ANOMALIES, DIMENSIONAL REGULARIZATION AND NONCOMMUTATIVE GEOMETRY: UNFINISHED DRAFT

ALAIN CONNES AND MATILDE MARCOLLI

ABSTRACT. In this paper we show that the Breitenlohner-Maison prescription for treating the presence of chiral symmetry in Dimensional Regularization fits remarkably well with the framework of noncommutative geometry. In fact, it corresponds to taking the cup product of spectral triples, with a specific spectral triple X_z whose dimension spectrum is a single complex number z. We give a realization of X_z using the space of $\mathbb{Q}$-lattices. We introduce a formalism of "evanescent gauge potentials" and relate the computation of anomalous graphs in dimension 2 and 4 to local index cocycles. We draw a dictionary of analogies between evanescent gauge potentials and vanishing cycles in algebraic geometry.

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1. INTRODUCTION

This is an unfinished and unpolished draft, written during a stay of the authors at the Kavli Institute for Theoretical Physics in Santa Barbara, as guests of the program "Mathematical Structures in String Theory" in the fall semester of 2005.

Dimensional regularization (Dim-Reg) is the most efficient of the regularization methods used in quantum field theory to start dealing with the divergences. It has so far been used at a purely "formal" level in which the basic formula

$$\int e^{-\lambda\, k^2}\, d^d k \,=\, \pi^{d/2}\, \lambda^{-d/2}\,, \tag{1.1}$$

is used to "define" the meaning of the integral in d-dimensions.
The main advantage of this procedure is that it is so "canonical" that it respects all the symmetries such as space-time or gauge symmetries. This advantage breaks down in the presence of a chiral symmetry where the γ_5 matrix cannot be handled naively but requires the more sophisticated prescription of t'Hooft-Veltman and Breitenlohner-Maison (*cf.* [26], [5]).
We shall show in this paper that this prescription actually fits remarkably well with the framework of noncommutative geometry. A noncommutative geometry is given by a spectral triple

$$(\mathcal{A}, \mathcal{H}, D) \tag{1.2}$$

where besides the algebra $\mathcal{A}$ concretely represented in the Hilbert space $\mathcal{H}$ the essential ingredient is the self-adjoint operator D in $\mathcal{H}$ which encodes the "metric" on the spectrum of $\mathcal{A}$. The *dimension* of a spectral triple is governed by a subset $\Sigma \subset \mathbb{C}$ called the dimension spectrum (*cf.* [18], [12]), which is specified as the set of singularities of analytic continuations of zeta functions of the form

$$\zeta_P(s) = \mathrm{Tr}(P\,|D|^{-s}) \tag{1.3}$$

where P varies in a suitable algebra of operators of pseudodifferential type generated by $\mathcal{A}$ and D ([18]). Typically (say for an ordinary manifold M) this dimension spectrum is a set of integers

$$\{n \in \mathbb{N}\,|\, 0 \leq n \leq \dim(M)\} \tag{1.4}$$

Additional structures such as the $\mathbb{Z}/2$ grading γ of the Hilbert space $\mathcal{H}$ arise when dealing with *even* spectral triples. By construction the $\mathbb{Z}/2$ grading γ anticommutes with the operator D and makes it possible to define the *product* with any other spectral triple $(\mathcal{A}', \mathcal{H}', D')$ as follows

$$\mathcal{A}'' = \mathcal{A} \otimes \mathcal{A}'\,, \quad \mathcal{H}'' = \mathcal{H} \otimes \mathcal{H}'\,, \quad D'' = D \otimes 1 \,+\, \gamma \otimes D'\,. \tag{1.5}$$

We shall show that the prescription of t'Hooft-Veltman and Breitenlohner-Maison corresponds to taking the product in the above sense of the standard geometry of (Euclidean) space-time by a very specific spectral triple X_z of dimension $z \in \mathbb{C}$ with $\Re(z) > 0$, *i.e.* whose dimension spectrum is reduced to the complex number z. The effect of taking the product by X_z is to shift by z the dimension spectrum of the original geometric space thus removing the singularities at the integral points.
A first adjustment of the general theory of spectral triples will be used to simplify the computations (but it is not essential). It consists in allowing for "type II" situations (*cf.* [3], [6], [7]) where the trace used is no longer the traditional type I trace on $\mathcal{L}(\mathcal{H})$ but is the trace on a type II_∞ von-Neumann algebra. This passage from type I to type II makes it possible to include the proper "infrared" behaviour at little expanse.
Another more serious modification is that when $z \in \mathbb{C}$ is no longer a real number it is necessary to drop the hypothesis that D is self-adjoint.
After describing the spaces X_z we shall show that the anomalies of the gauge theory canonically associated to a spectral triple (*cf.* [11], [12]) are finite and can be explicitely computed with the same terms as in the local index formula in noncommutative geometry of [18].
The compatibility of anomalies with NCG has been known and exploited for quite some time but so far the lack of a conceptual meaning for DimReg prevented people from making direct contact with the computations as performed in physics.
Our approach makes it possible, in particular, to apply DimReg in the general framework of noncommutative geometry.

1.1. **The cubic anomaly and DimReg.**

The cubic anomaly is central to the problem of renormalizability of a theory like the standard model, which involves a gauge theory. The problem is that, in such a theory, one needs to ensure that the renormalized Lagrangian is still invariant under the same group of local gauge symmetries that leave

the bare Lagrangian invariant. Thus, it is necessary, in order to have renormalizability and unitarity, that gauge invariance is preserved at each order in the renormalized perturbation series.
In general the Ward identities determine relations between the Green functions arising from symmetries of the Lagrangian. These influence the renormalizability of theories with nontrivial symmetries, by ensuring cancellations of divergences between different sectors of the theory. There is a complicated interplay of symmetry and renormalization, whereby the Ward identities can be affected by higher order corrections, which can generate anomalous terms. In the case of gauge theories, the Ward identities (or Slavnov–Taylor identities) are related to the symmetries by BRS gauge transformations. In gauge theories, the problem arises primarily from fermions coupling to gauge fields. This manifests itself in the Adler–Bell–Jackiw (ABJ) anomaly for electro-weak interactions. The ABJ anomaly can be formulated in terms of path integrals as the fact that the measure is not invariant under the action of γ_5. Higher order terms do not contribute any further corrections.
Dimensional Regularization (DimReg) is considered the best regularization method, when dealing with the problem of gauge invariance, because it does not change the formal expression of the Ward identities, while analytically continuing the dimension $D = 4$ to complex points $D - z = d \in \mathbb{C}$.
In the DimReg scheme, the ABJ anomaly is closely related to the problem of defining γ_5 for complex dimensions. In fact, while for diagrams with open fermion lines it is possible to use a formal definition of γ_5 in arbitrary complex dimension, which satisfies anticommutation relations with all the γ_μ (*cf. e.g.* [9]), when one considers cases like the triangle diagram, with closed fermion loops, one runs into the further problem of making this formal definition of γ_5 compatible with the trace.
In the case of electro-weak interactions where the vector boson associated to the weak interaction acquires mass through the Higgs mechanism, renormalizability is obtained by showing that the theory is obtained via a formal transformation of the fields from a renormalizable gauge theory (the latter can have non-physical particles that may spoil unitarity, *cf.* [25]). For such a formal transformation to be possible, one needs to be able to renormalize the gauge theory in a gauge invariant way. This can be spoiled by the presence of the ABJ anomaly, as shown in [23].
The question of constructing gauge theories that are free of anomalies, both for electro-weakand strong interactions, was considered in [22], where conditions are given on which gauge groups and representations will satisfy the vanishing of the triangle anomaly. In the standard electro-weak theory with $SU(2) \times U(1)$, the condition implies relations between the hypercharges.
In fact, an interesting aspect of the equations imposed by the vanishing of the cubic anomalies is that it imposes constraints on the hypercharges and on the Weyl representations. For instance, it is shown in [21] and [28] that requiring the theory to be free of anomalies suffices to fix these data uniquely. This could be an interesting point if one wants to use the vanishing of the anomalies as a *constraint on the geometry.*

2. The spaces X_z

We look for a spectral triple whose $D = D_z$ fulfills the following basic equation,

$$\mathrm{Tr}(e^{-\lambda D^2}) = \pi^{z/2}\lambda^{-z/2}, \quad \forall \lambda \in \mathbb{R}_+^*\,, \tag{2.1}$$

corresponding, using (1.1), to

$$\mathrm{Tr}(e^{-\lambda D^2}) = \int e^{-\lambda\, k^2}\, d^z k \quad \forall \lambda \in \mathbb{R}_+^*\,. \tag{2.2}$$

Let then Z be a self-adjoint operator affiliated to a type II_∞ factor N and with spectral measure given by

$$\mathrm{Tr}_N(1_E(Z)) = \frac{1}{2}\int_E dy \tag{2.3}$$

for any interval $E \subset \mathbb{R}$, where 1_E is the characteristic function of E.
We then let, for any complex number $z \in \mathbb{C}$ with $\Re(z) > 0$,

$$D_z = \rho(z)\, F\, |Z|^{1/z} \tag{2.4}$$

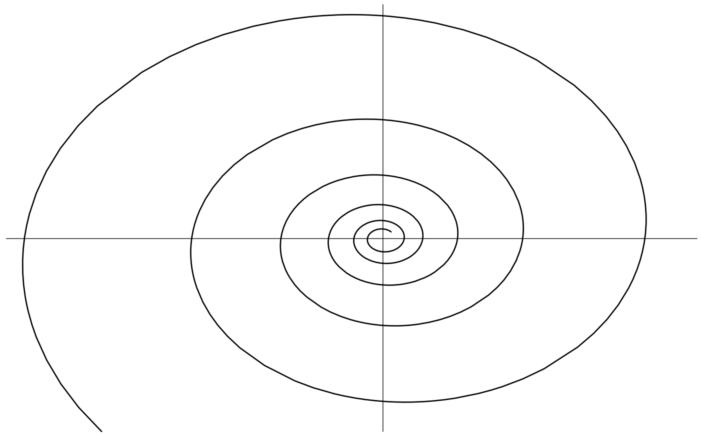

FIGURE 1. The spectrum of D_z^2 for $\Im(z) \neq 0$.

where F is the sign of the operator Z, $|Z|$ its absolute value, and the complex power $|Z|^{1/z}$ is taken by the usual functional calculus. We take the normalization constant $\rho(z)$ to be

$$\rho(z) = \pi^{-\frac{1}{2}} \, (\Gamma(\frac{z}{2} + 1))^{\frac{1}{z}}. \tag{2.5}$$

Proposition 2.1. *The operator D_z of* (2.4)*, with the normalization factor* (2.5) *satisfies* (2.1)*, after imposing an infrared cutoff on the Trace in the case $\Im(z) \neq 0$. The zeta function of D_z has a single (simple) pole at $s = z$ and is absolutely convergent in the half space $\Re(s/z) > 1$.*

Proof. We first check it in the case $z \in \mathbb{R}_+^*$. One has

$$\mathrm{Tr}_N(e^{-\lambda D_z^2}) = \frac{1}{2} \int e^{-\lambda \, \rho^2 \, |y|^{2/z}} \, dy.$$

Thus, by setting $\rho^2 \, |y|^{2/z} = u$, one gets $dy = \rho^{-z} \, \frac{z}{2} \, u^{\frac{z}{2}-1} \, du$ and

$$\mathrm{Tr}_N(e^{-\lambda D_z^2}) = \rho^{-z} \, \frac{z}{2} \int_0^\infty e^{-\lambda \, u} \, u^{\frac{z}{2}-1} \, du = \rho^{-z} \, \Gamma(\frac{z}{2} + 1) \, \lambda^{-z/2},$$

so that one obtains as required

$$\mathrm{Tr}_N(e^{-\lambda D_z^2}) = \pi^{z/2} \, \lambda^{-z/2}, \quad \forall \lambda \in \mathbb{R}_+^* \,. \tag{2.6}$$

Note that $\rho(z)$ is just a normalization factor and it is well behaved in the limit $z \to 0$, since we have

$$\frac{1}{z} \, \log \Gamma(\frac{z}{2} + 1) = -\frac{\gamma}{2} + \frac{\pi^2}{48} \, z + \cdots + (-1)^n \frac{\zeta(n)}{n \, 2^n} \, z^{n-1} + \cdots \tag{2.7}$$

In particular, we have $\rho(0) = e^{-\gamma/2}$. (Here γ denotes the Euler constant, not to be confused with the γ used everywhere else for the grading!)
The above computation of $\mathrm{Tr}_N(e^{-\lambda D_z^2})$ makes perfect sense when $z \in \mathbb{R}_+^*$, but it requires more care when the imaginary part of z is non-zero. In fact, in that case the spectrum of D_z^2 is a spiral in $\mathbb{C}$ (*cf.* Figure 1), so that $e^{-\lambda D_z^2}$ no longer belongs to the domain of the trace Tr_N.
One can define the trace by means of a suitable regularization, but it is simpler to directly deal with the zeta function

$$\mathrm{Tr}_N((D_z^2)^{-s/2}) \tag{2.8}$$

instead of the theta function

$$\mathrm{Tr}_N(e^{-\lambda D_z^2}). \tag{2.9}$$

We need to impose an infrared cutoff, that is, we perform the integral

$$\mathrm{Tr}_N((D_z^2)^{-s/2}) = \frac{1}{2}\int (\rho^2\,|y|^{2/z})^{-s/2}\,dy \tag{2.10}$$

in the region outside $|y| < 1$. This gives, using Tr'_N to indicate the cutoff,

$$\mathrm{Tr}'_N\left((D_z^2)^{-s/2}\right) = \rho^{-s}\int_1^\infty u^{-s/z}\,du = \rho^{-s}\frac{z}{s-z} \tag{2.11}$$

which, as a function of s, has a single (simple) pole at $s = z$ and is absolutely convergent in the half space $\Re(s/z) > 1$. □

The algebra $\mathcal{A}'$ for the spectral triple defining X_z will play no role below except for the unit element $1 \in \mathcal{A}'$. One could include in $\mathcal{A}'$ any operator a such that $[D_z, a]$ is bounded and both a and $[D_z, a]$ are smooth for the "geodesic flow"

$$T \mapsto e^{it|D_z|}\,T\,e^{-it|D_z|} \tag{2.12}$$

and it is a relevant question to show that a suitable dimension of the spectrum of such operators is bounded above by $1/\Re(1/z) = x + y^2/x$ for $z = x + iy$.

The dimension spectrum of X_z is reduced to the single point z as shown by (2.11).

In order to justify the use of the infrared cutoff in (2.10), one can make the following observation. Instead of the operator $|Z|$, consider an operator $f(|Z|)$, where $f \in C^\infty(\mathbb{R}^+)$ has the following properties: f is positive, bounded below by a strictly positive real number, one has $f(x) = x$ for $x > 1/2$, and $f(x)$ is smaller than $1/2$ for $x \leq 1/2$. Then one alters (2.6) with an error term

$$R(\lambda, z) = \int_0^{\frac{1}{2}} (e^{-\lambda\,\rho^2\,|y|^{2/z}} - e^{-\lambda\,\rho^2\,f(|y|)^{2/z}})\,dy. \tag{2.13}$$

This is an analytic function of λ. In the region $\Re(z) > 0$, $|z| < 1$ and $|\lambda| < 2^{\Re(2/z)}$, it satisfies the estimate

$$|R(\lambda, z)| < C\,|\lambda|\,2^{\Re(-2/z)}, \tag{2.14}$$

for some finite constant $C > 0$. This does not affect the pole parts in z for all computations involving DimReg, as we shall check below. With $\rho(z)Ff(|Z|^{1/z})$ instead of $D_z = \rho(z)F|Z|^{1/z}$ the zeta function $\mathrm{Trace}_N((D_z^2)^{-s/2})$ is well defined and it differs from (2.11) by an entire function of s which does not alter the dimension spectrum.

3. An arithmetic realization of X_z

To obtain a concrete realization of the spaces X_z we shall take for N the type II_∞ factor obtained from the noncommutative space $\mathcal{L}$ of commensurability classes of $\mathbb{Q}$-lattices in $\mathbb{R}$ (*cf.* [4], [16], [17]). The unbounded element Y will then be simply given by the inverse of the covolume.

