
then take $b = \sum_{n \geq 1} a^n$ $\quad a + ab = b$

$\Rightarrow \quad \varphi(b) = \varphi(a) + \varphi(a)\varphi(b) = \varphi(a)(1 + \varphi(b))$

if $\varphi(a) = 1$ would have $\varphi(b) = 1 + \varphi(b)$

so $\nexists$ such a $\quad \|\varphi\| \leq 1$ i.e. $\sup_{\|a\| \leq 1} \frac{|\varphi(a)|}{\|a\|} \leq 1 \Rightarrow |\varphi(a)| \leq \|a\|$ for $\|a\| \leq 1$

if > 1 $\exists\, a$ $\|a\| < 1$ $\varphi(a) = 1 \Rightarrow \varphi$ unbounded or $\|\varphi\| > 1$

$\Rightarrow \varphi(1) = 1 \Rightarrow \|\varphi\| = 1$

$M = \ker(\varphi)$ two sided ideal codim one; max ideal

$\mathcal{M}_A$ = max ideal space = set of all multiplicative linear functionals

Compact Hausdorff space (if A unital; loc comp. otherwise)

topology: weak*-topology

see elts of $\mathcal{M}_A$ as contin. lin. functionals
$\Rightarrow$ topology from top. on that space A^* dual (cont. lin. funct.)

- different choices: norm topology $\|L\| = \sup_{\|a\| \leq 1} \frac{|La|}{\|a\|}$

  or weak*-topol.: smallest making all functionals $a : L \mapsto La$ continuous

  (ptwise)

- $\mathcal{M}_A$ closed in weak*-topol. (multipl. property also a ptwise condition)

  weak*-closed subset of unit ball of dual space A^* (Banach - Alaoglu thm $\Rightarrow$ compact & Hausdor

Gelfand transform:

$$\Gamma : A \to C(M_A) \quad \text{contin functions on } M_A$$

$$\Gamma(a) = \hat{a} \qquad \hat{a}(\varphi) := \varphi(a)$$

(So far defined for arbitrary Banach algebras)
though typically M_A can be very small
if e.g. no two sided ideals
(no points in classical sense)

Gelfand-Naimark thm:

For A commutative C^*-algebra

$\Gamma : A \to C_0(M_A)$ Gelfand transform is isometric $*$-isomorphism

Pf: φ multipl. linear functional on A

$\varphi(a^*) = \overline{\varphi(a)}$: in fact check first for $a = a^*$ $\varphi(a)$ real:

take $U_t = e^{ita} = \sum_{n \geq 0} \frac{(itA)^n}{n!}$ unitary since $a = a^*$

$$\Rightarrow \|U_t\| = 1 \Rightarrow |\varphi(U_t)| \leq 1$$

$$|\varphi(U_t)| = \left| \sum_{n \geq 0} \frac{(it\varphi(a))^n}{n!} \right| = \left| e^{it\varphi(a)} \right| = e^{-t \, \mathrm{Im}\, \varphi(a)}$$

since ≤ 1 for all $t \in \mathbb{R}$ need $\mathrm{Im}\, \varphi(a) = 0$: $\varphi(a) \in \mathbb{R}$

Then for general case $a = h + ik$ h, k self adjoint and

$$\varphi(a^*) = \varphi(h - ik) = \varphi(h) - i\varphi(k) = \overline{\varphi(h+ik)} = \overline{\varphi(a)} \, .$$

So $\Gamma(a)^* = \Gamma(a^*)$ is a $*$-homomorphism

for $a = a^*$

$$\|\Gamma(a)\|_\infty = \sup_{\varphi \in M_A} |\varphi(a)| = r(a) \overset{a=a^*}{=} \|a\|$$

note: this is spectral radius of a $r(a)$ in fact

$$\sigma_A(a) = \sigma_{C_0(M_A)}(\Gamma(a)) = \{\varphi(a) : \varphi \in M_A\}$$

a invertible in A iff $\Gamma(a)$ invertible in $C_0(M_A)$ ie. does not vanish on M_A

$\Gamma(a) - \varphi(a)1$ not invertible : $\hat{a}(\varphi) - \varphi(a) = 0$

So Γ isometry : for general

$a = b^* b \qquad \| \hat{a} \|_\infty = \| a \|$

$\| b^* b \|_\infty = \| b \|_\infty^2 = \| b^* b \| = \| b \|^2$

Image of A under Γ (unital) norm-closed ~~self adjoint~~ involutive subalgebra of $C_0(\mathcal{M}_A)$ which separates points

↳ pts of $\mathcal{M}_A$ are distinct multipl. lin functionals on A
So $\varphi_1 \neq \varphi_2$ $\exists a \in A$ $\varphi_1(a) \neq \varphi_2(a)$

Stone-Weierstrass theorem
$\Rightarrow$ Γ surjective ($\Rightarrow$ $*$-isom. isometric)

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e.g. $x \in A$ normal $[x, x^*] = 0$ then $C^*(x) \cong C(\sigma(x))$
unital

Justification for NC geometry using C^*-algebras

* Every commutative C^*-algebra is $C_0(X)$ for a loc. compact Hausdorff space X.

* A non-commutative C^*-algebra A is the algebra of functions on a "non commutative" topological space.

idea then: notions of geometry on X rephrased solely in terms of the algebra $C_0(X)$
continue then to make sense also for noncommutative A and give corresponding geometric notions for n.c. spaces

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Similar idea: formulations of Quantum Mechanics

Schrödinger equation: $i\hbar \frac{\partial}{\partial t} \psi = H \psi$

$\psi = e^{\frac{-itH}{\hbar}} \psi_0 \qquad \left(\frac{\hbar^2}{2m} \Delta + V\right)$

Heisenberg:

$$\frac{dA}{dt} = -i\hbar \, [A, H] \qquad \left(+ \frac{\partial A}{\partial t}\right)$$

matrix mechanics : $\langle \psi, A \psi \rangle = \langle \psi_0, e^{\frac{it}{\hbar}H} A \, e^{\frac{-it}{\hbar}H} \psi_0 \rangle$