The Visual Cortex is a Contact Bundle

William C. Hoffman

P.O. Box 2005
Sierra Vista, Arizona 85636

ABSTRACT

It is now generally accepted among neuroscientists that the sensory cortex of the brain is arranged in a structure that is simultaneously "topographic" (a pointwise mapping), layered, and columnar. The microcolumns in the columnar structure exhibit both a directional and an areal response in addition to the pointwise one. It is shown that these directional-areal response fields are contact elements upon the visual manifold that generate visual contours as the "lifts" of the form stimuli upon the retina into a contact bundle embodied in the visual cortex itself. Those invariances of form perception termed the "psychological constancies"—shape, size, motion, color constancies, etc.—represent the action of the conformal group CO(1,3) upon the visual manifold. It follows that the connection for the cortical contact bundle is a Lie group connection. The paper closes with an axiomatic treatment of visual perception and expression of Riegel's dialectical psychology in terms of the symmetric difference operation to provide an association between perceptual and cognitive function.

---

1. INTRODUCTION

This paper is devoted to a mathematical model of the brain utilizing the structures of modern differential geometry and topology. To some this may seem a bit much. Yet what happens at the *local, neuronal* level in the brain is integrated somehow into our *global* percepts and cognitions at the *macroscopic, psychological* scale. This duality between local and global scales and the primacy of form in our perceptions suggest that the connection with differential geometry and topology may not be so remote after all.

The cortex is a thin surface layer of neuronal tissue that surrounds the brain. In it the higher brain functions reside. The German word for the cortex is *Rinde*—rind—an apt descriptor of this tissue, only a couple of millimeters thick, wherein much of what we see, hear, and feel is actually perceived.

It is generally accepted among neuroscientists nowadays that the cerebral cortex is organized in a topographic, laminar, and columnar structure.

APPLIED MATHEMATICS AND COMPUTATION 32:137–167 (1989)

655 Avenue of the Americas, New York, NY 10010 0096-3003/89/$03.50

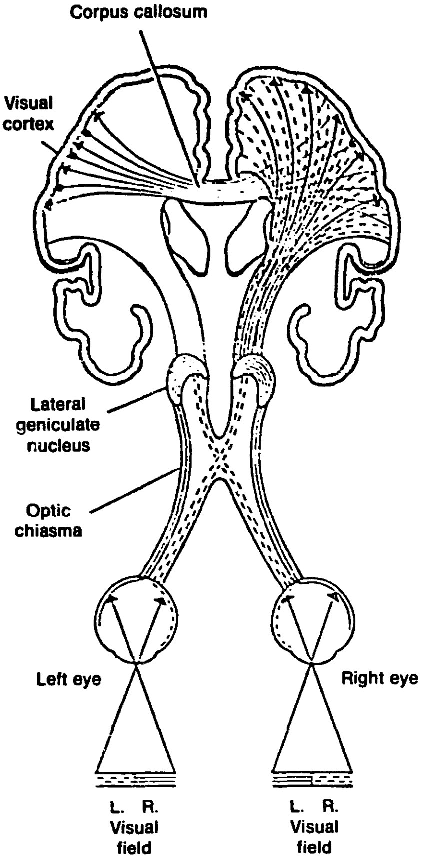


FIG. 1. The visual pathway: the visual field splits into right and left halves at the eyes, crosses over in part at the chiasm, becomes further sharpened and processed for color at the lateral geniculate bodies, and then flows onward, via the optic radiation, to the visual cortex at the back of the brain in humans.

"Topographic" here means something in the nature of the point-to-point retinotopic map from retina to visual cortex that Talbot and Marshall [1] discovered many years ago (Figure 1). The term "laminar" refers to the arrangement of the primary perceptual regions of the cortex in recognizable layers (Figure 2) that consist largely of neurons of a particular morphology [2]. A fuller description of these neuronal types is shown in Figure 3.

It is difficult for anyone versed in the topological—"qualitative"—theory of differential equations to gaze upon such Golgi-Cox preparations of cortical neurons without being reminded of a local phase portrait.[1] This aspect of the

---

[1] These Golgi-Cox preparations are *spatial* phase portraits of neuronal processes, *not* to be confused with such temporal phase portraits as those for the Hodgkin-Huxley equations for the propagation of the nerve impulse in a section of axon.