A $\mathbb{Q}$-lattice in $\mathbb{R}$ is a pair (Λ, ϕ), with Λ a lattice in $\mathbb{R}$, and $\phi : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}\Lambda/\Lambda$ a homomorphism of abelian groups. Two $\mathbb{Q}$-lattices (Λ_j, ϕ_j) are commensurable iff the lattices Λ_j are commensurable (*i.e.* $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$) and the maps ϕ_j are equal modulo $\Lambda_1 + \Lambda_2$.

We let $\mathbb{R}$ denote the equivalence relation of commensurability on the space of $\mathbb{Q}$-lattices in $\mathbb{R}$. It is by construction an étale groupoid with space of units $\mathcal{R}^{(0)}$ the space of $\mathbb{Q}$-lattices in $\mathbb{R}$. We identify the latter with $\hat{\mathbb{Z}} \times \mathbb{R}_+^*$ by means of the map

$$L(\rho, \lambda) = (\lambda^{-1}\mathbb{Z}, \lambda^{-1}\rho)), \tag{3.1}$$

where we use the canonical isomorphism

$$\hat{\mathbb{Z}} \sim \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}). \tag{3.2}$$

We can then label elements of the groupoid $\mathcal{R}$ by triples (r,ρ,λ) with $r\in\mathbb{Q}_+^*$ and $\rho\in\hat{\mathbb{Z}}$, such that $r\,\rho\in\hat{\mathbb{Z}}$, and $\lambda\in\mathbb{R}_+^*$. The map (3.1) extends to

$$L(r,\rho,\lambda)=((r^{-1}\,\lambda^{-1}\mathbb{Z},\,\lambda^{-1}\rho)\,,(\lambda^{-1}\mathbb{Z},\lambda^{-1}\rho)),\quad \forall (r,\rho,\lambda)\in\mathcal{R}. \tag{3.3}$$

The source and range maps are then given by

$$s(r,\rho,\lambda)=\,(\rho,\lambda)\,,\quad r(r,\rho,\lambda)=\,(r\,\rho,r\,\lambda), \tag{3.4}$$

and the composition by

$$(r_1,\rho_1,\lambda_1)\circ(r_2,\rho_2,\lambda_2)=(r_1\,r_2,\rho_2,\lambda_2),\quad \text{if}\quad r_2\rho_2=\rho_1,\quad r_2\lambda_2=\lambda_1. \tag{3.5}$$

It is convenient to use a more compact notation and consider the pair (ρ,λ) as an adele, *i.e.* as an element of the subset

$$\mathbb{A}_1^+=\,\hat{\mathbb{Z}}\times\mathbb{R}_+^*\subset\mathbb{A}_{\mathbb{Q}}, \tag{3.6}$$

where $\mathbb{A}_{\mathbb{Q}}=\mathbb{A}_{\mathbb{Q},f}\times\mathbb{R}$ denotes the adeles of $\mathbb{Q}$. Thus, we identify $\mathcal{R}$ with the groupoid obtained by restriction to $\mathbb{A}_1^+\subset\mathbb{A}_{\mathbb{Q}}$ of the groupoid $\mathbb{A}_{\mathbb{Q}}\rtimes\mathbb{Q}_+^*$ obtained from the action of $\mathbb{Q}_+^*$ by multiplication on $\mathbb{A}_{\mathbb{Q}}$.
In fact, it is slightly more convenient to use the equivalent groupoid

$$G_1=\mathbb{A}_1\rtimes\mathbb{Q}^* \tag{3.7}$$

obtained by restriction of the groupoid $\mathbb{A}_{\mathbb{Q}}\rtimes\mathbb{Q}^*$ (which arises in the construction of the adele class space in [14], [15]) to the subset

$$\mathbb{A}_1=\,\hat{\mathbb{Z}}\times\mathbb{R}^*\subset\mathbb{A}_{\mathbb{Q}}. \tag{3.8}$$

The convolution of functions is given by

$$f_1*f_2(r,a):=\sum f_1(rs^{-1},s\,a)\,f_2(s,a). \tag{3.9}$$

The adjoint of f is given by

$$f^*(r,a):=\overline{f(r^{-1},r\,a)}\,. \tag{3.10}$$

The restriction to $\mathbb{A}_1$ of the additive Haar measure da of adeles is preserved by the action of $\mathbb{Q}^*$, hence it gives rise to a natural trace Tr_N on the crossed product algebra. More explicitely this is of the form

$$\mathrm{Tr}_N(f):=\int_{\mathbb{A}_1}f(1,a)\,da. \tag{3.11}$$

We denote by N the type II_∞ factor obtained from this trace. It is dual (in the sense of the duality introduced in [10]) to the type III_1 factor associated to the "critical" (temperature $\beta=1$) KMS state on the BC system (*cf.* [4], [16]). In more geometric terms, the space $\mathbb{A}_1^+$ is the total space of a principal $\mathbb{R}_+^*$-bundle over $\hat{\mathbb{Z}}$ corresponding to the difference between considering $\mathbb{Q}$-lattices or $\mathbb{Q}$-lattices up to scale. Moreover, the equivalence relation of commensurability is compatible with the projection.

Let us then define an unbounded element of N by considering the function on G_1

$$Y(1,\rho,\lambda)=\,\lambda\,,\quad \text{and}\quad Y(r,\rho,\lambda)=\,0\quad\text{if}\quad r\neq 1\,. \tag{3.12}$$

One checks that (with a suitable normalization of da) the corresponding unbounded element of N has the correct spectral measure with respect to the trace Tr_N.

4. The "evanescent" gauge potentials

In noncommutative geometry the gauge bosons appear as derived objects through the simple issue of Morita equivalence. In fact, they appear as "inner fluctuations of the metric" induced by self Morita equivalences of the algebra *cf.* [13].

Indeed, let $\mathcal{E}$ be a finite, projective, hermitian right $\mathcal{A}$-module. In order to define the analogue of the operator D for the algebra of endomorphisms

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E}) \tag{4.1}$$

one needs the choice of a *hermitian connection* on $\mathcal{E}$. Such a connection ∇ is a linear map

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D,$$

where $\Omega^1_D \subset \mathcal{L}(\mathcal{H})$ is the $\mathcal{A}$-bimodule of operators of the form

$$A = \Sigma a_i[D, b_i], \quad a_i, b_i \in \mathcal{A}\,. \tag{4.2}$$

The connection ∇ satisfies the rules (*cf.* [11])

$$\nabla(\xi a) = (\nabla\xi)a + \xi \otimes da, \quad \forall \xi \in \mathcal{E}, \;\; \forall a \in \mathcal{A}. \tag{4.3}$$

$$(\xi, \nabla\eta) - (\nabla\xi, \eta) = d(\xi, \eta), \quad \forall \xi, \eta \in \mathcal{E}, \tag{4.4}$$

where $da = [D, a]$. The minus sign in (4.4) comes from the equality $d(a^*) = -(da)^*$.

Any algebra $\mathcal{A}$ is Morita equivalent to itself via $\mathcal{E} = \mathcal{A}$. When one applies the construction above in this context one gets the inner deformations of the spectral geometry. These replace the operator D by

$$D \mapsto D + A, \tag{4.5}$$

where $A = A^*$ is an arbitrary selfadjoint operator of the form (4.2). Here we disregard the real structure for simplicity, since the latter can then easily be taken care of by replacing the algebra $\mathcal{A}$ by the tensor product $\mathcal{A} \otimes \mathcal{A}^\circ$ of $\mathcal{A}$ by its opposite algebra $\mathcal{A}^\circ$.

The fact that one has such nontrivial perturbations of the spectral geometry reflects the lack of invariance of the operator D under the action of the unitary group $\mathcal{U}$ of $\mathcal{A}$. In fact, replacing D by $u\,D\,u^*$ for $u \in \mathcal{U}$ amounts to adding to D the gauge potential $u[D, u^*] \in \Omega^1_D$. More generally, the gauge transformations are given by

$$u\,(D + A)\,u^* = \; D + \; \alpha_u(A)\,, \quad \text{with} \quad \alpha_u(A) = \; u[D, u^*] + \; u\,A\,u^*, \quad \forall u \in \mathcal{U}\,. \tag{4.6}$$

4.1. **Chiral gauge transformations.**

When the basic algebra $\mathcal{A}$ is a $\mathbb{Z}/2$-graded algebra, all the above extends trivially, by replacing everywhere the commutators $[D, a]$ with *graded* commutators

$$[D, a]_- := \; D\,a - \; (-1)^{\deg(a)} a\,D. \tag{4.7}$$

Here the subscript $_-$ reminds one of the use of graded commutators.

This applies in particular when one replaces the original algebra $\mathcal{A}$ with the algebra $\tilde{\mathcal{A}}$ generated (in the even case) by $\mathcal{A}$ and γ. As an algebra, $\tilde{\mathcal{A}}$ is isomorphic to the direct sum of two copies of $\mathcal{A}$ by

$$a + \; b\,\gamma \mapsto (a + b, \, a - b).$$

We endow it with the $\mathbb{Z}/2$-grading $\theta \in \mathrm{Aut}(\tilde{\mathcal{A}})$ such that $\theta(\gamma) = -\gamma$, while θ is the identity on $\mathcal{A}$. The unitary group $\tilde{\mathcal{U}}$ of $\tilde{\mathcal{A}}$ now contains γ and it is the group of *chiral* gauge transformations. However, the anticommutation of γ with D shows that this procedure does not generate really new gauge potentials, but simply multiplies the existing ones by an arbitrary power of γ.

The situation becomes much more interesting when one keeps the algebra $\tilde{\mathcal{A}}$, but one replaces the original spectral triple by its product (1.5) with the space X_z.

Thus, we start with a spectral triple $(\mathcal{A},\mathcal{H},D)$ and form its product with the (type II_∞) spectral triple defining the space X_z. With the notations of (1.5) we therefore let $D'=D_z$, and we adopt the following shorthand notation (compatible with the physics notation of [9])

$$\bar{D}=D\otimes 1\,,\quad \hat{D}=\gamma\otimes D'\,,\quad D''=\bar{D}+\hat{D}\,. \tag{4.8}$$

The fact that the spectral triple defining the space X_z is of type II_∞ introduces some technical subtleties that can be ignored at first reading (*cf.* [3], [6], [7] for the detailed general theory).
We let the algebra $\mathcal{A}$ act on $\mathcal{H}\otimes\mathcal{H}'$ by

$$a\mapsto a\otimes 1 \tag{4.9}$$

Thus, one obtains a type II_∞ spectral triple for the algebra $\mathcal{A}$. We then extend the algebra to the graded algebra $\tilde{\mathcal{A}}$ generated by $\mathcal{A}$ and γ, with the action on $\mathcal{H}\otimes\mathcal{H}'$ still given by (4.9).
The odd element $\gamma\in\tilde{\mathcal{A}}$ no longer anticommutes (*i.e.* graded-commutes) with D'', since $\gamma\otimes 1$ commutes with $\hat{D}=\gamma\otimes D'$. Thus, one gets a new non-trivial gauge potential of the form

$$B=[D'',\gamma]_-=2\,\gamma\,\hat{D}\,. \tag{4.10}$$

This gauge potential accounts for the lack of invariance of D'' under the "chiral" gauge transformation $\gamma\in\tilde{\mathcal{U}}$ and it corresponds, in the language of QFT, to the divergence $\partial_\mu\, j_5^\mu$ of the axial current j_5^μ.
One needs special care in discussing this point, since it involves Euclidean Fermi fields ξ and η, which one needs to treat as independent integration variables in the Euclidean functional integral (see [8], section 5.2 of "The use of instantons").
Thus, the fermionic part of the Lagrangian in the action functional is of the form

$$\mathcal{L}_{\text{fermions}}=\langle\eta\,,\,D''\xi\rangle \tag{4.11}$$

When extending the theory to $\mathbb{Z}/2$-graded algebras, one defines the gauge transformations α_u on the fermions, for $u\in\tilde{\mathcal{U}}$ using the $\mathbb{Z}/2$-grading θ as

$$\alpha_u(\xi)=u\,\xi\,,\quad \alpha_u(\eta)=\theta(u)\,\eta\,. \tag{4.12}$$

It follows that the lack of invariance of the action (4.11) is still accounted for by the gauge potential given by the graded commutator with u. In particular if γ (which is odd) would anticommute with D'', then the action (4.11) would be invariant under the corresponding chiral transformation. This is not the case and the variation at order one of $\mathcal{L}_{\text{fermions}}$ under the chiral gauge transformation $u=e^{i\omega\gamma}$, for $\omega=\omega^*\in\mathcal{A}$, is given by

$$\delta\,\mathcal{L}_{\text{fermions}}=\langle\eta,\,(i\,[D,\omega]\,\gamma+i\,\omega\,B)\,\xi\rangle \tag{4.13}$$

with $B=2\gamma\,\hat{D}$ as in (4.10).
We shall use the physics terminology *evanescent* to qualify the gauge potentials of the form $E=\omega\,B$. The origin of the terminology is clear since they vanish when the extra dimension z is set to 0.

5. The local index cocycle

Let us recall briefly the local index formula in the context of noncommutative geometry [18], which we shall need here in the even case.
We let $(\mathcal{A},\mathcal{H},D)$ be a finitely summable even spectral triple. The analogue of the geodesic flow is given by the one parameter group of automorphisms of $\mathcal{L}(\mathcal{H})$,

$$t\mapsto F_t(T)=e^{it|D|}\,T\,e^{-it|D|}\,. \tag{5.1}$$

We shall say that an operator T on $\mathcal{H}$ is *smooth* iff the above map is smooth, *i.e.* if it belongs to $C^\infty(\mathbb{R},\mathcal{L}(\mathcal{H}))$. We also define

$$OP^0:=\{T\in\mathcal{L}(\mathcal{H})\,;\;T\text{ is smooth}\}\,. \tag{5.2}$$

We say that the spectral triple $(\mathcal{A},\mathcal{H},D)$ is *regular* if it satisfies the condition

$$a\;\text{ and }\;[D,a]\in OP^0,\quad\forall\,a\in\mathcal{A}\,. \tag{5.3}$$

(Notice that $\mathcal{A}$ denotes the dense subalgebra of elements that have bounded commutator with D and not the C^*-algebra of the spectral triple.)

As we already mentioned, in the context of spectral triples, the usual notion of *dimension* of a space is replaced by the *dimension spectrum* , which is the subset Σ of $\{z \in \mathbb{C}, \Re(z) \geq 0\}$ of singularities of the analytic functions

$$\zeta_b(z) = \mathrm{Tr}\,(b|D|^{-z}), \quad \Re(z) > p\ , \quad b \in \mathcal{B}\,. \tag{5.4}$$

Here p is the crude dimension provided by the rate of growth of the eigenvalues of D, and $\mathcal{B}$ denotes the algebra generated by $\delta^k(a)$ and $\delta^k([D,a])$, for $a \in \mathcal{A}$, with

$$\delta(T) = [|D|, T] \tag{5.5}$$

the derivation that generates the geodesic flow.

