Modern Portfolio Theory

Brief: Modern Portfolio Theory is a quantitative way for risk management in finance. It answers the question of how to balance between risk and return (i.e. profit). It starts with defining risk as the variance of the daily/weekly returns. And through matrix calculus, it develops a quantitative method of managing the risk.

Long:

1. Risk as variance:
   How to define risk quantitatively? Risk = Bad things? If you know for sure that a bad thing is going to happen, then it's unhappiness but not risk. Risk is considered as the lack of information about the future. Statistically speaking, it's the cloud of distribution function over an expected mean. If we only want a number, it's the variance/std deviation.
2. Development:
   First, a portfolio is defined as a set of assets (stock, bond, derivative, etc.). Each of the asset has its expected return and variance. Return is defined as the percentage change of the value of the assets over a day/week (Give examples here). Also we assume the money under our management is finite, we normalize it to 1. So $\boldsymbol{w'u} = 1$ (u is the unit n-vector of all ones, w is the vector of weights of assets, which is something we want to calculate).
   Suppose we know **m**=(E[X_1],E[X_2],…E[X_n])', the vector of arithmetic mean returns of all assets, then the mean return of our portfolio is $\boldsymbol{w'm}$ where **w** is the portfolio weight vector (Give examples here). Suppose we also know the covariance matrix C:

$$C = \begin{pmatrix} \sigma_{x1}^2 & \sigma_{x1,x2} & \cdots & \sigma_{x1,xn} \\ \sigma_{x2,x1} & & & \\ \cdots & & \cdots & \\ \sigma_{xn,x1} & \sigma_{xn,x2} & \cdots & \sigma_{xn}^2 \end{pmatrix}$$

   So $V = \boldsymbol{w'}C\boldsymbol{w}$ is the variance of the total portfolio.
   (Example of a portfolio of 2 assets is given. The notion of risk frontier is developed.)
   Generally for n assets, we want to minimize $V = \boldsymbol{w'}C\boldsymbol{w}$, subject to $\boldsymbol{w'm} = \mu \ and \ \boldsymbol{w'u} = 1$.
   Use Lagrange multiplier, $L = \boldsymbol{w'}C\boldsymbol{w} - \lambda_1(\boldsymbol{w'm} - \mu) - \lambda_2(\boldsymbol{w'u} - 1)$,
   Solution:

$$\lambda_2 = \frac{1-\left(u'C^{-1}m\right)\lambda_1}{\left(u'C^{-1}u\right)} \qquad (2.1)$$

$$w = \lambda_1\left(I - \frac{C^{-1}J}{u'C^{-1}u}\right)C^{-1}m + \frac{C^{-1}u}{u'C^{-1}u} \qquad (2.2)$$

$$w'Cw = \lambda_1^2\left(\frac{\left(u'C^{-1}u\right)\left(m'C^{-1}m\right)-\left(u'C^{-1}m\right)^2}{u'C^{-1}u}\right) + \frac{1}{u'C^{-1}u} \qquad (2.3)$$

   (Source: Kenneth Winston, Lecture slides of BEM111)
   ($J = \boldsymbol{uu'}$)

If we do not care about return and only want to minimize the variance, set $\lambda_1 = 0$

$$w = \frac{C^{-1}u}{u'C^{-1}u} \quad and \quad w'Cw = \frac{1}{u'C^{-1}u} \qquad (2.4)$$

3. Expansion:
   The mean-variance efficient frontier problem is often specified with the additional constraint that all weights must be non-negative. This changes the problem from one that can be solved in closed form to one that can be solved using a quadratic programming algorithm:
   E.g.:
   a) Minimize -λ**m'w** + ½**w'**C**w**
   b) Subject to each element of **w**≥0, **w'u**=1
4. Important Notes:
   Back in part 2, we use the covariance matrix to derive all the formulae. But how to get the covariance matrix?
   One way is to use historic data. But this might fail in the majority of situations. The problem is that we want to trade based on our solution, so the covariance matrix should be forward-looking variances and covariances, which are not necessarily equal to the variances and covariances in the past.
   Improvement: There is something called the option, which is the RIGHT to buy or sell the underlying asset at a fixed price at some time in the future (examples given). The options are of the most popular derivatives of stocks and indices. Under the assumption of normality for returns, people have developed models for calculating the fair value of the options:

$$C(S,t) = N(d_1)\ S - N(d_2)\ Ke^{-r(T-t)}$$

$$d_1 = \frac{\ln(\frac{S}{K}) + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(\frac{S}{K}) + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}.$$

   (Explanation given depending on the time) (Source: Wikipedia)
   So if we already have the price, we can calculate the "implied variance" and use it as the forward looking variance. But it's still hard to imply covariance.
5. Conclusion:
   MPT answers what the risk is, what the relationship between risk and return is and how we deal with it. But as the covariance matrix is hard to get, this theory is still under development and is generating tons of interesting thoughts recently.