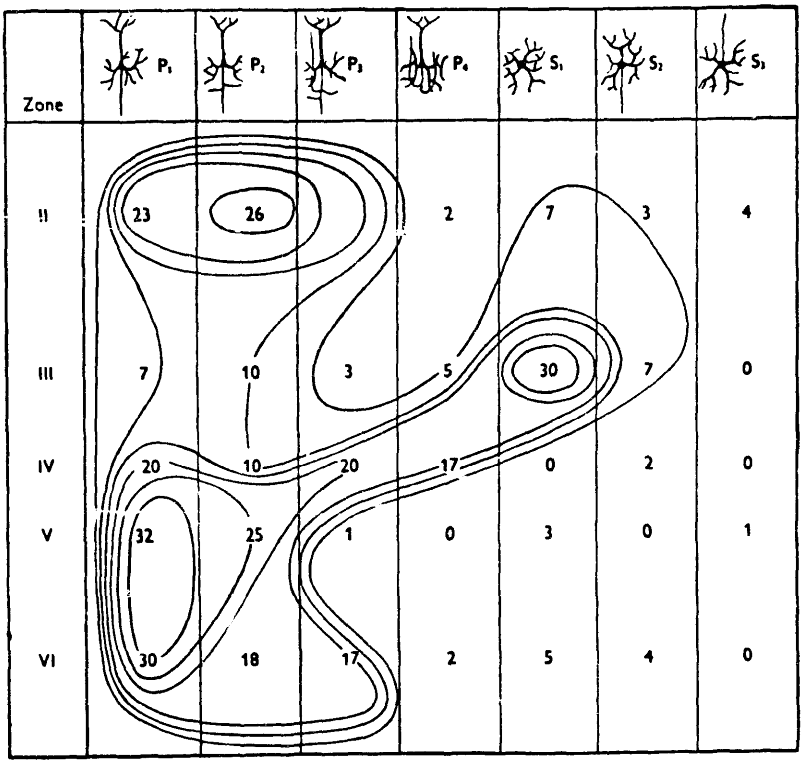


FIG. 2. A contour diagram for densities of 5, 10, 15, 20, 25, and 30 neurons per cubic centimeter of visual cortical tissue that shows the numbers of cells of different types contained in unit cortical volume in the several cortical layers. (See Figure 3 for identification of the seven types of cortical neurons.) The apparent "zoning" is one basis for the concept of a layered cortex; the other is that the layers are clearly discernible under the microscope. (After Sholl [2].)

matter is pointed up in the archetypal neurons studied by Colonnier [3] shown in Figure 4. The pyramidal cells, numbered 1, 2, and 3, possess a cylindrical symmetry. The stellate cells, 4 and 5, exhibit, on the other hand, a radial, spheroidal sort of symmetry.

The relative importance of the neuronal arborescence compared to the soma (the neuronal cell body) is indicated in Figure 5. What keeps pace with memory and learning throughout life is the growth and proliferation of the neuronal arborescence [4]. The proliferation process continues all through life unless terminated by senile dementia or Alzheimer's disease.

It is worthy of note that the basic neocortical circuit [5, 6] has all the characteristics of a hyperbolic dynamical system [7], as shown in Figure 6, which thus serves as a basis for Hebb's [8] cell assembly of reverberating neuronal circuits.

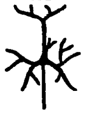
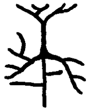
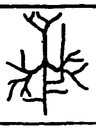
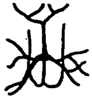
	Type	Description
	P_1	Pyramidal cell with unbranched axon to white matter
	P_2	Pyramidal cell with branched axon to white matter
	P_3	Pyramidal cell with branched axon to white matter and recurrent collaterals
	P_4	Pyramidal cell with axon forming recurrent collaterals and branches only
	S_1	Stellate cell with axon distributed within the dendritic field of the cell
	S_2	Stellate cell with axon to white matter
	S_3	Stellate cell with axon to outermost cortical zone

FIG. 3. The seven types of neuron in the visual cortex, classified according to their characteristic morphology. (After Sholl [2].)

"Columnar" in the cortical description refers to the latest dimension to be added to cortical structure, viz., the tiny cortical columns (hereafter *microcolumns*) found by Hubel and Wiesel [9–11], Mountcastle [12], and others. These microcolumns run transverse to the laminar structure. Several examples are shown in Figure 7. Most of the electrode tracks shown traverse several neighboring microcolumns (I, II, III, IV, and VI in Figure 7) owing to the curvature of the cortical convolutions. Every so often, however, conditions are just right, and so-called "pure penetrations" (V in Figure 7) occur. These "pure penetrations" are characterized by occurrence of the *same* orientation response throughout the entire microcolumn.

The term "orientation response" is employed by neuroscientists to describe the directionally sensitive response field of a single cortical neuron. As

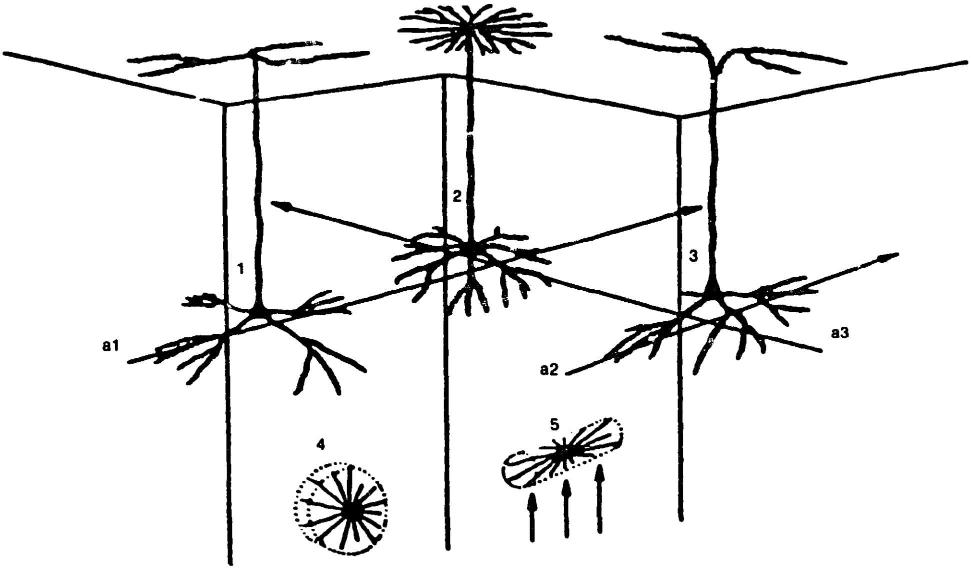


FIG. 4. Colonnier's archetypal cortical neurons. Types 1, 2, and 3 are pyramidal cells, exhibiting a cylindrical kind of symmetry. Types 4 and 5 are stellate cells, ones that possess primarily radial or spheroidal symmetry. (After Colonnier [3].)