The local index theorem is of the following form (*cf.* [18]):

Theorem 5.1. *Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular finitely summable spectral triple with simple dimension spectrum. The following holds.*

- *The equality*

$$⨍ P := \mathrm{Res}_{z=0}\,\mathrm{Tr}\,(P|D|^{-z}) \tag{5.6}$$

  defines a trace on the algebra generated by $\mathcal{A}$, $[D, \mathcal{A}]$ and $|D|^z$, with $z \in \mathbb{C}$.
- *Assuming $⨍\gamma\, a = 0$ for all $a \in \mathcal{A}$, the formula*

$$\varphi_0(a) = \lim_{z \to 0} \mathrm{Tr}(\gamma\, a\, |D|^{-z}), \quad \forall a \in \mathcal{A}, \tag{5.7}$$

  defines a linear form φ_0 on $\mathcal{A}$.
- *For $n > 0$ an even integer, there is only a finite number of non-zero terms in*

$$\varphi_n(a^0, \ldots, a^n) := \sum_k c_{n,k} ⨍ \gamma\, a^0 [D, a^1]^{(k_1)} \ldots [D, a^n]^{(k_n)}\, |D|^{-n-2|k|}, \quad \forall a^j \in \mathcal{A}. \tag{5.8}$$

  Here we are using the notation $T^{(k_i)} = \nabla^{k_i}(T)$ with $\nabla(T) = D^2 T - T D^2$. The summation index k is a multi-index with $|k| = k_1 + \ldots + k_n$, and the coefficients $c_{n,k}$ are given by the formulae[1]

$$c_{n,k} = \frac{(-1)^{|k|}}{2}\,(k_1! \ldots k_n!)^{-1}\,((k_1+1)\ldots(k_1+k_2+\ldots+k_n+n))^{-1}\,\Gamma\,(|k| + n/2)\,. \tag{5.9}$$

- *The expression* (5.8) *defines the even components $(\varphi_n)_{n=0,2,\ldots}$ of a cocycle in the (b, B)-bicomplex of $\mathcal{A}$.*
- *The pairing of the cyclic cohomology class $(\varphi_n) \in HC^*(\mathcal{A})$ with $K_0(\mathcal{A})$ gives the Fredholm index of D with coefficients in $K_0(\mathcal{A})$.*

One of the ingredients in the proof of Theorem 5.1 will be quite useful below and we recall it here (*cf.* [18]). For any $r \in \mathbb{R}$, one lets

$$OP^r = \{T\,;\ |D|^{-r}\,T \in OP^0\}\,. \tag{5.10}$$

We then have the following result (*cf.* [18]):

Lemma 5.2. *Let $T \in OP^0$ and $n \in \mathbb{N}$.*

(1) $\nabla^n(T) \in OP^n$
(2) $D^{-2}\,T = \sum_0^n (-1)^k\, \nabla^k(T)\, D^{-2k-2} + R_n$, *with*

$$R_n = \ (-1)^{n+1}\, D^{-2}\, \nabla^{n+1}(T)\, D^{-2n-2}\ \in OP^{-n-3}. \tag{5.11}$$

[1] the trace τ_0 of [18] Proposition II.1 is $\frac{1}{2}⨍$

6. Finiteness of anomalous graphs and relation with residues

Given a regular even spectral triple $(\mathcal{A},\mathcal{H},D)$ we denote by $OP(\mathcal{A},\mathcal{H},D)$ the algebra generated by $\mathcal{A}$, D and γ. The following lemma will suffice to prove the finiteness of the anomalous graphs.

Lemma 6.1. *Suppose given a regular spectral triple $(\mathcal{A},\mathcal{H},D)$. Let P be an element of $OP(\mathcal{A},\mathcal{H},D)$. For $n>k>0$ and in the limit $z\to 0$, one has*

$$\mathrm{Tr}\,(\hat D^{2k}\,(P\otimes 1)\,D''^{-2n}) = -\frac{1}{2}\,B(k,n-k)\,⨍ P\,D^{-2(n-k)}\,,\quad B(p,q)=\frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\,.$$

Proof. For $z\neq 0$ one has

$$\mathrm{Tr}\,(\hat D^{2k}\,(P\otimes 1)\,D''^{-2n}) = \frac{1}{\Gamma(n)}\int_0^\infty \mathrm{Tr}(\hat D^{2k}\,(P\otimes 1)\,e^{-t\bar D^2}\,e^{-t\hat D^2})\,t^{n-1}\,dt$$

We also have

$$\mathrm{Tr}_N(D_z^{2k}\,e^{-tD_z^2}) = \frac{z\,(z+2)\cdots(z+2k-2)}{2^k}\,\pi^{z/2}\,t^{-z/2-k},\quad \forall t\in\mathbb{R}_+^*\,, \tag{6.1}$$

while

$$\int_0^\infty e^{-tD^2}\,t^{n-1-z/2-k}\,dt = \Gamma(n-z/2-k)\,|D|^{z-2(n-k)}. \tag{6.2}$$

One has $\hat D^{2k}\,(P\otimes 1) = P\otimes D_z^{2k}$ thus using the factorization of the trace and (6.1) and (6.2), we get

$$\mathrm{Tr}\,(\hat D^{2k}\,(P\otimes 1)\,D''^{-2n}) = \pi^{z/2}\,\frac{\prod_0^{k-1}(z/2+j)}{\Gamma(n)}\,\Gamma(n-z/2-k)\,\mathrm{Tr}(P\,|D|^{z-2(n-k)})\,,$$

which gives the required result since when $z\to 0$ one has,

$$\pi^{z/2}\,\frac{\prod_0^{k-1}(z/2+j)}{\Gamma(n)}\,\Gamma(n-z/2-k)\sim B(k,n-k)\,\frac{z}{2}\,,$$

with the minus sign coming from the nuance between $|D|^z$ and $|D|^{-z}$. ☐

As a corollary of Lemma 6.1 one gets the following expression for the residue at the simple pole $z=0$ in the DimReg expression of one loop graphs with only fermionic internal lines, we work as above with a regular spectral triple with simple dimension spectrum.

Proposition 6.2. *Let $A\in\Omega_D^1$ be a gauge potential and $n>0$. Then the expression*

$$\mathrm{Tr}(((A\otimes 1)\,D''^{-1})^n) \tag{6.3}$$

has at most a simple pole at $z=0$ with residue given by

$$\mathrm{Res}_{z=0}\,\mathrm{Tr}(((A\otimes 1)\,D''^{-1})^n) = -⨍ (A\,D^{-1})^n \tag{6.4}$$

Proof. One has

$$D''^{-1} = D''\,D''^{-2} = (\bar D+\hat D)\,D''^{-2} \tag{6.5}$$

and

$$D''^2 = \bar D^2+\hat D^2. \tag{6.6}$$

Using Lemma 5.2 (2) to move the terms D''^{-2} to the right we can express (6.3) as a sum of terms of the form

$$\mathrm{Tr}((A\otimes 1)\,(\bar D+\hat D)\,\ldots(A^{(k_j)}\otimes 1)\,(\bar D+\hat D)\,)\ldots(A^{(k_n)}\otimes 1)\,(\bar D+\hat D)\,D''^{-2k}) \tag{6.7}$$

where we use the notation $A^{(m)} = \nabla^m(A)$ and $k = n + \sum k_j$. Only finitely many of these terms have a divergent trace at $z = 0$. By construction $\hat{D}$ anticommutes with the A's and commutes with all other terms. Thus one can expand (6.7) in terms of the form

$$\mathrm{Tr}\,(\hat{D}^{2q}\, P\, {D''}^{-2m})$$

By Lemma 6.1 any of these terms with $q > 0$ is regular at $z = 0$. Thus the only contribution to the pole comes from the case $q = 0$. This means that one only takes the terms $\bar{D}$ in the expansion of (6.7) and one is thus dealing with

$$\mathrm{Tr}((P \otimes 1)\, D''^{-2k})\,, \quad P = A\, D\, \ldots A^{(k_j)}\, D \ldots A^{(k_n)}\, D \tag{6.8}$$

As in the proof of Lemma 6.1 one has

$$\mathrm{Tr}\,((P \otimes 1)\, {D''}^{-2k}) = \frac{1}{\Gamma(k)} \int_0^\infty \mathrm{Tr}((P \otimes 1)\, e^{-t\bar{D}^2}\, e^{-t\hat{D}^2})\, t^{k-1}\, dt$$

Using the factorization of the trace and (2.1) while

$$\int_0^\infty e^{-tD2}\, t^{k-1-z/2}\, dt = \Gamma(k - z/2)\, |D|^{z-2k}$$

we get

$$\mathrm{Tr}\,((P \otimes 1)\, {D''}^{-2k}) = \pi^{z/2}\, \frac{\Gamma(k - z/2)}{\Gamma(k)}\, \mathrm{Tr}(P\, |D|^{z-2k})\,.$$

Thus the residue at $z = 0$ is given by

$$-\!\!\!\!\!\!\int P\, |D|^{-2k}$$

The computation of $-\!\!\!\!\!\!\int (A\, D^{-1})^n$ gives exactly the same result. Indeed, one again writes

$$D^{-1} = D\, D^{-2}$$

and uses Lemma 5.2 (2) to move the terms D^{-2} to the right, which allows to express $-\!\!\!\!\!\!\int (A\, D^{-1})^n$ as a sum of terms of the form

$$-\!\!\!\!\!\!\int A\, D\, \ldots A^{(k_j)}\, D \ldots A^{(k_n)}\, D\, D^{-2k}$$

which match exactly with the above terms. □

7. The simplest anomalous graphs

We shall now analyze the graphs with only fermionic internal lines which involve linearly the evanescent gauge potential B, starting from the simplest, which is the tadpole, and continuing to more involved cases.

The data we start with are, as above, an even spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with simple dimension spectrum, *i.e.* we use exactly the same setup as in the local index formula of [18] that we recalled above in §5.

7.1. **The Tadpole.**

We start with the simplest graph. This has one loop with a single external leg to which is assigned an evanescent gauge potential $E = \gamma a \hat{D}$ (see Figure 2).

The analytic expression for the tadpole is of the form

$$\mathrm{Tr}(E\, {D''}^{-1}), \tag{7.1}$$

where ${D''}^{-1}$ plays the role of the fermionic propagator.

We assume $-\!\!\!\!\!\!\int \gamma\, a = 0$ for all $a \in \mathcal{A}$ and obtain the following result.

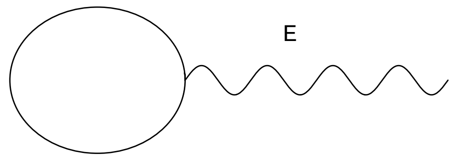

Figure 2. The tadpole.

Lemma 7.1. *Let φ_0 be the linear form (5.7), which is the zeroth order component of the local index cocycle, as in Theorem 5.1. In the limit $z \to 0$, one obtains*

$$\mathrm{Tr}\,(E\,D''^{-1}) = -\,\varphi_0(a), \quad \forall a \in \mathcal{A}. \tag{7.2}$$

Proof. One has

$$D''^{-2} = \int_0^\infty e^{-t\bar{D}^2}\, e^{-t\hat{D}^2}\, dt. \tag{7.3}$$

Thus, for fixed z, one gets using (6.5) and (6.6),

$$\mathrm{Tr}(E\,D''^{-1}) = \mathrm{Tr}(\gamma\, a\hat{D}\,(\bar{D} + \hat{D})\,D''^{-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\,D''^{-2}). \tag{7.4}$$

In fact, the terms with an odd number of $\hat{D}$'s have zero trace since one can find an involution in $\mathcal{H}'$ which anticommutes with D_z. Using (7.3) this gives

$$\mathrm{Tr}(E\,D''^{-1}) = \int_0^\infty \mathrm{Tr}(\gamma\, a\hat{D}^2\, e^{-t\bar{D}^2}\, e^{-t\hat{D}^2})\,dt \tag{7.5}$$

and the trace factorizes to give

$$\mathrm{Tr}(\gamma\, a\hat{D}^2\, e^{-tD^2}\, e^{-t\hat{D}^2}) = \mathrm{Tr}(\gamma\, a\, e^{-tD^2})\mathrm{Tr}(D_z^2\, e^{-tD_z^2}). \tag{7.6}$$

By the basic equality (2.6) one has

$$\mathrm{Tr}_N(e^{-tD_z^2}) = \pi^{z/2} t^{-z/2}, \quad \forall t \in \mathbb{R}_+^*. \tag{7.7}$$

After differentiating in t this gives

$$\mathrm{Tr}_N(D_z^2\, e^{-tD_z^2}) = \frac{z}{2}\,\pi^{z/2}\, t^{-z/2-1} \quad \forall t \in \mathbb{R}_+^*. \tag{7.8}$$

Thus, (7.5) and (7.6) give

$$\mathrm{Tr}(E\,D''^{-1}) = \frac{z}{2}\,\pi^{z/2} \int_0^\infty \mathrm{Tr}(\gamma\, a\, e^{-tD^2})\, t^{-z/2-1}\; dt \tag{7.9}$$

Moreover, we have

$$\int_0^\infty e^{-tD^2}\, t^{-z/2-1}\; dt = \Gamma(-z/2)\,|D|^z, \tag{7.10}$$

while the limit for $z \to 0$ of $\mathrm{Tr}(\gamma\, a\,|D|^z)$ is $\varphi_0(a)$. Thus, we get the required equality since, for $z \to 0$, we have

$$\frac{z}{2}\,\pi^{z/2}\Gamma(-z/2) \to -1.$$

□

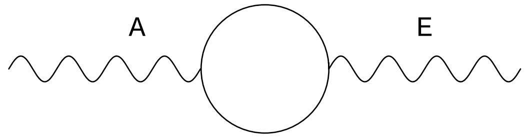

FIGURE 3. The self energy graph.

7.2. **The self-energy graph.**

The next term linear in E now involves a usual gauge potential $A = \sum a_i\,[D'', b_i]$. (see Figure 3). Its analytic expression is given by

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}). \tag{7.11}$$

One has the following result.

Lemma 7.2. *In the limit $z \to 0$, one obtains*

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = \sum_0^\infty (-1)^{n+1}\frac{1}{2n+2}\,⨍\,\gamma\,a\,\nabla^n(B)\,D^{-2n-2} \tag{7.12}$$

where $B = dA + A'$ with

$$dA = \sum [D, a_i]\,[D,\, b_i]\,,\qquad A' = \sum a_i\,\nabla(b_i)\,.$$

and only finitely many terms in the infinite sum are non-zero.

Proof. Using (6.5), one obtains

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = \mathrm{Tr}(\gamma\,a\hat{D}\,(\bar{D}+\hat{D})\,D''^{-2}\,A\,(\bar{D}+\hat{D})\,D''^{-2})\,.$$

Since the terms with an odd number of $\hat{D}$ do not contribute, this gives

$$\mathrm{Tr}(\gamma\,a\hat{D}^2\,D''^{-2}\,A\,\bar{D}\,D''^{-2}) + \mathrm{Tr}(\gamma\,a\hat{D}\,\bar{D}\,D''^{-2}\,A\,\hat{D}\,D''^{-2})$$

In the second term the second appearance of $\hat{D}$ can be put in front provided one cares about signs, since $\hat{D}$ commutes with D''^{-2} and anticommutes with both $A = \sum a_i\,[D'', b_i] = \sum a_i\,[\bar{D}, b_i]$ and $\bar{D}$. Thus, we can write the above as

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = \mathrm{Tr}(\gamma\,a\hat{D}^2\,D''^{-2}\,(A\,\bar{D}\,+\bar{D}\,A)\,D''^{-2})\,.$$

One has

$$(A\,\bar{D}\,+\bar{D}\,A) = B\otimes 1\,,\qquad B = dA + A'\,, \tag{7.13}$$

where

$$dA = \sum [D, a_i]\,[D,\, b_i]\,,\qquad A' = \sum a_i\,\nabla(b_i)\,. \tag{7.14}$$

Thus we can use Lemma 5.2 to move the first D''^{-2} across and get (only finitely many terms will contribute)

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = \sum_0^\infty (-1)^n\,\mathrm{Tr}(\gamma\,a\hat{D}^2\,(\nabla^n(B)\otimes 1)\,D''^{-2n-4})\,.$$

Since $\hat{D}^2$ commutes with $a \in \mathcal{A}$ we then conclude using Lemma 6.1.

