A generalization the Ehrhart Polynomial for simplices dilated by polynomials

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Abstract

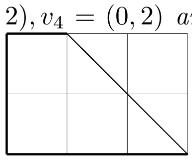
In this talk, I prepare to introduce some basic concepts about polytopes, lattice points and couting lattice points. Then I'm going to introduce the Ehrhart polynomials, which are quasi-polynomials that count the number of lattice points in a rational polytope. Finally, I'm going to talk about my investigation on a slight generalization of the Ehrhart polynomial, which is a problem my professor gave me during a summer research program.

Definition 0.1 *A (convex) polytope is the convex hull of finitely many points in* $\mathbb{R}^d$. *That is, given a (finite) set of vectors* $\{v_1, v_2, \cdots, v_n\} \subset \mathbb{R}^d$*, the polytope* $\mathcal{P}$ *is the* ***convex hull*** *of these vectors:*
$\mathcal{P} = conv(v_1, v_2, \cdots, v_n) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n | \lambda_i \geq 0$ *and* $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}$.

Also, polytopes have another representation, called the facet representation.

Theorem 0.2 *(Weyl-Minkowski) The following statements are equivalent:*
(1) $\mathcal{P}$ *is the convex hull of a finite set of vectors:* $\mathcal{P} = conv(v_1, v_2, \cdots, v_n) = \{\lambda_1 v_1 + \lambda_2 v_2 + \cdots + \lambda_n v_n | \lambda_i \geq 0$ *and* $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1\}$.
(2) $\mathcal{P}$ *can be described by* $\mathcal{P} = \{\boldsymbol{x} \in \mathbb{R}^d | \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}\}$ *for some* $k \times d$ *matrix* $\boldsymbol{A}$ *and vector* $\boldsymbol{b}$*, for which the solution space is bounded.*

Example 0.3 *Let* $v_1 = (0,0), v_2 = (3,0), v_3 = (1,2), v_4 = (0,2)$ *and*



$\mathcal{P} = conv(v_1, v_2, v_3, v_4)$.

Once we have a polytope, we can talk about lattice points in it. A lattice point is a point with integral coordinates. Why do we care about lattice points? Because they relate to very important questions: The SAT problem asks whether a system of boolean formula has a satisfying assignment, which can be translated to asking whether a given polytope with boundaries described by the formulae intersect the unit cube nontrivially. Since SAT problem is hard (indeed, NP-complete), we don't expect an easy way to compute the number of lattice points. But we are interested in the behavior of the number of lattice points in a polytope.

A rational polytope is simply a polytope that has all vertices $v_1, \cdots, v_n$ having all rational coordinates. A **t-dilate** of a polytope with vertices $v_1, \cdots, v_n$ is simply $conv(tv_1, \cdots, tv_n)$, where $t \in \mathbb{N}^+$. The Ehrhart polynomials describe $L_{\mathcal{P}}(t)$, the number of lattice points in the $t-$dilates of a polytope. First of all, we need a notion of a **quasi-polynomial**.

Definition 0.4 *A **quasi-polynomial** of degree d is an expression of the form $q(t) = c_d(t)t^d + c_{d-1}(t)t^{d-1} + \cdots + c_0(t)$ where each $c_i(t)$ is a periodic function with period T, which shall be a positive integer. $q(t)$ is said to have period T.*

Then we have our results from Ehrhart.

Theorem 0.5 *(*Ehrhart*) Let $\mathcal{P} \subset \mathbb{R}^d$ be an rational polytope. There exists a quasi-polynomial $p(t)$ with degree d such that $p(t) = L_{\mathcal{P}}(t)\ \forall t \in \mathbb{Z}$. Moreover, if the vertices of $\mathcal{P}$ are integral, there is a polynomial that describes $L_{\mathcal{P}}(t)$.*

The main theorem I proved was the following.

Theorem 0.6 *Define the integral polytope $\mathcal{P}(t)$ by $\mathcal{P}(t) = conv(v_1(t), v_2(t), v_3(t))$ where $v_i(t) = (p_{i,1}(t), p_{i,2}(t)) \in \mathbb{R}^2$ for all i, $p_{i,j}(t)$ is an **integer-valued** polynomial for all i,j and $v_1(t), v_2(t), v_3(t)$ are only collinear for finitely many t's. Then there exists a **quasi-polynomial** l(t) such that $l(t) = L_{\mathcal{P}}(t)$ for $n \gg 0$.*

This results holds for general polytopes in higher dimensions but I present the triangle in a plane here. The proof involves some number theory and **Pick's Theorem**. If I have time, I will present my proof.