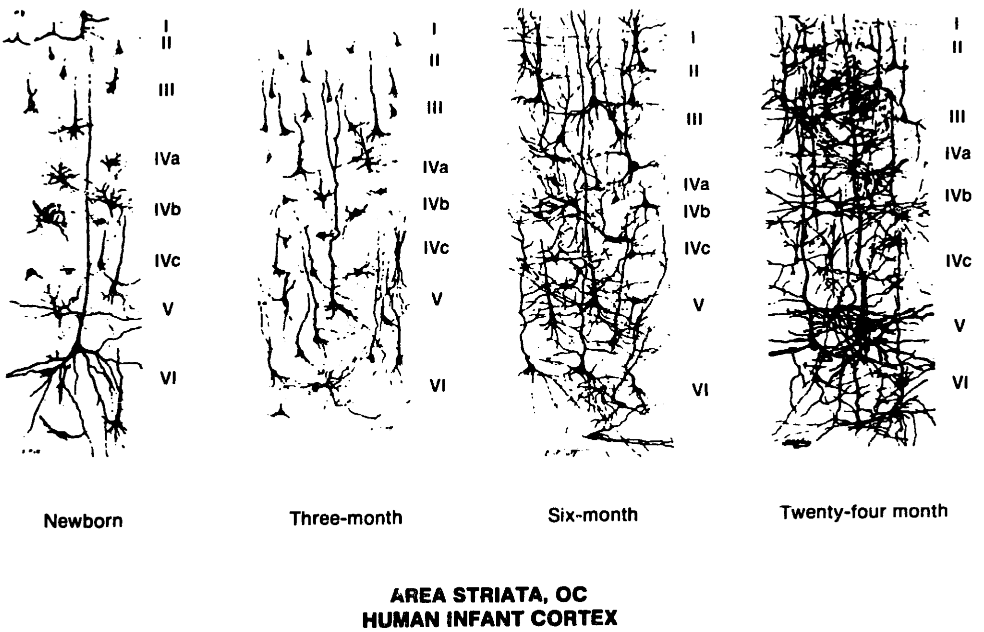


FIG. 5. Golgi-Cox preparations of neurons from the human visual cortex at the following stages of development: neonatal, 3 months, 6 months, 24 months. Neonatally, cortical neurons are mainly unipolar or bipolar, but shortly after birth their arborescences begin to branch and proliferate. This process continues all through life and keeps pace with the growth of memory and learning [4]. (After Pribram [51].)

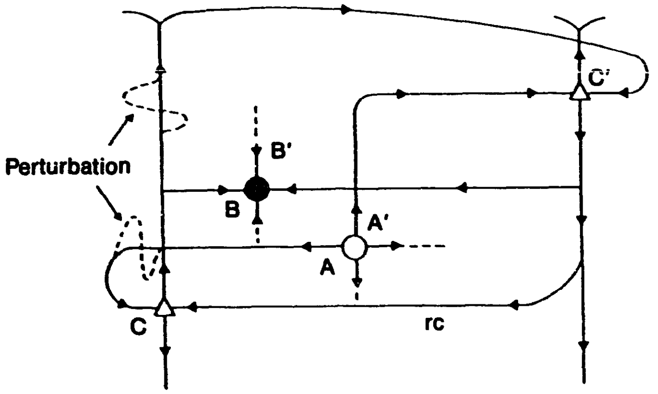


FIG. 6. Interpretation as a hyperbolic dynamical system of the Bishop-Coombs-Henry and Shepherd models for the basic neocortical circuit. *A* and *A'* denote excitatory stellate cells (sources), and *B* and *B'* are inhibitory stellate cells (sinks). *C* and *C'* are pyramidal cells (saddles). rc denotes a "recurrent collateral" type of axon. The dashed curves represent perturbations of the stationary system arriving in the form of an afferent volley of nerve impulses.

is clear from Figure 7, any particular cortical neuron has both a direction-field-like response and an areal one. In the experimental preparation, a dark bar or a slit of light is moved about on a screen in front of the subject until the neuron begins to fire. The stimulus element is then rotated until the maximum response is obtained, as shown in Figure 8. The direction of maximal response is said to represent the *orientation response* of the cell, indicated by an arrow in Figure 7.

We emphasize the dual nature of these response fields: directional *and* areal. The latter aspect is represented in Figure 7 by the associated rectangles. The essential features are depicted at length in Figure 9.

Hubel and Wiesel classified the orientation response fields as one of three types: simple, complex, and hypercomplex. Simple cells are those that encompass the least area. Complex cells were postulated by Hubel and Wiesel to be those representing the combined output of several simple cells. Hypercomplex cells (2 and III in Figure 7), in contrast to the length-independent response of simple and complex cells, represent orientation responses that are limited by the length or angular character of the stimulus element.

These three aspects of cortical neuron function—the microcolumnar structure, the linear nature of the response field, and the areal character of the latter—will all play key roles in the formal model adduced below for the way in which visual percepts are reconstructed from such microscopic orientation response fields.