□

Notice that, since the regularized trace is a trace outside the dimension spectrum, the order of the terms $E\,D''^{-1}$ and $A\,D''^{-1}$ is irrelevant.
It is worthwhile to double check this in this example by redoing the calculation with the other order. One has

$$\begin{aligned}\mathrm{Tr}(A\,D''^{-1}\,E\,D''^{-1}) &= \mathrm{Tr}(A\,(\bar D+\hat D)\,D''^{-2}\,\gamma\,a\hat D\,(\bar D+\hat D)\,D''^{-2})\\ &= \mathrm{Tr}(A\,\hat D\,D''^{-2}\,\gamma\,a\hat D\,\bar D\,D''^{-2})+\mathrm{Tr}(A\,\bar D\,D''^{-2}\,\gamma\,a\hat D^2\,D''^{-2})\\ &= -\mathrm{Tr}(\gamma\,\hat D^2\,A\,D''^{-2}\,a\,\bar D\,D''^{-2})+\mathrm{Tr}(\gamma\,\hat D^2\,A\,D''^{-2}\,\bar D\,a\,D''^{-2})\\ &= \sum_0^\infty(-1)^n\,\mathrm{Tr}(\gamma\,\hat D^2\,A\,(\nabla^n([D,a])\otimes 1)\,D''^{-2n-4})\\ &= \sum_0^\infty(-1)^{n+1}\frac{1}{2n+2}⨍\gamma\,A\,\nabla^n([D,a])\,D^{-2n-2}\,.\end{aligned}$$

We thus get

$$\mathrm{Tr}(A\,D''^{-1}\,E\,D''^{-1})=\sum_0^\infty(-1)^{n+1}\frac{1}{2n+2}⨍\gamma\,A\,\nabla^n([D,a])\,D^{-2n-2}\,. \tag{7.15}$$

It might seem at first sight that one passes from (7.12) to (7.15) by a simple integration by parts *i.e.* using

$$⨍\gamma\,\nabla^n(B)\,C=(-1)^n⨍\gamma\,B\,\nabla^n(C) \tag{7.16}$$

but it is a bit more subtle. One uses the equality

$$D^{-2n-2}\,a=\sum_0^\infty(-1)^k\,c(n,k)\,\nabla^k(a)\,D^{-2n-2k-2} \tag{7.17}$$

with

$$c(n,0)=1\,,\quad c(n,k)=\frac{\prod_1^k(n+j)}{k!}\,, \tag{7.18}$$

to write

$$⨍\gamma\,a\,\nabla^n(B)\,D^{-2n-2}=\sum_0^\infty(-1)^k\,c(n,k)⨍\gamma\,\nabla^n(B)\,\nabla^k(a)\,D^{-2n-2k-2}$$

which after integration by parts yields

$$⨍\gamma\,a\,\nabla^n(B)\,D^{-2n-2}=\sum_0^\infty(-1)^{n+k}\,c(n,k)⨍\gamma\,B\,\nabla^{(n+k)}(a)\,D^{-2n-2k-2}$$

The desired equality between (7.12) and (7.15) then follows using

$$\sum_0^m(-1)^k\,\frac{c(m-k,k)}{m-k+1}=\int_0^1(t-1)^m\,dt=(-1)^m\,\frac{1}{m+1}$$

and the other integration by parts

$$⨍\gamma\,(DA+AD)\,C=⨍\gamma\,A\,[D,C]\,. \tag{7.19}$$

To conclude we thus have the two equivalent expressions (7.12) and (7.15) for the self-energy graph. We can express both in a more compact notation using the derivation (*cf.* [18]),

$$\Theta(T)=\sum_1^\infty\frac{(-1)^{n+1}}{n}\,\nabla^n(T)\,D^{-2n} \tag{7.20}$$

which generates the one parameter group Ad D^{2z}. Indeed one has

$$D^2\,T\,D^{-2}=T+\nabla(T)\,D^{-2}=(1+\epsilon)(T)$$

and by construction

$$\Theta = \log(1+\epsilon), \quad e^{\Theta} = 1+\epsilon\,. \tag{7.21}$$

We also need the basic "Planck" function

$$\pi(z) = \frac{z}{e^z - 1} \tag{7.22}$$

which is familiar in index theory in the formulas giving the characteristic classes. We can then rewrite the above formulas as

Proposition 7.3. *One has, in the limit $z \to 0$,*

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = -\frac{1}{2}\,⨍\,\gamma\,a\,\pi(\Theta)(B)\,D^{-2} = -\frac{1}{2}\,⨍\,\gamma\,A\,\pi(\Theta)([D,a])\,D^{-2}$$

Proof. From the above it is enough to check that

$$\pi(\Theta) = \sum_0^\infty \frac{(-1)^n}{n+1}\,\epsilon^n \tag{7.23}$$

which follows from (7.21) and (7.22). □

It also follows from the above discussion that for X and Y in the algebra generated by $\mathcal{A}$, γ and D one has the formula of integration by parts

$$⨍\,X\,\pi(\Theta)(Y)\,D^{-2} = ⨍\,Y\,\pi(\Theta)(X)\,D^{-2}\,. \tag{7.24}$$

One can understand this formula as an instance of the KMS condition fulfilled by the functional

$$\varphi_1(X) = ⨍\,X\,D^{-2} \tag{7.25}$$

with respect to the one-parameter group of automorphisms $\sigma_t = e^{i\,t\,\Theta}$. Indeed one has, as required by the KMS_1 condition

$$\varphi_1(X\,\sigma_i(Y)) = \varphi_1(Y\,X)$$

and (7.24) follows from the simple equality

$$\pi(-z) = e^z\,\pi(z)\,,$$

combined with the invariance of φ_1 under σ_t which implies that for any formal series f,

$$\varphi_1(f(\Theta)(X)\,Y) = \varphi_1(X\,f(-\Theta)(Y))\,. \tag{7.26}$$

Given a Hochschild cochain φ of dimension n on an algebra $\mathcal{A}$, it defines (*cf.* [11]) a functional on the universal n-forms $\Omega^n(\mathcal{A})$ by the equality

$$\int_\varphi a_0\,da_1 \cdots da_n = \varphi(a_0,\, a_1, \cdots, a_n) \tag{7.27}$$

When φ is a Hochschild cocycle one has

$$\int_\varphi a\,\omega = \int_\varphi \omega\,a, \quad \forall a \in \mathcal{A} \tag{7.28}$$

The boundary operator B_0 defined on normalized cochains by

$$(B_0\varphi)(a_0,\, a_1, \cdots, a_{n-1}) = \varphi(1, a_0,\, a_1, \cdots, a_{n-1}) \tag{7.29}$$

is defined in such a way that

$$\int_\varphi d\,\omega = \int_{B_0\varphi} \omega \tag{7.30}$$

Proposition 7.3 allows to express the self energy graph in terms of the following two cochain

$$\psi_2(a_0, a_1, a_2) = \frac{1}{4}\,⨍\,\gamma\,a_0\,[D,a_1]\,\pi(\Theta)([D,a_2])\,D^{-2} \tag{7.31}$$

so that

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = -2\int_{\psi_2} A\,da \tag{7.32}$$

Proposition 7.4. (1) *The cochain $B_0\,\psi_2$ is cyclic : $A\,B_0\,\psi_2 = 2\,B_0\,\psi_2$.*
(2) $B\,\psi_2 + b\,\varphi_0 = 0$
(3) *If $b\,\varphi_0 = 0$ one has*

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = -2\int_{\psi_2} dA\,a \tag{7.33}$$

Proof. 1) One has

$$B_0\,\psi_2(a_0,a_1) = \frac{1}{4}⨍\gamma\,[D,a_0]\,\pi(\Theta)([D,a_1])\,D^{-2}$$

which is antisymmetric by (7.24) (γ anticommutes with $[D,a]$).
2) Since D anticommutes with γ and $D[D,a]+[D,a]D = \nabla(a)$ one gets

$$B\,\psi_2(a_0,a_1) = -\frac{1}{2}⨍\gamma\,a_0\,\pi(\Theta)(\nabla(a_1))\,D^{-2} = -\frac{1}{2}⨍\gamma\,a_0\,\Theta(a_1)$$

The coboundary $b\varphi_0$ is given by

$$b\varphi_0(a,b) = \mathrm{Lim}_{s\to 0}\,\mathrm{Tr}(\gamma\,a(b\,|D|^{-s} - |D|^{-s}\,b)) =$$
$$\mathrm{Lim}_{s\to 0}\,\mathrm{Tr}(\gamma\,a\,(1-e^{-\frac{s}{2}\Theta})(b)\,|D|^{-s}) = \frac{1}{2}⨍\gamma\,a\,\Theta(b)$$

3) In that case one has, by 1) and 2), $B_0(\psi_2)=0$ and

$$⨍_{\psi_2}(dA)a - ⨍_{\psi_2} A\,da = ⨍_{\psi_2} d(Aa) = 0$$

by (7.30) thus the result follows from (7.32). ∎

7.3. **Anomalous graphs in dimension 2.**

We shall first deal with the two dimensional case and explain how the result of Proposition 7.4 simplifies in this case. The main result of this section is Theorem 7.8 which shows that the sum of the anomalous graphs is given by the pairing with the local index cocycle.
In the two dimensional case, the top component of the local index cocycle is given by the following Hochschild 2-cocycle

$$\varphi_2(a_0,\,a_1,a_2) = \frac{1}{4}⨍\gamma\,a_0\,[D,a_1]\,[D,a_2]\,D^{-2},\quad \forall a_j\in\mathcal{A}. \tag{7.34}$$

Since one is in dimension 2 all the terms with $n>0$ in (7.23) vanish and one gets $\varphi_2 = \psi_2$.

In case the coboundary of the tadpole vanishes, *i.e.* assuming that $b\varphi_0 = 0$ (7.2), one gets by Theorem 5.1 that φ_2 is a cyclic cocycle and hence $\int_{\varphi_2}$ is a closed graded trace [11]. We thus get,

Corollary 7.5. *In dimension ≤ 2, and if $b\varphi_0 = 0$, φ_2 is a cyclic cocycle and*

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = -2\int_{\varphi_2} a\,dA$$

Next, we consider the original ABJ triangle graph. The computation below will prepare the ground for the four dimensional case and we thus do it in the required generality.
The analytic expression for the triangle graph of Figure 4 is given by

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}\,A\,D''^{-1}). \tag{7.35}$$

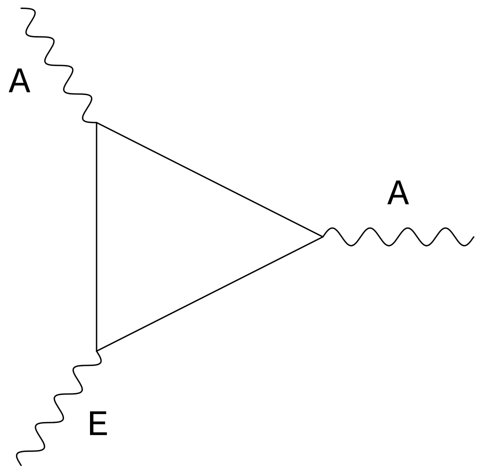

Figure 4. The triangle graph.

This can be polarized in the following manner:

$$\mathrm{Tr}(E\, D''^{-1}\, A_1\, D''^{-1}\, A_2\, D''^{-1}), \tag{7.36}$$

where one can now assume that A_j are monomials of the form

$$A_j = \, a_j\, [D,\, b_j]\,.$$

We first prove the following lemma independent of the dimension,

Lemma 7.6. *Let $P_j \in OP(\mathcal{A}, \mathcal{H}, D)$, for $k = 1$ and $k = 2$ one has in the limit $z \to 0$,*

$$\mathrm{Tr}(\hat{D}^{2k}\, P_0\, D''^{-2}\, P_1\, D''^{-2}\, P_2\, D''^{-2}) =$$
$$\frac{1}{2} \sum (-1)^{a+b+1} \frac{(k-1)!(a+b+2-k)!}{(a+b+2)\, b!\, (a+1)!} ⨍ P_0 \nabla^a(P_1)\, \nabla^b(P_2)\, D^{-2(a+b+3-k)}$$

Proof. One has, using Lemma 5.2, for $P_j \in OP(\mathcal{A}, \mathcal{H}, D)$,

$$D''^{-2}\, P_1\, D''^{-2}\, P_2 \sim \sum d(a,b)\, \nabla^a(P_1)\, \nabla^b(P_2)\, D''^{-2(a+b+2)}$$

where the coefficients are given by

$$d(a,b) = \, (-1)^{a+b} \sum_{0 \le c \le b} \frac{(a+c)!}{a!\, c!} = \, (-1)^{a+b} \frac{(a+b+1)!}{b!\, (a+1)!}$$

Thus using Lemma 6.1 for $n = a + b + 3$ (since there is a remaining term D''^{-2} and the equality

$$\frac{1}{2} \frac{(a+b+1)!}{b!\, (a+1)!} \frac{(k-1)!(a+b+2-k)!}{(a+b+2)!} = \frac{1}{2} \frac{(k-1)!(a+b+2-k)!}{(a+b+2)\, b!\, (a+1)!}$$

one gets the required formula. □

Note that one has

$$D^{-2}\, P_1\, D^{-2}\, P_2 \sim \sum d(a,b)\, \nabla^a(P_1)\, \nabla^b(P_2)\, D^{-2(a+b+2)}$$

but this does not allow to undo the above reordering under the residue since different powers of D'' occur in applying Lemma 6.1.

Proposition 7.7. *In the two dimensional case one has in the limit $z \to 0$ (with $E = \gamma a \hat{D}$)*

$$\mathrm{Tr}(E\, D''^{-1}\, A\, D''^{-1}\, A\, D''^{-1}) = 2 \int_{\varphi_2} a\, A^2 \,.$$

Proof. Using (6.5), one gets

$$\mathrm{Tr}(E\, D''^{-1}\, A\, D''^{-1}\, A\, D''^{-1}) = \mathrm{Tr}(\gamma\, a\hat{D}\, (\bar{D} + \hat{D})\, D''^{-2}\, A\, (\bar{D} + \hat{D})\, D''^{-2}\, A\, (\bar{D} + \hat{D})\, D''^{-2}) \,.$$

Since the terms with an odd number of $\hat{D}$ do not contribute we get terms with 4 occurences of $\hat{D}$, *i.e.*

$$T_4 = \mathrm{Tr}(\gamma\, a\hat{D}^2\, D''^{-2}\, A\, \hat{D}\, D''^{-2}\, A\, \hat{D}\, D''^{-2}) = -\mathrm{Tr}(\gamma\, a\hat{D}^4\, D''^{-2}\, A\, D''^{-2}\, A\, D''^{-2})$$

(where the minus sign comes from the anticommutation of $\hat{D}$ with A) and terms with two occurences of $\hat{D}$ which give the following terms T_j, $j \in \{1, 2, 3\}$

$$T_1 = \mathrm{Tr}(\gamma\, a\hat{D}^2\, D''^{-2}\, A\, \bar{D}\, D''^{-2}\, A\, \bar{D}\, D''^{-2})$$

$$T_2 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, D''^{-2}\, A\, \hat{D}\, D''^{-2}\, A\, \bar{D}\, D''^{-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\, \bar{D}\, D''^{-2}\, A\, D''^{-2}\, A\, \bar{D}\, D''^{-2})$$

$$T_3 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, D''^{-2}\, A\, \bar{D}\, D''^{-2}\, A\, \hat{D}\, D''^{-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\, \bar{D}\, D''^{-2}\, A\, \bar{D}\, D''^{-2}\, A\, D''^{-2})$$

Let us first compute T_4 and T_2. By Lemma 7.6, we get

$$T_4 = \frac{1}{4} ⨍ \gamma\, a\, A^2\, D^{-2}$$

since all the terms with non-zero powers of ∇ give zero since we are in dimension 2.
Next, by Lemma 7.6, we get

$$T_2 = -\frac{1}{4} ⨍ \gamma\, a\, D\, A^2\, D^{-3}$$

We can replace $a\, D \to D\, a$ since the commutator $[D, a]$ is bounded. Thus since the sign changes when we permute D and γ we get using the trace property of the residue,

$$T_2 = \frac{1}{4} ⨍ \gamma\, a\, A^2\, D^{-2}$$

We thus get

$$T_2 + T_4 = 2 \int_{\varphi_2} a\, A^2$$

Next one has by Lemma 7.6,

$$T_1 = -\frac{1}{4} ⨍ \gamma\, a\, A\, D\, A\, D^{-3}$$

Since one is in dimension 2 one can permute the factor D^{-2} which gives

$$T_1 = -\frac{1}{4} ⨍ \gamma\, a\, A\, D^{-1}\, A\, D^{-1}$$

If we let

$$\psi(a_0, a_1) = ⨍ \gamma\, a_0\, [D, a_1]\, D^{-1}$$

we get

$$b\psi(a_0, a_1, a_2) = -⨍ \gamma\, a_0\, [D, a_1]\, [a_2, D^{-1}] = -⨍ \gamma\, a_0\, [D, a_1]\, D^{-1}\, [D, a_2]\, D^{-1}$$

Moreover one has

$$\int_{b\psi} a\, A^2 = -⨍ \gamma\, a\, A\, D^{-1}\, A\, D^{-1}$$

as one checks replacing the first A by $a_1[D, b_1]$ and the second by $a_2[D, b_2]$, and showing that

$$⨍ \gamma\, a\, a_1[D, b_1]\, D^{-1}\, a_2[D, b_2]\, D^{-1} = ⨍ \gamma\, a\, a_1[D, b_1] a_2\, D^{-1}\, [D, b_2]\, D^{-1}$$

$$= -\int_{b\psi} a\, a_1\, db_1\, a_2\, db_2$$

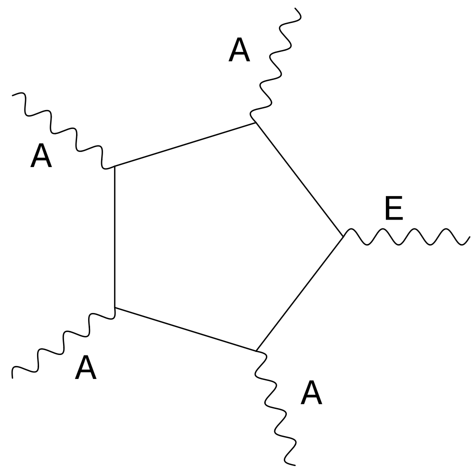

FIGURE 5. The five legs graph.

We have thus shown that

$$T_1 = \frac{1}{4}\int_{b\psi} a\,A^2$$

Since $b\psi$ is a coboundary this ensures that T_1 is not a significant term but in fact we shall now show that it is exactly canceled by T_3, indeed one has

$$T_3 = -\frac{1}{4}\,⨍\,\gamma\,a\,D\,A\,D\,A\,D^{-4} = -\frac{1}{4}\,⨍\,\gamma\,D\,a\,A\,D\,A\,D^{-4} = \frac{1}{4}\,⨍\,\gamma\,a\,A\,D\,A\,D^{-3} = -T_1$$

where we used the boundedness of $[D, a]$ to permute D and a and the minus sign comes from permuting D with γ. □

We can now summarize the relation with the index cocycle in the two dimensional case as follows,

Theorem 7.8. *Assume that the dimension is* ≤ 2 *and that the coboundary of the tadpole vanishes,* $b\varphi_0 = 0$*, then*

(1) φ_2 *is a cyclic two cocycle.*
(2) *For any* $a \in \mathcal{A}$ *and any gauge potential* A *one has*

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}\,A\,D''^{-1}) - \mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}) = 2\int_{\varphi_2} a\,(dA + A^2)$$

In other words the alternate sum of the anomalous graphs has a simple interpretation in terms of the pairing of the curvature $F = dA + A^2$ and of the local index cocycle.

Proof. This follows from Proposition 7.7 and Corollary 7.5. □

7.4. **The five legs graph in dimension** 4.

The analytic expression for the pentagon graph of Figure 5 is given by

$$\mathrm{Tr}(E\,D''^{-1}\,A\,D''^{-1}\,A\,D''^{-1}\,A\,D''^{-1}\,A\,D''^{-1}), \tag{7.37}$$

and we can proceed as above and rewrite this as

$$\mathrm{Tr}(\gamma\,a\hat{D}\,(\bar{D}+\hat{D})\,D''^{-2}\,A\,(\bar{D}+\hat{D})\,D''^{-2}\,A\,(\bar{D}+\hat{D})\,D''^{-2}\,A\,(\bar{D}+\hat{D})\,D''^{-2}\,A\,(\bar{D}+\hat{D})\,D''^{-2}). \tag{7.38}$$

Since we work in dimension 4 we can move the $D^{''-2}$ to the right in each of the obtained monomials, since the contributions from all additional terms obtained from lemma 5.2 all vanish at $z=0$. We are thus dealing with,

$$\mathrm{Tr}(\gamma\, a\hat{D}\,(\bar{D}+\,\hat{D})\,A\,(\bar{D}+\,\hat{D})\,A\,(\bar{D}+\,\hat{D})\,A\,(\bar{D}+\,\hat{D})\,A\,(\bar{D}+\,\hat{D})\,D^{''-10})\,, \tag{7.39}$$

and the various terms are given by even powers of $\hat{D}$. The first

$$\mathrm{Tr}(\gamma\, a\hat{D}^6\,A^4\,D^{''-10}) = -\frac{1}{24}\,⨍\,\gamma\, a\,A^4\,D^{-4}$$

is the only term of degree 6 in $\hat{D}$, the others are :

Terms in $\hat{D}^4$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\bar{D}\,A\,\bar{D}\,A\,\hat{D}\,A\,\hat{D}\,A\,\hat{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,D\;A\,D\,A^3\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,D\;A\,D\,A^3\;D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\bar{D}\,A\,\hat{D}\,A\,\bar{D}\,A\,\hat{D}\,A\,\hat{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,D\,A^2\,D\,A^2\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,D\;A^2\,D\,A^2\;D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\bar{D}\,A\,\hat{D}\,A\,\hat{D}\,A\,\bar{D}\,A\,\hat{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,D\,A^3\,D\,A\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,D\,A^3\,D\,A\;D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\bar{D}\,A\,\hat{D}\,A\,\hat{D}\,A\,\hat{D}\,A\,\bar{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,D\,A^4\,D\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,D\,A^4\,D\,D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\hat{D}\,A\,\bar{D}\,A\,\bar{D}\,A\,\hat{D}\,A\,\hat{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,A\,D\,A\,D\,A^2\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,A\,D\,A\,D\,A^2\,D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\hat{D}\,A\,\bar{D}\,A\,\hat{D}\,A\,\bar{D}\,A\,\hat{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,A\,D\,A^2\,D\,A\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,A\,D\,A^2\,D\,A\,D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\hat{D}\,A\,\bar{D}\,A\,\hat{D}\,A\,\hat{D}\,A\,\bar{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,A\,D\,A^3\,D\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,A\,D\,A^3\,D\,D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\hat{D}\,A\,\hat{D}\,A\,\bar{D}\,A\,\bar{D}\,A\,\hat{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,A^2\,D\,A\,D\,A\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,A^2\,D\,A\,D\,A\,D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\hat{D}\,A\,\hat{D}\,A\,\bar{D}\,A\,\hat{D}\,A\,\bar{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,A^2\,D\,A^2\,D\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,A^2\,D\,A^2\,D\,D^{-6}$$

$$\mathrm{Tr}(\gamma\, a\hat{D}\,\hat{D}\,A\,\hat{D}\,A\,\hat{D}\,A\,\bar{D}\,A\,\bar{D}\,D^{''-10}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\,A^3\,D\,A\,D\,D^{''-10}) = \frac{1}{24}\,⨍\,\gamma\, a\,A^3\,D\,A\,D\,D^{-6}$$

Using the boundedness of $[D,a]$ and the anticommutation of D with γ (and the trace property) one gets three pairwise cancelations and a term in

$$\frac{1}{24}\,⨍\,\gamma\, a\,D\,A^4\,D\,D^{-6} = -\,\frac{1}{24}\,⨍\,\gamma\, a\,A^4\,D^{-4}$$

There remains three terms which add up to

$$\frac{1}{24}\,⨍\,\gamma\, a\,(A^2\,D\,A\,D\,A+\,A\,D\,A^2\,D\,A+\,A\,D\,A\,D\,A^2)\,D^{-6}$$

which can be written as

$$\frac{1}{24}\,⨍\,\gamma\, a\,A\,(AD+DA)^2\,A\,D^{-6} - \,\frac{1}{24}\,⨍\,\gamma\, a\,A^4\,D^{-4} \tag{7.40}$$

We can then use the decomposition

$$D\,A+\,A\,D = \,dA\,+\,A'$$

where dA is bounded and replace the contribution (7.40) by

$$\frac{1}{24}\,⨍\,\gamma\, a\,A\,A^{'\,2}\,A\,D^{-6} - \,\frac{1}{24}\,⨍\,\gamma\, a\,A^4\,D^{-4} \tag{7.41}$$

Thus adding up the terms we got so far we get

$$\frac{1}{24} ⨍ \gamma\, a\, A\, A^{'\,2}\, A\, D^{-6} - \frac{1}{8} ⨍ \gamma\, a\, A^4\, D^{-4} \tag{7.42}$$

Terms in $\hat{D}^2$

$$t_1 = \mathrm{Tr}(\gamma\, a\hat{D}\, \hat{D}\, A\, \bar{D}\, A\, \bar{D}\, A\, \bar{D}\, A\, \bar{D}\, D^{''-10}) = -\frac{1}{8} ⨍ \gamma\, a\, (AD)^4\, D^{-8}$$

$$t_2 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, A\, \hat{D}\, A\, \bar{D}\, A\, \bar{D}\, A\, \bar{D}\, D^{''-10}) = -\frac{1}{8} ⨍ \gamma\, a\, DA^2D(AD)^2\, D^{-8}$$

$$t_3 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, A\, \bar{D}\, A\ \hat{D}\, A\, \bar{D}\, A\, \bar{D}\, D^{''-10}) = -\frac{1}{8} ⨍ \gamma\, a\, (DA)^2(AD)^2\, D^{-8}$$

$$t_4 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, A\, \bar{D}\, A\, \bar{D}\, A\, \hat{D}\, A\, \bar{D}\, D^{''-10}) = -\frac{1}{8} ⨍ \gamma\, a\, (DA)^3(AD)\, D^{-8}$$

$$t_5 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, A\, \bar{D}\, A\, \bar{D}\, A\, \bar{D}\, A\, \hat{D}\, D^{''-10}) = -\frac{1}{8} ⨍ \gamma\, a\, (DA)^4\, D^{-8}$$

Note that $t_1 + t_5 = 0$ using the anticommutation of D with γ and the boundedness of $[D, a]$ which allows to permute a with D. For the same reason we can move the front D to the end in the remaining three terms, which then add up to three times (7.40) and thus contribute by

$$\frac{1}{8} ⨍ \gamma\, a\, A\, A^{'\,2}\, A\, D^{-6} - \frac{1}{8} ⨍ \gamma\, a\, A^4\, D^{-4} \tag{7.43}$$

Thus adding up all the terms we get

$$\mathrm{Tr}(E\, D^{''-1}\, (A\, D^{''-1})^4) = \frac{1}{6} ⨍ \gamma\, a\, A\, A^{'\,2}\, A\, D^{-6} - \frac{1}{4} ⨍ \gamma\, a\, A^4\, D^{-4} \tag{7.44}$$

We can thus summarize the above computation,

Proposition 7.9. *In the 4 dimensional case one has in the limit $z \to 0$ (with $E = \gamma a\hat{D}$)*

$$\mathrm{Tr}(E\, D^{''-1}\, (A\, D^{''-1})^4) = \int_\varphi a\, A^4\,,$$

where $\varphi = -12\,\varphi_4 + \frac{1}{12} b\psi$ where

$$\varphi_4(a_0, a_1, a_2, a_3, a_4) = \frac{1}{48} ⨍ \gamma\, a_0\, [D, a_1]\, [D, a_2]\, [D, a_3]\, [D, a_4]\, D^{-4}$$

and

$$\psi(a_0, a_1, a_2, a_3) = ⨍ \gamma\, a_0\, [D, a_1]\, \nabla^2(a_2)\, [D, a_3]\, D^{-6}$$

Proof. Using

$$\nabla^2(a\, b) = \nabla^2(a)\, b + 2\, \nabla(a)\nabla(b) + a\, \nabla^2(b)$$

one gets

$$b\psi(a_0, a_1, a_2, a_3, a_4) = 2 ⨍ \gamma\, a_0\, [D, a_1]\, \nabla(a_2)\, \nabla(a_3)\, [D, a_4]\, D^{-6}$$

It follows that

$$\int_{b\psi} a\, A^4 = 2 ⨍ \gamma\, a\, A\, A^{'\,2}\, A\, D^{-6}$$

while one has by construction

$$-12 \int_{\varphi_4} a\, A^4 = -\frac{1}{4} ⨍ \gamma\, a\, A^4\, D^{-4}$$

Thus the conclusion follows from (7.44). ∎

Let us now compute $B\psi$, one has

$$B\psi(a_0,a_1,a_2) = -2 \fint \gamma\, a^0[D,a^1]^{(1)}[D,a^2]^{(1)}\ D^{-6} - 2 \fint \gamma\, a^0[D,a^1]^{(2)}[D,a^2]\ D^{-6} + \fint \gamma\, a_0\,(\nabla(a_1)\,\nabla^2(a_2) - \nabla^2(a_1)\,\nabla(a_2))\,D^{-6} \tag{7.45}$$

To check this one treats separately the three terms in $B\psi = AB_0\psi = B_0\psi + (B_0\psi)^\lambda + (B_0\psi)^{\lambda^2}$ with

$$B_0\psi(a_0,a_1,a_2) = \fint \gamma\,[D,a_0]\,\nabla^2(a_1)\,[D,a_2]\,D^{-6}$$

One has

$$B_0\psi(a_0,a_1,a_2) = -\fint \gamma\, a_0\,(D\,\nabla^2(a_1)\,[D,a_2] + \nabla^2(a_1)\,[D,a_2]\,D)\,D^{-6}$$
$$= -\fint \gamma\, a^0[D,a^1]^{(2)}[D,a^2]\ D^{-6} - \fint \gamma\, a_0\,\nabla^2(a_1)\,\nabla(a_2))\,D^{-6}$$
$$B_0\psi(a_1,a_2,a_0) = \fint \gamma\,[D,a_1]\,\nabla^2(a_2)\,[D,a_0]\,D^{-6} = \fint \gamma\,[D,a_1]\,\nabla^2(a_2)\,D^{-6}\,[D,a_0]$$

since the commutator of $[D,a_0]$ with D^{-6} is of lower order. Thus

$$B_0\psi(a_1,a_2,a_0) = -\fint \gamma\,[D,a_0]\,[D,a_1]\,\nabla^2(a_2)\,D^{-6}$$
$$= \fint \gamma\, a_0\,(D\,[D,a_1]\,\nabla^2(a_2) + [D,a_1]\,\nabla^2(a_2)\,D)\,D^{-6}$$
$$= \fint \gamma\, a^0[D,a^1]\,[D,a^2]^{(2)}\ D^{-6} + \fint \gamma\, a_0\,\nabla(a_1)\,\nabla^2(a_2))\,D^{-6}$$

Finally one has

$$B_0\psi(a_2,a_0,a_1) = \fint \gamma\,[D,a_2]\,\nabla^2(a_0)\,[D,a_1]\,D^{-6} = -\fint \gamma\,\nabla^2(a_0)\,[D,a_1]\,[D,a_2]\,D^{-6}$$

which gives

$$B_0\psi(a_2,a_0,a_1) = -\fint \gamma\, a_0\,[D,a_1]^{(2)}\,[D,a_2]\,D^{-6} - 2\fint \gamma\, a_0\,[D,a_1]^{(1)}\,[D,a_2]^{(1)}\,D^{-6} - \fint \gamma\, a_0\,[D,a_1]\,[D,a_2]^{(2)}\,D^{-6}$$

Thus adding up the three terms in $B\psi = AB_0\psi = B_0\psi + (B_0\psi)^\lambda + (B_0\psi)^{\lambda^2}$, one gets (7.45).