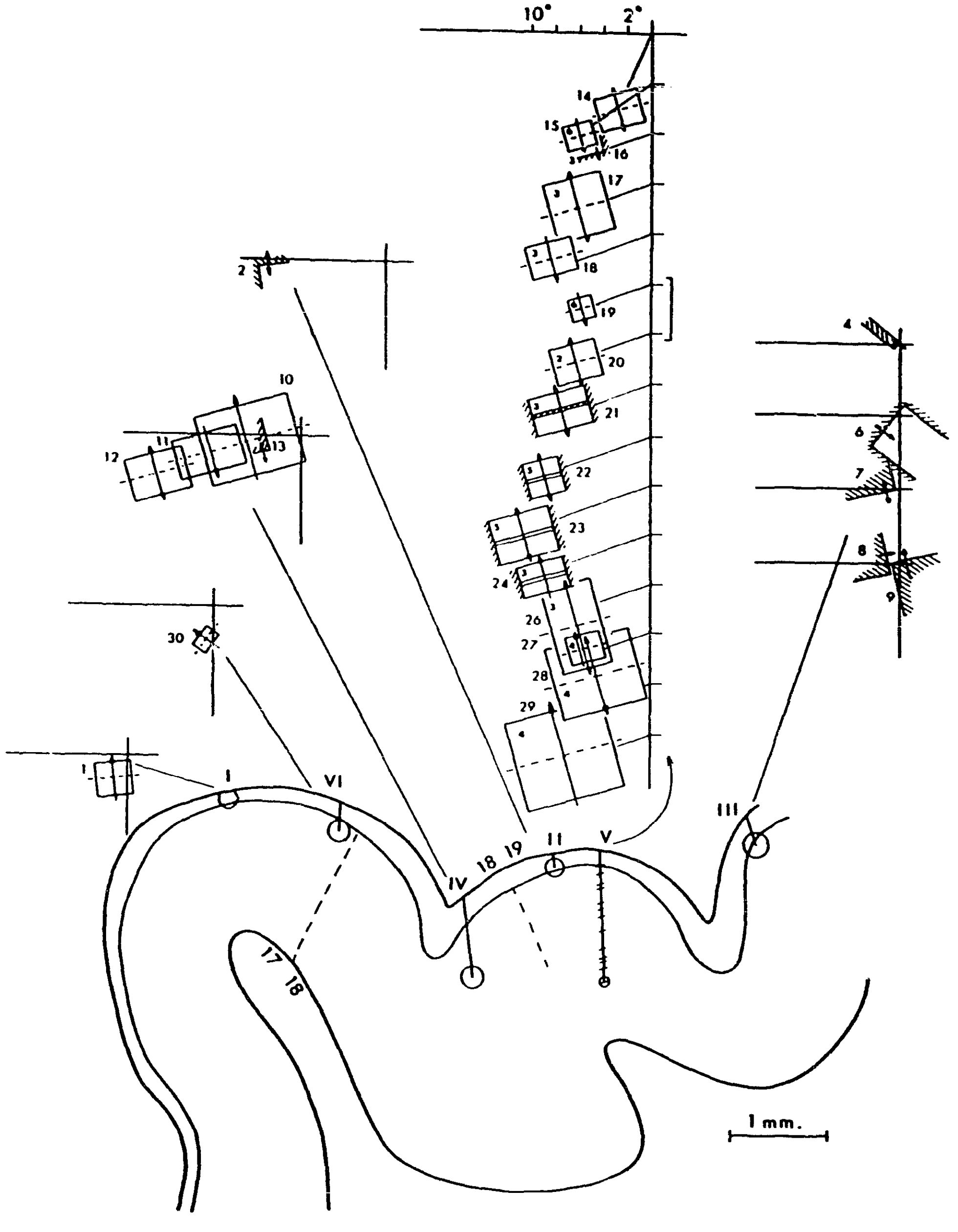


FIG. 7. Six microelectrode penetrations through the visual cortex (area 17) and the psychovisual cortex (areas 18 and 19) of a cat. The circles at the end of each track indicate electrolytic lesions made to locate the terminus of the track. Track *V* is a "pure penetration," i.e., one transverse to the cortical layers in which the microelectrode remained in the same microcolumn throughout. Penetrations I–IV and VI, on the other hand, are oblique penetrations which cross several microcolumns. The directional response at each stage of a penetration is indicated by an arrow; the position and size of the receptive field is indicated by a rectangle. In the upper right corner appear response fields characteristic of hypercomplex cells. (After Hubel and Wiesel [11].)

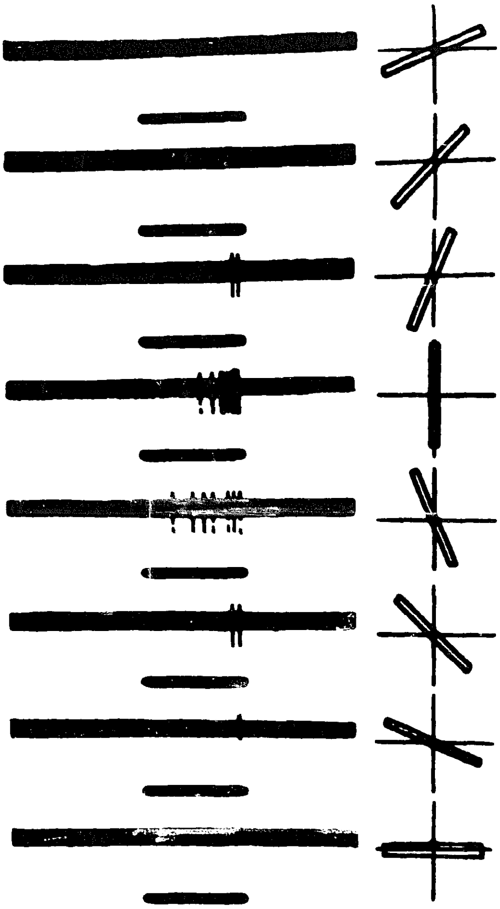


FIG. 8. A typical orientation response field (ORF) in the visual cortex: the neuronal firing rate response to shining a rectangular 1° × 8° slit of light on the receptive field of a neuron whose "orientation" (i.e., directional) response is maximal in the vertical direction.