7.5. **The four legs graph in dimension 4.**

The analytic expression for the square graph of Figure 6 is given by

$$\mathrm{Tr}(E\,D^{''-1}\,A\,D^{''-1}\,A\,D^{''-1}\,A\,D^{''-1}). \tag{7.46}$$

and we proceed as above and rewrite this as

$$\mathrm{Tr}(\gamma\,a\hat{D}\,(\bar{D} + \hat{D})\,D^{''-2}\,A\,(\bar{D} + \hat{D})\,D^{''-2}\,A\,(\bar{D} + \hat{D})\,D^{''-2}\,A\,(\bar{D} + \hat{D})\,D^{''-2})\,, \tag{7.47}$$

and write the various terms with an even number of $\hat{D}$ as follows,

Terms in $\hat{D}^4$

$$S_1 = \mathrm{Tr}(\gamma\,a\hat{D}\,\bar{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}) = -\,\mathrm{Tr}(\gamma\,a\hat{D}^4\,\bar{D}\,D^{''-2}\,A\,D^{''-2}\,A\,D^{''-2}\,A\,D^{''-2})$$
$$S_2 = \mathrm{Tr}(\gamma\,a\hat{D}\,\hat{D}\,D^{''-2}\,A\,\bar{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}) = -\,\mathrm{Tr}(\gamma\,a\hat{D}^4\,D^{''-2}\,A\,\bar{D}\,D^{''-2}\,A\,D^{''-2}\,A\,D^{''-2})$$
$$S_3 = \mathrm{Tr}(\gamma\,a\hat{D}\,\hat{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}\,A\,\bar{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}) = -\,\mathrm{Tr}(\gamma\,a\hat{D}^4\,D^{''-2}\,A\,D^{''-2}\,A\,\bar{D}\,D^{''-2}\,A\,D^{''-2})$$
$$S_4 = \mathrm{Tr}(\gamma\,a\hat{D}\,\hat{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}\,A\,\hat{D}\,D^{''-2}\,A\,\bar{D}\,D^{''-2}) = -\,\mathrm{Tr}(\gamma\,a\hat{D}^4\,D^{''-2}\,A\,D^{''-2}\,A\,D^{''-2}\,A\,\bar{D}\,D^{''-2})$$

Terms in $\hat{D}^2$

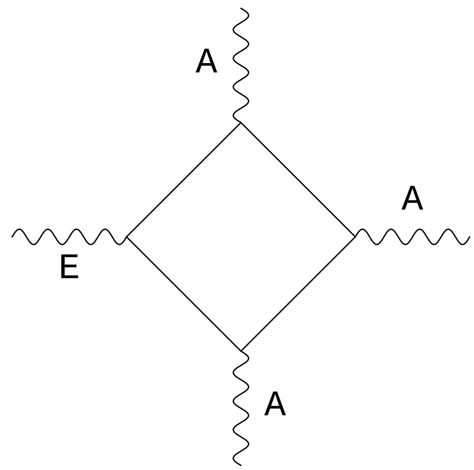

FIGURE 6. The four legs graph.

$$R_1 = \mathrm{Tr}(\gamma\, a\hat{D}^2\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2})$$

$$R_2 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, D^{''-2}\, A\, \hat{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\, \bar{D}\, D^{''-2}\, A\; D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2})$$

$$R_3 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \hat{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, D^{''-2}\, A\, \bar{D}\, D^{''-2})$$

$$R_4 = \mathrm{Tr}(\gamma\, a\hat{D}\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \hat{D}\, D^{''-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, \bar{D}\, D^{''-2}\, A\, D^{''-2})$$

Let us now compute S_1 in dimension 4. We need to move the $D^{''-2}$ to the right. To get some feeling about the order we first look at the term (using Lemma 6.1)

$$-\,\mathrm{Tr}(\gamma\, a\hat{D}^4\, \bar{D}\, A^3\, D^{''-8}) = \frac{1}{12} ⨍ \gamma\, a\, D\, A^3\, D^{-4}$$

Thus the terms with more than one occurence of ∇ can all be ignored in the process of moving the $D^{''-2}$ to the right. Thus we can use the following replacements

$$D^{''-2}\, P_1\, D^{''-2}\, P_2\, D^{''-2}\, P_3\, D^{''-2} \sim \tag{7.48}$$

$$P_1\, P_2\, P_3\, D^{''-8} - \nabla(P_1)\, P_2\, P_3\, D^{''-10} - 2\, P_1\, \nabla(P_2)\, P_3\, D^{''-10} - 3\, P_1\, P_2\, \nabla(P_3)\, D^{''-10}$$

when we compute the S terms. We thus get

$$S_1 = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\, \bar{D}\, D^{''-2}\, A\, D^{''-2}\, A\, D^{''-2}\, A\, D^{''-2}) = -\,\mathrm{Tr}(\gamma\, a\hat{D}^4\, \bar{D}\, A^3\, D^{''-8}) +$$

$$\mathrm{Tr}(\gamma\, a\hat{D}^4\, \bar{D}\, \nabla(A)\, A^2\, D^{''-10}) + 2\mathrm{Tr}(\gamma\, a\hat{D}^4\, \bar{D}\, A\, \nabla(A)\, A\, D^{''-10}) + 3\,\mathrm{Tr}(\gamma\, a\hat{D}^4\, \bar{D}\, A^2\, \nabla(A)\, D^{''-10}) =$$

$$\frac{1}{12} ⨍ \gamma\, a\, D\, A^3\, D^{-4} - \frac{1}{24} ⨍ \gamma\, a\, D\, (\nabla(A)\, A^2 + 2\, A\, \nabla(A)\, A + 3\, A^2\, \nabla(A))\, D^{-6}$$

and in a similar manner

$$S_2 = \frac{1}{12} ⨍ \gamma\, a\, A\, D\, A^2\, D^{-4} - \frac{1}{24} ⨍ \gamma\, a\, (\nabla(A)\, D\, A^2 + 2\, A\, D\, \nabla(A)\, A + 3\, A\, D\, A\, \nabla(A))\, D^{-6}$$

$$S_3 = \frac{1}{12} ⨍ \gamma\, a\, A^2\, D\, A\, D^{-4} - \frac{1}{24} ⨍ \gamma\, a\, (\nabla(A)\, A\, D\, A + 2\, A\, \nabla(A)\, D\, A + 3\, A^2\, D\, \nabla(A))\, D^{-6}$$

$$S_4 = \frac{1}{12} ⨍ \gamma\, a\, A^3\, D^{-3} - \frac{1}{24} ⨍ \gamma\, a\, (\nabla(A)\, A^2 + 2\, A\, \nabla(A)\, A + 3\, A^2\, \nabla(A))\, D^{-5}$$

Using the boundedness of $[D, a]$ and the anticommutation of D with γ one gets

$$S_1 + S_4 = \frac{1}{12} ⨍ \gamma\, a\, (D\, A^3 +\, A^3\, D)\, D^{-4} \tag{7.49}$$

Using the equality $DA + AD = dA + A'$ and the boundedness of dA one gets

$$S_2 + S_3 = \frac{1}{12} ⨍ \gamma\, a\, A\,(D\,A + A\,D)\,A\,D^{-4} - \frac{1}{24} ⨍ \gamma\, a\,(\nabla(A)\,A'\,A + 2\,A\,\nabla(A')\,A + 3\,A\,A'\,\nabla(A))\,D^{-6}$$

One has

$$D\,A^3 + A^3\,D = (D\,A + A\,D)\,A^2 - A\,(D\,A + A\,D)\,A + A^2\,(D\,A + A\,D)$$

We can thus collect together all the S terms and get,

$$S_1 + S_2 + S_3 + S_4 = \frac{1}{12} ⨍ \gamma\, a\,((dA + A')\,A^2 + A^2\,(dA + A'))\,D^{-4} - \frac{1}{24} ⨍ \gamma\, a\,(\nabla(A)\,A'\,A + 2\,A\,\nabla(A')\,A + 3\,A\,A'\,\nabla(A))\,D^{-6} \tag{7.50}$$

Let us now look at the R-terms. The first term in R_1 after moving the D''^{-2} to the right, is

$$\mathrm{Tr}(\gamma\, a\hat{D}^2\,(A\,\bar{D})^3\,D''^{-8}) = -\frac{1}{6} ⨍ \gamma\, a\,(A\,D)^3\,D^{-6}$$

and the above simplification applies also in computing the R terms. One gets

$$R_1 = \mathrm{Tr}(\gamma\, a\hat{D}^2\,D''^{-2}\,A\,\bar{D}\,D''^{-2}\,A\,\bar{D}\,D''^{-2}\,A\,\bar{D}\,D''^{-2}) = \mathrm{Tr}(\gamma\, a\hat{D}^2\,(A\,\bar{D})^3\,D''^{-8})$$

$$-\mathrm{Tr}(\gamma\, a\hat{D}^2\,\nabla(A)\,\bar{D}\,(A\,\bar{D})^2\,D''^{-10}) - 2\,\mathrm{Tr}(\gamma\, a\hat{D}^2\,A\,\bar{D}\,\nabla(A)\,\bar{D}\,A\,\bar{D}\,D''^{-10}) - 3\,\mathrm{Tr}(\gamma\, a\hat{D}^2\,(A\,\bar{D})^2\,\nabla(A)\,\bar{D}\,D''^{-10})$$

$$= -\frac{1}{6} ⨍ \gamma\, a\,(A\,D)^3\,D^{-6} + \frac{1}{8} ⨍ \gamma\, a\,(\nabla(A)\,D\,(AD)^2 + 2A\,D\,\nabla(A)\,D\,A\,D + 3\,(AD)^2\,\nabla(A)\,D)\,D^{-8}$$

Similarly

$$R_2 = -\frac{1}{6} ⨍ \gamma\, a\,D\,A^2\,D\,A\,D^{-5} + \frac{1}{8} ⨍ \gamma\, a\,(D\,\nabla(A)A\,D\,A\,D + 2D\,A\,\nabla(A)\,D\,A\,D + 3\,D\,A^2\,D\,\nabla(A)\,D)\,D^{-8}$$

$$R_3 = -\frac{1}{6} ⨍ \gamma\, a\,D\,A\,D\,A^2\,D^{-5} + \frac{1}{8} ⨍ \gamma\, a\,(D\,\nabla(A)\,D\,A^2\,D + 2D\,A\,D\nabla(A)\,A\,D + 3\,D\,A\,D\,A\,\nabla(A)\,D)\,D^{-8}$$

$$R_4 = -\frac{1}{6} ⨍ \gamma\, a\,(D\,A)^3\,D^{-6} + \frac{1}{8} ⨍ \gamma\, a\,(D\,\nabla(A)\,(D\,A)^2 + 2D\,A\,D\nabla(A)\,D\,A + 3\,(D\,A)^2\,D\,\nabla(A))\,D^{-8}$$

Using the boundedness of $[D, a]$ and the anticommutation of D with γ one gets

$$R_1 + R_4 = -\frac{1}{6} ⨍ \gamma\, a\,((A\,D)^3 + (D\,A)^3)\,D^{-6}$$

Next,

$$R_2 + R_3 = -\frac{1}{6} ⨍ \gamma\, a\,D\,A(dA + A')\,A\,D^{-5} + \frac{1}{8} ⨍ \gamma\, a\,(D\,\nabla(A)A'\,A\,D + 2D\,A\,\nabla(A')\,A\,D + 3\,D\,A\,A'\,\nabla(A)\,D)\,D^{-8}$$

$$= -\frac{1}{6} ⨍ \gamma\, a\,D\,A(dA + A')\,A\,D^{-5} - \frac{1}{8} ⨍ \gamma\, a\,(\nabla(A)A'\,A + 2A\,\nabla(A')\,A + 3\,A\,A'\,\nabla(A))\,D^{-6}$$

We thus get

$$R_1 + R_2 + R_3 + R_4 = -\frac{1}{6} ⨍ \gamma\, a\,((A\,D)^3 + (D\,A)^3)\,D^{-6} - \frac{1}{6} ⨍ \gamma\, a\,D\,A(dA + A')\,A\,D^{-5} - \frac{1}{8} ⨍ \gamma\, a\,(\nabla(A)\,A'\,A + 2\,A\,\nabla(A')\,A + 3\,A\,A'\,\nabla(A))\,D^{-6} \tag{7.51}$$

and adding with the S-terms gives the following sum

$$\Sigma = \frac{1}{12} ⨍ \gamma\, a\,((dA + A')\,A^2 + A^2\,(dA + A'))\,D^{-4} - \frac{1}{6} ⨍ \gamma\, a\,((A\,D)^3 + D\,A(dA + A')\,A\,D + (D\,A)^3)\,D^{-6} - \frac{1}{6} ⨍ \gamma\, a\,(\nabla(A)\,A'\,A + 2\,A\,\nabla(A')\,A + 3\,A\,A'\,\nabla(A))\,D^{-6}$$

We need to simplify the term

$$T = -\frac{1}{6} ⨍ \gamma\, a\,((A\,D)^3 + D\,A^2\,D\,A\,D + D\,A\,D\,A^2\,D + (D\,A)^3)\,D^{-6}$$

Using $DA + AD = dA + A'$ one can rewrite it as

$$-\frac{1}{6}⨍\gamma\, a\,((dA + A')(ADAD) + (DADA)(dA + A'))\, D^{-6}$$

which gives using $ADAD = -A^2 D^2 + A\, dA\, D + A\, A'\, D$ and $DADA = -D^2 A^2 + D\, dA\, A + D\, A'\, A$ gives

$$\frac{1}{6}⨍\gamma\, a\,((dA + A')\, A^2 + A^2\,(dA + A'))\, D^{-4} + \frac{1}{6}⨍\gamma\, a\, \nabla(A^2\, A')\, D^{-6}$$
$$-\frac{1}{6}⨍\gamma\, a\,((dA + A')(A\, dA\, D + A\, A'\, D) + (D\, dA\, A + D\, A'\, A)(dA + A'))\, D^{-6}$$

The second line gives using dimension 4,

$$-\frac{1}{6}⨍\gamma\, a\,(dA\, A\, A'\, D + A'\, A\, dA\, D + A'\, A\, A'\, D + D\, dA\, A\, A' + D\, A'\, A\, dA + D\, A'\, A\, A')\, D^{-6}$$
$$= -\frac{1}{6}⨍\gamma\, a\,(A'\, A\, A'\, D + D\, A'\, A\, A')\, D^{-6}$$

we can thus collect these terms and get

$$T = \frac{1}{6}⨍\gamma\, a\,((dA + A')\, A^2 + A^2\,(dA + A'))\, D^{-4} + \frac{1}{6}⨍\gamma\, a\, \nabla(A^2\, A')\, D^{-6} \tag{7.52}$$
$$-\frac{1}{6}⨍\gamma\, a\,(A'\, A\, A'\, D + D\, A'\, A\, A')\, D^{-6}$$

Thus when we replace T in the sum Σ we get

$$\Sigma = \frac{1}{4}⨍\gamma\, a\,((dA + A')\, A^2 + A^2\,(dA + A'))\, D^{-4} + \frac{1}{6}⨍\gamma\, a\, \nabla(A^2\, A')\, D^{-6} \tag{7.53}$$
$$-\frac{1}{6}⨍\gamma\, a\,(A'\, A\, A'\, D + D\, A'\, A\, A')\, D^{-6} - \frac{1}{6}⨍\gamma\, a\,(\nabla(A)\, A'\, A + 2\, A\, \nabla(A')\, A + 3\, A\, A'\, \nabla(A))\, D^{-6}$$

Using (7.48) in the form

$$D^{-2}\, P_1\, D^{-2}\, P_2\, D^{-2}\, P_3 \sim \tag{7.54}$$
$$P_1\, P_2\, P_3\, D^{-6} - \nabla(P_1)\, P_2\, P_3\, D^{-8} - 2\, P_1\, \nabla(P_2)\, P_3\, D^{-8} - 3\, P_1\, P_2\, \nabla(P_3)\, D^{-8}$$

one gets

$$⨍\gamma\, a\,(\nabla(A)\, A'\, A + 2\, A\, \nabla(A')\, A + 3\, A\, A'\, \nabla(A))\, D^{-6} = ⨍\gamma\, a\, A\, A'\, A\, D^{-4} - ⨍\gamma\, a\, D^{-2}\, A\, D^{-2}\, A'\, D^{-2}\, A\, D^2$$

Thus

$$\Sigma = \frac{1}{4}⨍\gamma\, a\,((dA + A')\, A^2 + A^2\,(dA + A'))\, D^{-4} + \frac{1}{6}⨍\gamma\, a\, \nabla(A^2\, A')\, D^{-6} \tag{7.55}$$
$$-\frac{1}{6}⨍\gamma\, a\,(A'\, A\, A'\, D + D\, A'\, A\, A')\, D^{-6} - \frac{1}{6}⨍\gamma\, a\, A\, A'\, A\, D^{-4} + \frac{1}{6}⨍\gamma\, a\, D^{-2}\, A\, D^{-2}\, A'\, D^{-2}\, A\, D^2$$

Let us gather a certain number of these terms in the form

$$\int_\psi da\, A^3 \tag{7.56}$$

Using integration by parts one has

$$\frac{1}{6}⨍\gamma\, a\, \nabla(A^2\, A')\, D^{-6} = -\frac{1}{6}⨍\gamma\, \nabla(a)\, A^2\, A'\, D^{-6}$$

and this is given in the form (7.56) by the contribution

$$\psi_1(a_0, a_1, a_2, a_3, a_4) = -\frac{1}{6}⨍\gamma\, a_0\, \nabla(a_1)\, [D, a_2]\, [D, a_3]\, \nabla(a_4)\, D^{-6}$$

Next, one has

$$-\frac{1}{6}\fint \gamma\, a\,(A'\,A\,A'\,D + \,D\,A'\,A\,A')\,D^{-6} = \,\frac{1}{6}\fint \gamma\,[D,a]\,A'\,A\,A'\,D^{-6}$$

and this is given in the form (7.56) by the contribution

$$\psi_2(a_0,a_1,a_2,a_3,a_4) = \,\frac{1}{6}\fint \gamma\, a_0\,[D,a_1]\,\nabla(a_2)\,[D,a_3]\,\nabla(a_4)\,D^{-6}$$

Next one has

$$\frac{1}{6}\fint \gamma\, a\,D^{-2}\,A\,D^{-2}\,A'\,D^{-2}\,A\,D^2 = \,\frac{1}{6}\fint \gamma\,\nabla(a)\,A\,A'\,A\,D^{-6} + \,\frac{1}{6}\fint \gamma\, a\,A\,D^{-2}\,A'\,D^{-2}\,A$$

where the first term of the rhs corresponds to

$$\psi_3(a_0,a_1,a_2,a_3,a_4) = \,\frac{1}{6}\fint \gamma\, a_0\,\nabla(a_1)\,[D,a_2]\,\nabla(a_3)\,[D,a_4]\,D^{-6}$$

and one has

$$\frac{1}{6}\fint \gamma\, a\,A\,D^{-2}\,A'\,D^{-2}\,A = \frac{1}{6}\fint \gamma\, a\,A\,A'\,A\,D^{-4} - \,\frac{1}{6}\fint \gamma\, a\,(A\,\nabla(A')\,A + 2\,A\,A'\,\nabla(A))\,D^{-6}$$

so that we can write the last two terms in the form

$$-\,\frac{1}{6}\fint \gamma\, a\,A\,A'\,A\,D^{-4} + \,\frac{1}{6}\fint \gamma\, a\,D^{-2}\,A\,D^{-2}\,A'\,D^{-2}\,A\,D^2 = \,\frac{1}{6}\fint \gamma\,\nabla(a)\,A\,A'\,A\,D^{-6}$$
$$-\,\frac{1}{6}\fint \gamma\, a\,(A\,\nabla(A')\,A + 2\,A\,A'\,\nabla(A))\,D^{-6}$$

Proposition 7.10. *In the* 4 *dimensional case one has in the limit* $z \to 0$ *(with* $E = \gamma a\hat{D}$*)*

$$\mathrm{Tr}(E\,D^{''-1}\,(A\,D^{''-1})^3) = \,-\,\int_{\varphi} a\,(dA\,A^2 + A^2\,dA)\,,$$

where $\varphi = \,-\,12\,\varphi_4 + \,\frac{1}{12}b\psi$ *is defined in Proposition 7.9.*

Proof. Let us compute the right hand side. The contribution coming from $-\,12\,\varphi_4$ just gives

$$\frac{1}{4}\fint \gamma\, a\,(dA\,A^2 + \,A^2\,dA)\,D^{-4} \tag{7.57}$$

which already appears in the sum (7.53).

To compute the next term we use the notation $A = \,u\,dv$ and forget about the summation symbol. No ambiguity will arise since the ordering of the terms will never change. With this notation one has

$$-\,\int_{\frac{1}{12}b\psi} a\,(dA\,A^2 + A^2\,dA) = \,-\frac{1}{6}\fint \gamma\, a\,([D,u]\,\nabla(v)\,u\,\nabla(v)\,u\,[D,v] + u\,[D,v]\,u\,\nabla(v)\,\nabla(u)\,[D,v])\,D^{-6}$$

Moreover in all the above expressions under the $\fint$ we can use the replacements

$$A \to u[D,v]\,,\quad dA \to \,[D,u]\,[D,v]\,,\quad A' \to \,u\nabla(v)\,,\quad \nabla(A) \to \nabla(u)\,[D,v] + \,u\,\nabla([D,v]) \tag{7.58}$$

We now collect all the terms from the above computation, except for those already used in 7.57 and use (7.58). We get

$$\frac{1}{4}\fint \gamma\, a\,(u\nabla(v)\,u[D,v]u[D,v] + \,u[D,v]u[D,v]\,u\nabla(v))\,D^{-4} + \frac{1}{6}\fint \gamma\, a\,\nabla(u[D,v]\,u[D,v]\,u\nabla(v))\,D^{-6}$$
$$-\frac{1}{6}\fint \gamma\, a\,(u\nabla(v)\,u[D,v]\,u\nabla(v)\,D + \,D\,u\nabla(v)\,u[D,v]\,u\nabla(v))\,D^{-6}$$
$$-\,\frac{1}{6}\fint \gamma\, a\,(\nabla(u[D,v])\,u\nabla(v)\,u[D,v] + 2\,u[D,v]\,\nabla(u\nabla(v))\,u[D,v] + 3\,u[D,v]\,u\nabla(v)\,\nabla(u[D,v]))\,D^{-6}$$

□

7.6. **The triangle graph and anomalous graphs in dimension 4.**

In dimension 4 the two dimensional component of the local index cocycle is given by

$$\varphi_2(a_0,a_1,a_2) = \frac{1}{4}⨍\gamma\, a^0[D,a^1][D,a^2]\; D^{-2} - \frac{1}{12}⨍\gamma\, a^0[D,a^1]^{(1)}[D,a^2]\; D^{-4}$$
$$-\frac{1}{6}⨍\gamma\, a^0[D,a^1][D,a^2]^{(1)}\; D^{-4} + \frac{1}{24}⨍\gamma\, a^0[D,a^1]^{(2)}[D,a^2]\; D^{-6}$$
$$+\frac{1}{8}⨍\gamma\, a^0[D,a^1]^{(1)}[D,a^2]^{(1)}\; D^{-6} + \frac{1}{8}⨍\gamma\, a^0[D,a^1][D,a^2]^{(2)}\; D^{-6}$$

We look for a formula of the form

$$\sum (-1)^n \operatorname{Tr}(E\, D^{''-1}\,(A\, D^{''-1})^n) = -\varphi'_0(a) + 2\int_{\varphi'_2} a\,(dA + A^2) - 12\int_{\varphi'_4} a\,(dA + A^2)^2$$

where the φ'_n are the components of a cocycle cohomologous to the local one φ_n. The point is that the pairing between cocycles and elements of K_0 *i.e.* idempotents $e \in \mathcal{A}$ is given by the combination

$$\langle \varphi, e\rangle = \varphi_0(e-\frac{1}{2}) - 2\,\varphi_2(e-\frac{1}{2},e,e) + 12\,\varphi_4(e-\frac{1}{2},e,e,e,e) \tag{7.59}$$

The two dimensional case tells us that we should not have terms of the form

$$⨍\gamma\, a_0\, \nabla(a_1)\, \nabla(a_2)\, D^{-4}$$

in the correction of φ_2, since such terms would already show up in $D = 2$. But we should expect terms like

$$⨍\gamma\, a_0\, (\nabla(a_1)\, \nabla^2(a_2) - \nabla^2(a_1)\, \nabla(a_2))\, D^{-6}$$

8. Evanescent gauge potentials and vanishing cycles

We want to present here a suggestive analogy between dimensional regularization and deformations of singularities. This can be thought of as an analogy between the deformation of the singularities of the special fiber of a geometric degeneration over a disk and the process of removal of singularities of the Feynman integrals in the deformation to complex dimension of DimReg.

More precisely, the two geometric setting we will compare are the following. On the algebro-geometric side, we consider the case of a geometric degeneration of a family $\mathfrak{X}$ of smooth algebraic varieties over a disk $\Delta \subset \mathbb{C}$. Here $\mathfrak{X}$ is a complex analytic manifold of dimension $\dim_{\mathbb{C}} \mathfrak{X} = n+1$ and $f : \mathfrak{X} \to \Delta$ is a flat, proper morphism with projective fibers. We assume that the map f is smooth on $\mathfrak{X}^* = \mathfrak{X} \smallsetminus Y$ and that Y is a divisor with normal crossings in $\mathfrak{X}$. On the side of noncommutative geometry, we consider the noncommutative spaces $(\mathcal{A}'', \mathcal{H}'', D'')$ obtained by taking the cup product of a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ with a noncommutative space X_z in "complexified dimension" $z \in \Delta$.

We shall see that the complex of gauge potentials on $(\mathcal{A}'', \mathcal{H}'', D'')$ behaves in many ways like a complex of forms with logarithmic poles associated to the family $\mathfrak{X}$ of smooth algebraic varieties over the disk $z \in \Delta$. In particular, in the algebro-geometric case, it is known that the special fiber $f^{-1}(0)$ of $f : \mathfrak{X} \to \Delta$ carries a mixed Hodge structure, [30], [24]. This structure also appears in the cohomological theory of the fibers over the archimedean places of an arithmetic variety, [19], [20].

To describe the analogous structure associated to the complex of gauge potentials in the case of a noncommutative space and its deformation to complexified dimension, we take the point of view of Saito's polarized Hodge–Lefschetz modules (*cf.* [29] and [24]).

8.1. **Hodge–Lefschetz modules.**

We recall the following notions from [29], *cf.* also [24]. Let $L = \oplus_{i,j\in\mathbb{Z}} L^{i,j}$ be a finite dimensional bigraded real vector space. Let ℓ_1, ℓ_2 be endomorphisms of L, with $[\ell 1, \ell_2] = 0$ and

$$\ell_1 : L^{i,j} \to L^{i+2,j}, \qquad \ell_2 : L^{i,j} \to L^{i,j+2}. \tag{8.1}$$

The data (L, ℓ_1, ℓ_2) define a *bigraded Lefschetz module* if

$$\ell_1^i : L^{-i,j} \to L^{i,j} \quad \text{and} \quad \ell_2^j : L^{i,-j} \to L^{i,j} \tag{8.2}$$

are isomorphisms for $i > 0$ ($j > 0$, resp.).

In the case of a geometric degeneration, with the generic fiber a compact Kähler manifold, one can obtain such structure from the action of the Lefschetz on the primitive part of the cohomology (induced by wedging with the Kähler form) and of the monomdromy: the Lefschetz and the log of the monodromy give the endomorphisms ℓ_1, ℓ_2.

Bigraded Lefschetz modules correspond bijectively to finite dimensional representations of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. We will use the following notation

$$\chi(\lambda) := \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \lambda \in \mathbb{R}^*, \qquad u(s) := \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} s \in \mathbb{R}, \qquad w := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{8.3}$$

and

$$u = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \frac{d}{ds} u(s)|_{s=0}.$$

In these terms, the representation $\sigma = \sigma_{(L,\ell_1,\ell_2)}$ satisfies

$$d\sigma(u,1) = \ell_1, \quad d\sigma(1,u) = \ell_2, \quad \sigma(\chi(\lambda),\chi(t)) = \lambda^i t^j x, \ \ \forall x \in L^{i,j}.$$

The data (L, ℓ_1, ℓ_2) define a *bigraded Hodge–Lefschetz module* if all the $L^{i,j}$ have a real Hodge structure and ℓ_1, ℓ_2 are morphisms of Hodge structures.

A *polarization* on (L, ℓ_1, ℓ_2) is a bilinear form

$$\psi : L \otimes L \to \mathbb{R} \tag{8.4}$$

which is compatible with the Hodge structure, satisfies

$$\psi(\ell_k x, y) + \psi(x, \ell_k y) = 0, \qquad k = 1, 2, \tag{8.5}$$

and is such that

$$\psi(\cdot, C\ell_1^i \ell_2^j \cdot) \tag{8.6}$$

is symmetric and positive definite on $L^{-i,-j}$. (Here C is the Weil operator.) The data $(L, \ell_1, \ell_2, \psi)$ then define a bigraded polarized Hodge–Lefschetz module.

It is also convenient (*e.g.* when working at the level of forms) to consider *differential bigraded polarized Hodge–Lefschetz modules* $(L, \ell_1, \ell_2, \psi, d)$, where $d : L^{i,j} \to L^{i+1,j+1}$ is a differential ($d^2 = 0$) satisfying $[\ell_1, d] = 0 = [\ell_2, d]$ and $\psi(dx, y) = \psi(x, dy)$. In this case, the cohomology $H^*(L, d)$ inherits the structure of a bigraded polarized Hodge–Lefschetz module. Moreover, H^* is identified with $Ker\Box$, where $\Box = d^*d + dd^*$, where $d^* = \sigma(w,w)^{-1} \circ d \circ \sigma(w,w)$, for w as in (8.3) and $\sigma = \sigma_{(L,\ell_1,\ell_2)}$, the representation of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$.

8.2. **A bigraded complex of gauge potentials.**

We now introduce an analog of this structure for noncommutative spaces. We let $(\mathcal{A},\mathcal{H},D)$ be an even finitely summable spectral triple. Let γ be the grading operator. We consider the graded algebra $\tilde{\mathcal{A}}$ generated by $\mathcal{A}$ and by γ. Let $(\mathcal{A}'',\mathcal{H}'',D'')$ be the noncommutative space obtained as the product of $(\tilde{\mathcal{A}},\mathcal{H},D)$ with the noncommutative space X_z in "complexified dimension" z. Namely, we have $\mathcal{A}''=\tilde{\mathcal{A}}$, $\mathcal{H}''=\mathcal{H}\otimes\mathcal{H}'$, and $D''=D''_z=D\otimes 1+\gamma\otimes D'_z$, where D'_z is the Dirac operator on the space X_z. We let $\Omega^*_D(\mathcal{A})$ denote the complex of gauge potentials of a triple $(\mathcal{A},\mathcal{H},D)$, and $H^*_D(\mathcal{A})$ its cohomology.

We begin now by considering the complex $\Omega^{m,r,k}$, with differentials $\delta:\Omega^{m,r,k}\to\Omega^{m+1,r,k}$ and $\delta':\Omega^{m,r,k}\to\Omega^{m,r+1,k}$. Here we take $\Omega^{m,r,k}$ to be the span of elements of the form

$$\nabla^k(\omega)D^{2r}, \tag{8.7}$$

with $\omega\in\Omega^m_D(\tilde{\mathcal{A}})$. Here we define $\nabla(a)=[D^2,a]$ as before, for an element $a\in\mathcal{A}$ or $a\in[D,\mathcal{A}]$, while we set $\nabla(a\gamma):=[D^2,a]\gamma$, for all $a\in\mathcal{A}$ or $a\in[D,\mathcal{A}]$.