2. THE VISUAL MANIFOLD

A basic property of visual perception for present purposes is the *figure-ground relation*, which generates the visual contours that bound a perceived object. The figure-ground relation appears to be intrinsic to visual perception. In patients blind from birth who have had their sight restored surgically, it is the first phenomenon to be observed [13]. We thus take as the *states* of the visual system these *visual contours*. Audition has a similar representation in terms of the trajectories of relaxation oscillations upon the cochlea [14], and haptic perception in the "Roman-soldier leggings" found by Werner [15] and his colleagues.

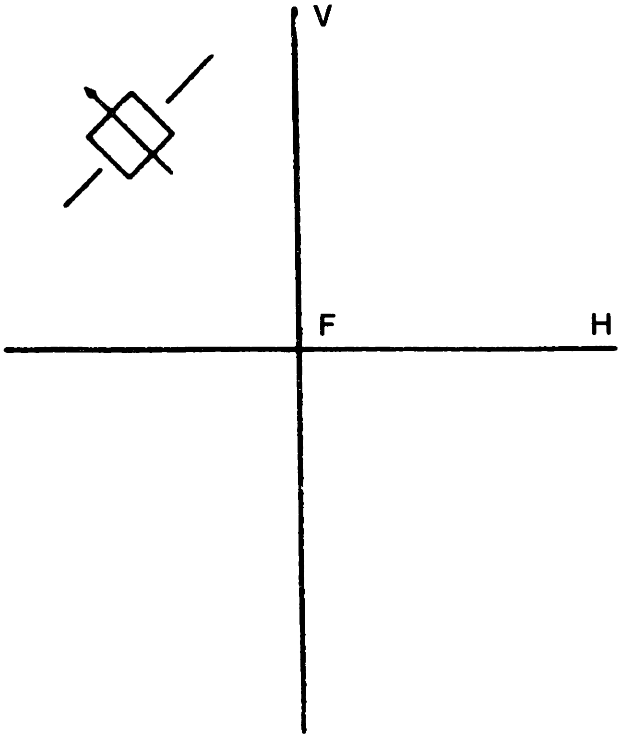


FIG. 9. The receptive field of a complex cell in visual cortex. The V and H axes represent respectively the vertical and horizontal meridians of the visual field, intersecting at the center of gaze F (the retinotopic projection of the fovea). This cell was in the right hemisphere; hence the receptive field—represented by the rectangle—lies in the left visual field. When this region is illuminated with diffuse light, no response is observed. Similar shining of small circular spots of light produces at best only weak or flickering responses. But an appropriately oriented line element source yields strong, reproducible responses. The particular cell recorded here responds best to a long, narrow slit of light oriented in the direction of the lines shown on either side of the response field. Movement of the line element in the transverse direction indicated by the arrow yields an especially strong response. (After Hubel and Wiesel [11].)

Referring again to Figure 1, we see that these paths, the visual contours, are "lifted" along the visual pathway from retina to visual cortex, where the patterns characteristic of the psychological constancies—shape, size, motion, color, etc.—are first "seen," and thence to the psychovisual cortex (areas 18 and 19 in the human visual cortex), wherein higher forms are perceived. Mathematically speaking, such a projection as this from retina to cortex constitutes the *path lifting property* (PLP) [16, 17]. A path on one manifold is "lifted" via a fibering to another manifold in coherent fashion. Here the first manifold is the visual field of view as imaged upon the retina; the second is what Colonnier termed the "cortical retina" within the occipital cortex, upon which all the contours, gradients, and textures of the visual image are somehow registered by the actions of countless tiny neurons. Both are manifolds in the technical, mathematical sense, a statement which we now proceed to justify.

The "center-surround" response fields of the cells of both retina and cortex constitute the *charts*, or "local coordinate patches," that cover the retina and so provide an atlas for the manifold. Adjacent, perhaps overlapping charts (response fields U_i and U_j) are connected in transition-function fashion by bipolar, horizontal, and ganglion cells of the retina. This center-surround type of response field is preserved along the visual pathway—the optic nerve to lateral geniculate body to optic radiation—and is again encountered at the cortical level. However, as was noted above, at the cortical level something new makes its appearance, namely, the directionally sensitive orientation response that is superimposed upon the center-surround response. At the cortical level we again have a manifold, but now one with a direction field also present. In the presence of a Lie transformation group such a direction field becomes oriented and so constitutes a vector field.

PRINCIPLE 2.1. *The cortical retina is a PL manifold.*

ARGUMENT. As demonstrated above, the retina $\mathscr{R}$ and the cortical retina V are manifolds; in the presence of the orientation response direction fields, the vector fields on V make it PL.

Figure 10 summarizes the essential features of the cortical structure of perceptual systems. The six-layer cortex is traversed by an ensemble of cortical microcolumns, each of which has a characteristic directional response, except in layer IVc, where incoming stimulus volleys terminate upon stellate cells. As indicated in Figure 10, the directional response varies smoothly in small increments across adjoining microcolumns. This structure is very suggestive of the mathematical entities of fibre bundle and cross section thereof, regarding the latter as a vector field.