We consider a descending filtration $F^p\supset F^{p+1}$ on $\Omega^*_D(\tilde{\mathcal{A}})$ defined by setting

$$F^p\Omega^m_D(\tilde{\mathcal{A}})=\oplus_{t+s=m,t\geq p}\,\Omega^{t,s}_D(\tilde{\mathcal{A}}), \tag{8.8}$$

where $\Omega^{t,s}_D(\tilde{\mathcal{A}})=\Omega^t_D(\mathcal{A})\theta^s$ is the span of $\omega\,\theta^s$, with $\omega\in\Omega^t_D(\mathcal{A})$ and $\theta=\gamma\hat{D}$. We take

$$F^{m+r-k}\Omega^m_D(\tilde{\mathcal{A}})=\oplus_{t+s=m,s+r\leq k}\,\Omega^t_D(\mathcal{A})\,\theta^s. \tag{8.9}$$

This way, we can endow the complex $\Omega^{m,r,k}$ with a tensor product of "Hodge structures" (the second 1-dim with $F^{-r}_D(\mathbb{C}):=\mathbb{C}\cdot D^{2r}$ and $F^{-(r+1)}_D(\mathbb{C})=0$)

$$\Omega^{m,r,k}=F^{m+r-k}\Omega^m_D(\tilde{\mathcal{A}})\otimes_{\mathbb{C}}F^{-r}_D(\mathbb{C}). \tag{8.10}$$

The index of the resulting filtration is $i-k$, where $i=2r+m$, hence we take then the $\Omega^{m,r,k}$ with the conditions $k\geq 0$ and $k\geq 2r+m$. The differential $d=\delta+\delta'$ on this complex is the same described above, induced by $da=[D'',a]$ for $a\in\nabla^k(\mathcal{A})$, when decomposing $D''=D\otimes 1+\gamma\otimes D'_z$, so that $\delta' a=[\bar{D},a]$ and $\delta''(a)=[\hat{D},a]$.

Notice that the decomposition of the total differential $d=\delta+\delta'$, where δ is essentially the original de Rham differential on $\Omega^*_D(\mathcal{A})$ and δ' acts by wedging with the differential $\theta=\gamma\hat{D}$, resembles very closely the case of geometric degenerations, where one also has a total differential $d=\delta+\delta'$, with δ the usual de Rham differential and δ' given by wedging with the form $\theta=f^*(dz/z)$, for $f:\mathfrak{X}\to\Delta$ and z the coordinate on the base Δ (*cf.* [30]).

8.3. **Representations.**

We introduce endomorphisms $\ell_1:\Omega^{m,r,k}\to\Omega^{m,r+1,k+1}$ and $\ell_2:\Omega^{m,r,k}\to\Omega^{m+2,r-1,k}$ of $(\Omega,d=\delta+\delta')$ defined as follows:

$$\ell_1(\nabla^k(\omega)D^{2r})=\epsilon(\nabla^k(\omega)D^{2r})=\nabla^{k+1}(\omega)D^{2(r-1)} \tag{8.11}$$

$$\ell_2(\nabla^k(\omega)D^{2r})=\sqrt{-1}\,\nabla^k(\omega)\wedge\theta^2\,D^{2(r+1)}=\sqrt{-1}\,\nabla^k(\omega\,(\gamma\hat{D})^2)\,D^{2(r+1)}, \tag{8.12}$$

with $\theta=\gamma\hat{D}$.

We then have the following result.

Lemma 8.1. *The endomorphisms ℓ_1 and ℓ_2 satisfy $[\ell_1,\ell_2]=0$ and are compatible with the differential, namely, $[\ell_1,d]=0$ and $[\ell_2,d]=0$.*

Proof. It is immediate to verify that $[\ell_1, \ell_2] = 0$. We check compatibility with the differential. We have $[\ell_1, d] = 0$. In fact, we can check the result on elements of the form $\nabla^k(a)D^{2r}$ or $\nabla^k(\gamma a)D^{2r}$ for $a \in \mathcal{A}$. We have

$$(\ell_1\delta - \delta\ell_1)\nabla^k(a)D^{2r} = [D^2, [D, \nabla^k(a)]]D^{2(r-1)} - [D, \nabla^{k+1}(a)]D^{2(r-1)} = 0,$$

since $[D^2, [D, b]] = -[D, [D^2, b]]$, for all $b \in \nabla^k(\mathcal{A})$. We also have

$$(\ell_1\delta - \delta\ell_1)\nabla^k(\gamma a)D^{2r} = [D^2, [\hat{D}, \gamma\nabla^k(a)]]D^{2(r-1)} - [\hat{D}, \gamma\nabla^{k+1}(a)]D^{2(r-1)} = 0,$$

where we use the fact that we have set $\nabla(\gamma a) := \gamma\nabla(a)$ and we get $[D^2, [\hat{D}, \gamma b]] = [D^2, 2\gamma b\hat{D}] = 2[D^2, b]\gamma\hat{D}$ and $[\hat{D}, \gamma[D^2, b]] = 2[D^2, b]\gamma\hat{D}$. Similarly, we verify that $[\ell_2, d] = 0$. This can be seen easily on elements of the form bD^{2r}, since $[D, b(\gamma\hat{D})^2] = -[D, b](\gamma\hat{D})^2$, and on elements γbD^{2r}, where $\ell_2[\hat{D}, \gamma b] = 2\sqrt{-1}\, b\theta^3 = -[\hat{D}, \ell_2(\gamma b)]$.

□

Thus, ℓ_1, ℓ_2 induce endomorphisms on the cohomology of the double complex.

We introduce an $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ representation (σ_1, σ_2) on the complex introduced above, which is associated to the operators ℓ_1, ℓ_2.

Definition 8.2. *Assume that the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is N-summable with $N = 2n$. Consider σ_1 and σ_2 defined by*

$$\sigma_1(\chi(\lambda)) = \lambda^{2r+m}, \qquad \sigma_1(u(s)) = \exp(s\,\ell_1), \qquad \sigma_1(w) = S_1. \tag{8.13}$$

$$\sigma_2(\chi(\lambda)) = \lambda^{n-m}, \qquad \sigma_2(u(s)) = \exp(s\,\ell_2), \qquad \sigma_2(w) = S_2. \tag{8.14}$$

Here the operators S_1 and S_2 are defined by powers of ℓ_1, ℓ_2, in the following way. We consider involutions

$$\hat{S}_1 : \Omega^{m,r,k} \to \Omega^{m,-(r+m),k-(2r+m)} \quad \text{and} \quad \hat{S}_2 : \Omega^{m,r,k} \to \Omega^{2n-m,r-(n-m),k}$$

of the form

$$\hat{S}_1(\nabla^k(\omega)D^{2r}) = \ell_1^{-(2r+m)}(\nabla^k(\omega)D^{2r}) \quad \text{and} \quad S_2(\nabla^k(\omega)D^{2r}) = \ell_2^{n-m}(\nabla^k(\omega)D^{2r}), \tag{8.15}$$

for $\omega \in \Omega^m_D(\tilde{\mathcal{A}})$. We set $S_1 = \sqrt{-1}^m \hat{S}_1$.

These satisfy the following.

Proposition 8.3. *The data specified in Definition 8.2 define a representation of* $\mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})$ *on (Ω, d).*

Proof. In order to show that $\sigma = \sigma_k$, for $k = 1, 2$, defined by (8.13) and (8.14) is indeed a representations of $\mathrm{SL}(2, \mathbb{R})$ it is sufficient ([27] §XI.2) to check that it satisfies the relations

$$\begin{aligned} \sigma(w)^2 = {} & \sigma(\chi(-1)), \\ \sigma(\chi(\lambda))\sigma(u(s))\sigma(\chi(\lambda^{-1})) = {} & \sigma(u(s\lambda^2)). \end{aligned} \tag{8.16}$$

The first relation is clearly satisfied and the second can be verified easily on elements $\nabla^k(\omega)D^{2r}$, with $\omega \in \Omega^m_D(\mathcal{A})$, where we have

$$\sigma_1(\chi(\lambda))\,\sigma_1(u(s))\,\sigma_1(\chi(\lambda^{-1}))\ \nabla^k(\omega)D^{2r} = \sigma_1(\chi(\lambda))\left(1 + s\ell_1 + \frac{s^2}{2}\ell_1^2 + \cdots\right)\lambda^{-(2r+m)}\ \nabla^k(\omega)D^{2r} =$$

$$\left(1 + \lambda^{2(r+1)+m}s\ell_1\lambda^{-(2r+m)} + \lambda^{2(r+2)+m}\frac{s^2}{2}\ell_1^2\lambda^{-(2r+m)} + \cdots\right)\nabla^k(\omega)D^{2r} = \exp(s\lambda^2\,\ell_1)\ \nabla^k(\omega)D^{2r},$$

and

$$\sigma_2(\chi(\lambda))\,\sigma_2(u(s))\,\sigma_2(\chi(\lambda^{-1}))\ \nabla^k(\omega)D^{2r} = \sigma_2(\chi(\lambda))\left(1 + s\ell_2 + \frac{s^2}{2}\ell_2^2 + \cdots\right)\lambda^{n-m}\ \nabla^k(\omega)D^{2r} =$$

$$\left(1+\lambda^{-n+m+2}s\ell_2\lambda^{n-m}+\lambda^{-n+m+4}\frac{s^2}{2}\ell_2^2\lambda^{n-m}+\cdots\right)\nabla^k(\omega)D^{2r}=\exp(s\lambda^2\,\ell_2)\;\nabla^k(\omega)D^{2r}.$$

□

Notice that, in the case of the representation associated to the "Lefschetz" ℓ_2, the analog of the Hodge * on forms, which appears in $\sigma_1(w)$, is realized by a power of ℓ_2, by analogy to what happens in the classical case, where the Hodge * can be realized (on the primitive cohomology) by the $(n-m)$-th power of the Lefschetz operator.

This construction parallels exactly what happens in the construction of the archimedean cohomology of [19] for the fibers at infinity of arithmetic varieties, in the form presented in [20]. This defines a structure that is analogous to the differential bigraded Hodge–Lefschetz modules, by setting $L^{i,j}=\oplus_k\Omega^{m,r,k}$ with $i=2r+m$ and $j=-n+m$.

8.4. **Polarization.**

We now discuss the polarization. Define the bilinear form ψ by setting

$$\psi(\nabla^k(\omega)D^{2r},\nabla^{k'}(\omega')D^{2r'}):=⨍\gamma\nabla^k(\eta^*)\nabla^{k'}(\eta')D^{2(r+r')} \tag{8.17}$$

where $\omega=\eta\theta^s$ and $\omega'=\eta'\theta^{s'}$.

Lemma 8.4. *The bilinear form* (8.17) *satisfies the relation* (8.5).

Proof. For $a=\nabla^k(\eta)\theta^sD^{2r}$ and $b=\nabla^{k'}(\eta')\theta^{s'}D^{2r'}$, we have

$$\psi(\ell_1(a),b)=⨍\gamma\nabla^{k+1}(\eta^*)\nabla^{k'}(\eta')\,D^{2(r-1+r')},$$

while

$$\psi(a,\ell_1(b))=⨍\gamma\nabla^k(\eta)\nabla^{k'+1}(\eta')\,D^{2(r+r'-1)}.$$

Thus, integration by parts gives $\psi(\ell_1(a),b)+\psi(a,\ell_1(b))=0$.
We also have

$$\psi(\ell_2(a),b)=⨍\gamma(-\sqrt{-1}\,\nabla^k(\eta^*))\nabla^{k'}(\eta')\,D^{2(r+1+r')},$$

and

$$\psi(a,\ell_2(b))=⨍\gamma\nabla^k(\eta^*)\,\sqrt{-1}\,\nabla^{k'}(\eta')\,D^{2(r+r'+1)}.$$

Thus, we also have $\psi(\ell_2(a),b)+\psi(a,\ell_2(b))=0$.

□

We also have the following result, analogous to the requirement (8.6) for polarizations of Hodge–Lefschetz modules.

Lemma 8.5. *The bilinear form* (8.17) *has the property that*

$$\langle a^*,b\rangle:=\psi(a,\ell_1^i\ell_2^jb)$$

agrees on $L^{-i,-j}$ with the integral with respect to the volume form of $(\mathcal{A},\mathcal{H},D)$. Under the assumptions of "tameness" for $(\mathcal{A},\mathcal{H},D)$, it agrees with the inner product on the complex of gauge potentials.

Proof. For $b=\nabla^{k'}(\eta')\theta^sD^{2r}$ in $L^{-i,-j}$, we have

$$\ell_2^j(b)=\nabla^{k'}(\eta')\theta^{s'+2j}D^{2r+2j}.$$

We then have

$$\ell_1^i\ell_2^j(b)=\nabla^{k'+i}(\eta')\theta^{s'+2j}D^{2r+2j-2i}.$$

This gives

$$\ell_1^i\ell_2^j(b)=\nabla^{k'+2r+m}(\eta')\theta^{s'+2(m-n)}D^{-2r-2n}.$$

Thus, for $a = \nabla^k(\eta')\theta^s D^{2r}$ in $L^{-i,-j}$, we obtain

$$\psi(a, \ell_1^i \ell_2^j(b)) = ⨍ \gamma \nabla^k(\eta^*) \nabla^{k'+2r+m}(\eta') D^{-2n}.$$

Recall that $\int a := ⨍ a D^{-2n}$ is the integration with respect to the volume form D^{-2n} of the spectral triple $(\mathcal{A}, \mathcal{H}, D)$. Under the assumption of "tameness" (*cf.* [31]) this satisfies $\int ab = \int ba$ and $\int a^* a \geq 0$, so that $\langle a^*, b\rangle$ agrees with the inner product of forms in $\Omega_D^*(\tilde{\mathcal{A}})$.

□

The fact that the structure described here in terms of Hodge–Lefschetz modules parallels very closely the construction of the archimedean cohomology of [19], in the form presented in [20], suggests that the formalism of DimReg via noncommutative geometry may be useful also to describe a "neighborhood" of the fibers at infinity of an arithmetic variety.

8.5. **Vanishing and nearby cycles.**

In the algebro-geometric setting of a degeneration over a disk, the local monodromy plays an important role in determining the limiting mixed Hodge structure. In fact, geometrically, the difference between the cohomology of the generic fiber and of the special fiber is measured by the *vanishing cycles*. These span the reduced cohomology $\tilde{H}^*(M_z, \mathbb{C})$ of the Milnor fiber $M_z := B_P \cap f^{-1}(z)$, defined for $P \in Y$, B_P a small ball around P, and a sufficiently small $z \in \Delta^*$. The *nearby cycles* span the cohomology $H^*(M_z, \mathbb{C})$. To eliminate the non-canonical dependence of everything upon the choice of z, one usually considers all choices by passing to the universal cover $\tilde{\Delta}^* = \mathbb{H}$ of the punctured disk and replacing M_z by $\tilde{\mathfrak{X}}^* = \mathfrak{X} \times_\Delta \tilde{\Delta}^*$. The identification (*cf.* [30])

$$H^m(\tilde{\mathfrak{X}}^*, \mathbb{C}) \stackrel{\simeq}{\to} \mathbb{H}^m(Y, \Omega^{\cdot}_{\mathfrak{X}/\Delta}(\log Y) \otimes_{\mathcal{O}_{\mathfrak{X}}} \mathcal{O}_Y) \tag{8.18}$$

shows that, when working with nearby cycles, one can use a complex of forms with logarithmic differentials. This approach, with an explicit resultion of the complex, was used ([30], [24]) to obtain a mixed Hodge structure on $H^m(\tilde{\mathfrak{X}}^*, \mathbb{C})$ determined by $(H^m(\tilde{\mathfrak{X}}^*, \mathbb{C}), L_\cdot, F^\cdot)$, where $F^\cdot$ is the Hodge filtration and $L_\cdot$ is the Picard–Lefschetz filtration associated to the local monodromy.

The analogy described above with the structure of polarized Hodge–Lefschetz modules allows one to think of the operator

$$\Theta(a) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \nabla^n(a) D^{-2n}. \tag{8.19}$$

as the *logarithm of the local monodromy*, with ℓ_1 satisfying $e^\Theta = 1 + \ell_1$, and with the induced action of $(2\pi\sqrt{-1})\ell_1$ on the cohomology corresponding to the residue of the Gauss–Manin connection in the algebro-geometric setting.

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A. Connes: Collège de France, 3, rue d'Ulm, Paris, F-75005 France
E-mail address: alain@connes.org

M. Marcolli: Max–Planck Institut für Mathematik, Vivatsgasse 7, Bonn, D-53111 Germany
E-mail address: marcolli@mpim-bonn.mpg.de