PRINCIPLE 2.2. *The cortical retina is embedded in a "smooth"* (C^∞) *manifold.*

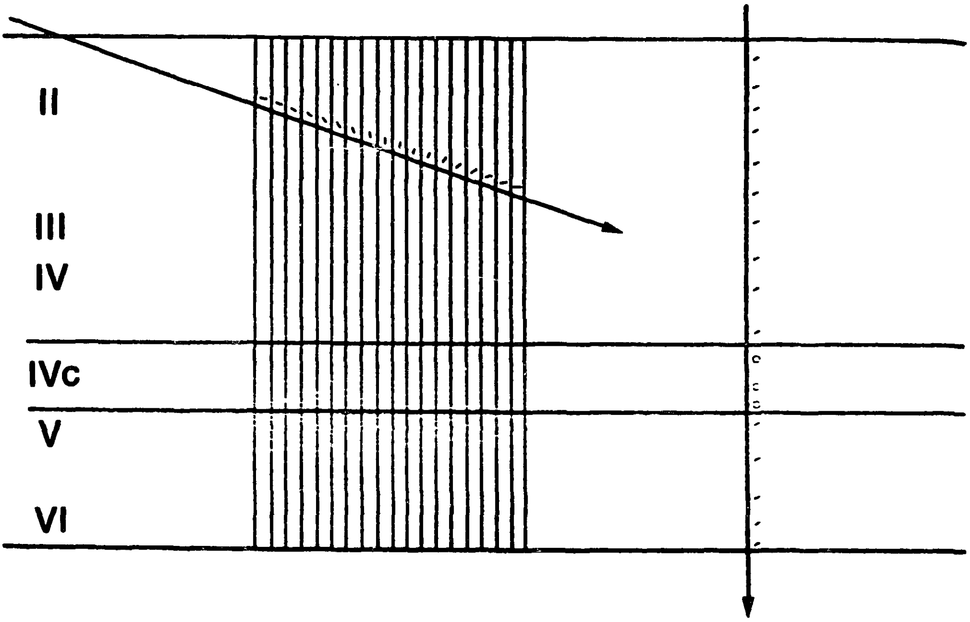


FIG. 10. A diagrammatic representation of the directionally sensitive microcolumns in the visual cortex of the monkey. Two penetrations are illustrated, one vertical ("pure"), the other oblique. In the vertical penetration the directional response remains constant from cell to cell within cortical layers above and below layer IVc. In layer IVc, wherein the afferent volleys terminate largely upon stellate cells, which then relay the stimulus on to pyramidal cells in other layers, the response fields exhibit circular symmetry without any directional dependence. In the oblique penetration one sees a systematic rotation of the ORF in increments of about 10° or less at every 50 μm or so. This variation may be continuous. (After Hubel and Wiesel [11].)

ARGUMENT. Quoting from Anderson [18],

> ...Not every space has a finite triangulation; those spaces which do are called polyhedra. Included among the polyhedra are such spaces as compact differential manifolds.

3. THE PATH LIFTING PROPERTY

Recall the figure-ground relation: primitive for form perception is the perception of objects defined by their bounding curves. Thus visual contours on the visual manifold V constitute the basic data for visual form perception, and similarly for the auditory and sensorimotor modalities. These paths, the visual contours, are "lifted" along the visual pathway by the actions of countless tiny neurons which, acting in concert, deliver a visual stimulus as such to our conscious awareness. So the problem is how to construct visual contours out of these microscopic orientation response fields (hereafter ORF's).

The thing one first thinks of is that the visual contours are integral curves of the cortical vector field embodied in the ORFs. In other words, the visual map is a tangent bundle $\pi: T\mathscr{R} = V \to \mathscr{R}$, where $\mathscr{R}$ denotes the retinal manifold and V the cortical "manifold of perceptual consciousness." But the ORFs unfortunately do line up head to tail as in an Euler line approximation to an integral curve. Furthermore, the ORFs have an areal character as well as a line-element one.

A second thought—the basis for which we shall proceed to establish: The path on the retina is lifted into a contact bundle $V = \mathscr{C}\mathscr{R}$, and this in turn is related in standard ways to a symplectic manifold.

To define a connection on an n-manifold, one needs either n vectors $X_i = \partial/\partial x_i - p_i \partial/\partial y$ or else n 1-forms $\theta = dy + \Sigma_i p_i dx_i$. In the present context of neuropsychology, these 1-forms are the ORFs, and the p_i correspond to the orientations.

Principle 3.1. *The visual pathway is a contact bundle $V = \mathscr{C}\mathscr{R}$, and the cortical ORFs represent contact forms defined by a connection 1-form, with an associated 2-form $\Omega = d\theta$ defined over a symplectic manifold.*

Before arguing this proposition we proceed to establish the existence of a connection on the basis of the following lemma.

Lemma 3.2. *The visual pathway has the path lifting property (PLP); the latter determines a law of parallel transport.*

To establish the existence of parallel transport and so the presence of a parallel connection one needs to verify the following five axioms [19]: (i) the existence of a unique parallel field within the vector bundle along any given path γ, with specified initial values; (ii) that the parallel transport map along γ which sends $\xi = P_\gamma \xi(0)$ to a parallel translate $P_\gamma \xi(u)$ is an isomorphism of linear vector spaces; (iii) that the parallel transport is independent of parametrization; (iv) that the parallel transport depends smoothly on the initial conditions; (v) that the initial tangent vectors for two different parallel transport flows are equal.

We are thus concerned with a phenomenon like inertial navigation, wherein a gyroscopically fixed direction is used to generate parallel transport of a frame of reference along the world line of an aircraft. The line element

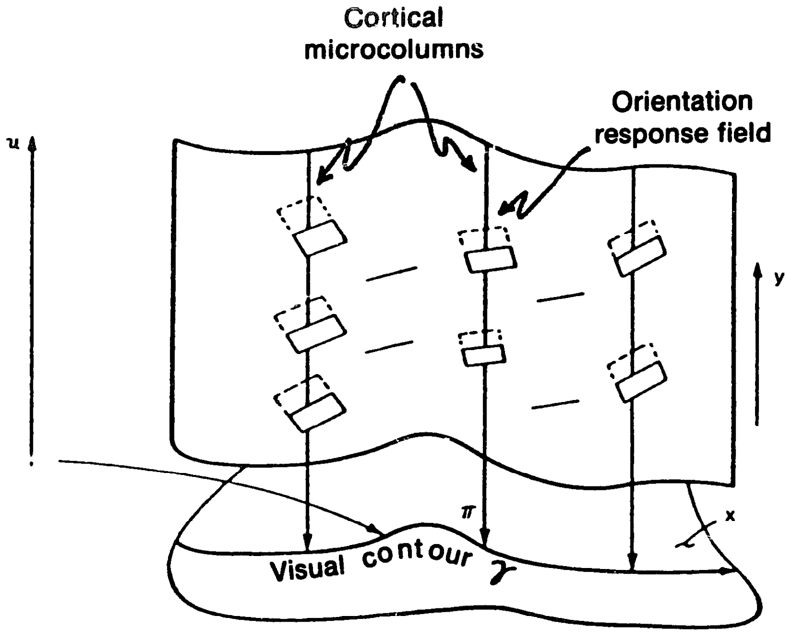


FIG. 11. A connection in a fibre bundle is fully determined by a law of parallel transport. At any given point in the bundle the operation of parallel transport may be used to lift all the curves γ that pass through the base point, thus specifying a vector field that is constant along the curve. Given one vector along the curve, all the rest are determined by the connection. The lifted curves have a unique contact element, viz., the horizontal subspace of the connection. (After Burke [21].)

system, starting from a given initial value, determines a field that is constant along the path γ.

We argue Lemma 3.2 in terms of the abstract ORF configuration shown in Figure 10. With reference to axiom (i) above, visual inspection indicates the presence of such a unique parallel field. The initial value cited in requirements (iv) and (v) above corresponds to some distinguished point referred to the center of vision on whatever contour is presented, the tangent vectors being determined by different connection 1-forms. Figure 11 depicts the essential features of parallel transport induced by such a connection in the fibre bundle V and defines the coordinates on the right hand side of the expression $\theta = dy + \Sigma_i p_i \, dx_i$. The association between Figures 11 and 10 seems clear.

V, being a vector bundle, must have vector field cross sections consisting of linear vector spaces, and the uniqueness and invertibility required by axiom (iii) are assured. Any particular realization of a perceptual system is of course an associated fibre bundle for some archetypal principal fibre bundle that represents such idealized bundle structures as those of Figure 10. Hence

(iii) follows; the neuropsychological embodiment of a visual contour γ is coordinate-free.

An important instance of the mapping expressed in axiom (iv) is the exponential map from the Lie algebra (of a vector field) to a Lie transformation group (the corresponding orbitstructure). The significant role of Lie transformation groups in the present context will be made clear farther on.

4. CONTACT BUNDLES AND THE ARGUMENT FOR THE MAIN PRINCIPLE 3.1

Three local structures are defined upon a manifold M: the tangent bundle TM; the cotangent bundle T^*M, dual to TM; and a family of contact bundles $\mathscr{C}M$. The latter are subsumed under the general rubric of cotangent bundles but possess more properties than the latter and greater flexibility in their definition. A contact bundle is defined as the collection of ordered pairs

$$\mathscr{C}M = \{(m, p) | \forall m \in M\}, \tag{1}$$

where p is a line element at m.

Contact is a weaker notion than tangency. Both are preserved under smooth maps but in the contact case the maps, though agreeing in direction, need not agree in rate, as would be the case for a vector field. The dimension of $\mathscr{C}M$ is also one less than that of TM, which expresses a relative degree of simplicity. Further, a number of contact bundles can be defined over the same manifold, while such is not the case for TM and T^*M.

The essential point for present purposes is that [20] a contact bundle singles out those special submanifolds—the integral manifolds—that are lifted into $\mathscr{C}M$ from the base space M. Though contact elements *need not necessarily* be integrable, when they are they generate an integral submanifold. The case for the plane $\mathbb{R}^2$ is shown in Figure 12, where the two-dimensional contact elements in the upper part of the figure do not "tile" together to generate a surface embedded in 3-space, while those in the lower part do. In such a case a lifted submanifold is characterized by the relations

$$p = f_{,x'}, \qquad q = f_{,y'}, \qquad f_{,x} + fy_{,f} = 0 \tag{2}$$

that govern the fundamental 1-form on $\mathbb{R}^2$:

$$\theta = df - p\,dx - q\,dy. \tag{3}$$

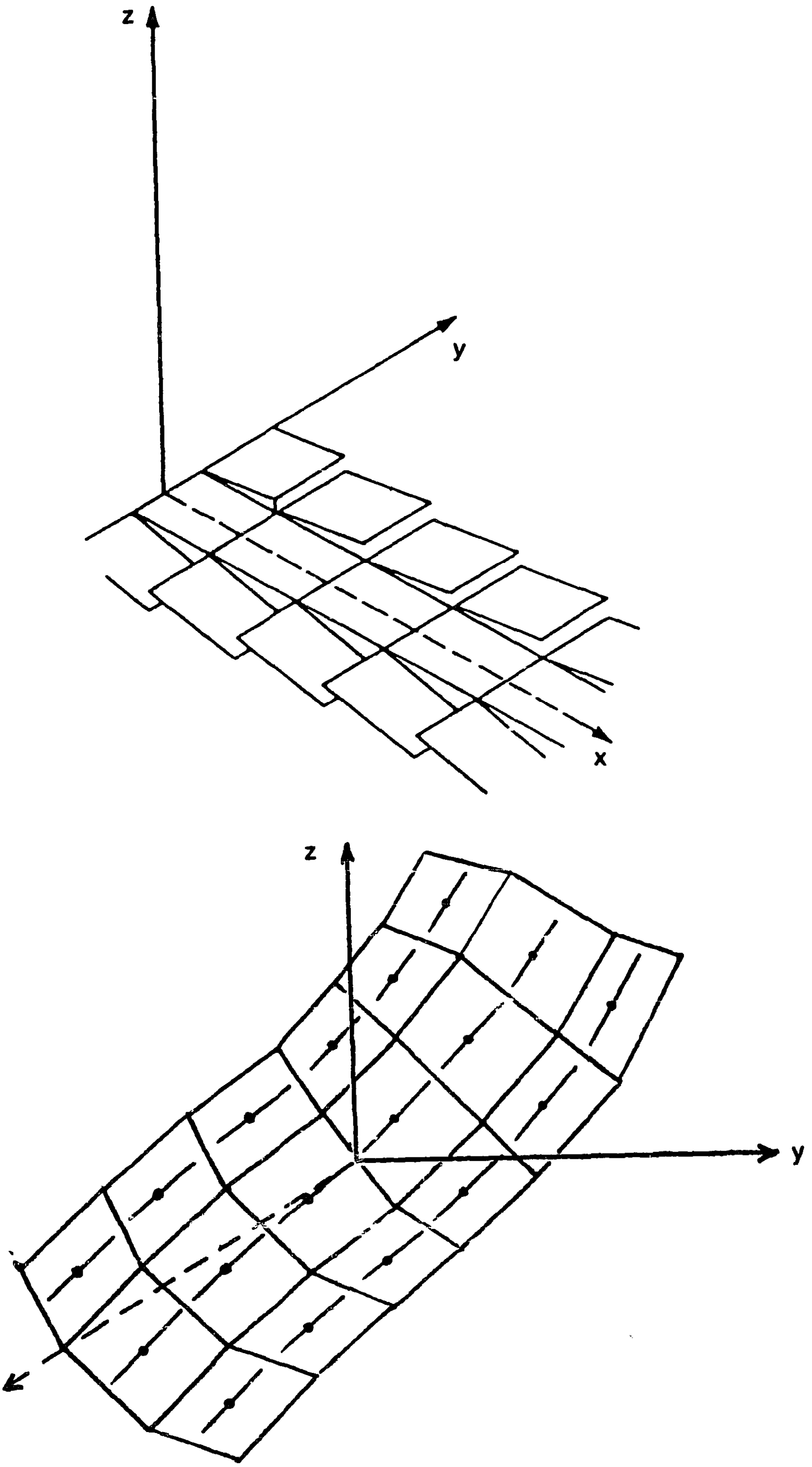


FIG. 12. (a) A nonintegrable field of contact 2-elements. (Such fields are the rule rather than the exception, in the real world. After Burke [21].) (b) An integrable field of contact elements that "tile together" to generate a surface as an integral manifold.

Theorem [21]. *For any submanifold in $\mathscr{R}$ to be a lift (to $V=\mathscr{C}\mathscr{R}$), p_i must equal $p_{,i}=\partial y/\partial x_i$; in other words, a contact structure picks out those special submanifolds that lift as integral curves of $\mathscr{C}\mathscr{R}$.*

Theorem [22]. *Every compact orientable 3-manifold carries a contact structure.*

Application. The visual manifold V is certainly compact and 3-dimensional. It is also orientable, for it carries a Lie transformation group, that of the psychological constancies, as discussed below. It follows that V does possess a contact structure.

Argument for Principle 3.1. A visual contour, which is a path upon the retina $\mathscr{R}$, is lifted into a principal contact bundle

$$V=\mathscr{C}\mathscr{R}=\{T^*\mathscr{R},\pi,\mathscr{R},G\}, \tag{4}$$

that is determined by the ORFs on V, regarding the latter as contact forms

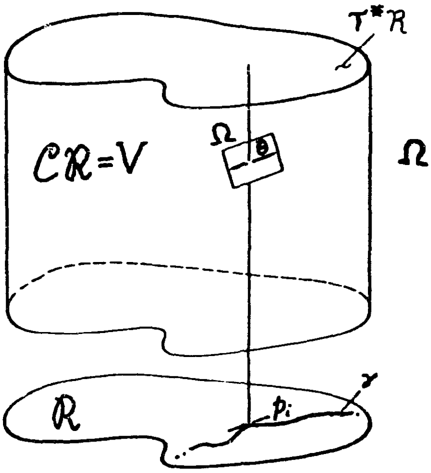


Fig. 13. The geometric pattern of a connection for the lifted curve γ. The vector field is represented by a section of the cotangent bundle $T^*\mathscr{R}$ on the retina $\mathscr{R}$. The local affine approximation to a section is a contact element Ω associated with the line element θ. The essence of a connection lies in the specification of a contact element Ω for each point in the contact bundle $\mathscr{C}\mathscr{R}=V$ (the visual manifold). Such a contact element represents at that particular point the idea of a constant field. (After Burke [21].)

defined by a connection 1-form θ (Figure 13). In local coordinates the connection 1-form is

$$\theta = dy + \sum_i p_i \, dx_i \tag{5}$$

and $\theta \wedge d\theta \neq 0$ in V.

When the p_i do match up with the direction field elements that are locally tangent to the visual contour, the latter will lift to the cortical contact bundle. The essential point is that the contact transformations involved in $\mathscr{C}\mathscr{R}$ take lifted curves to other lifted curves in diffeomorphic fashion.

5. LIE TRANSPORT: THE CONTACT BUNDLE IS HOMOGENEOUS

Contact transformations are those that preserve the contact structure of a contact bundle. In the present instance the transformations that act are those of a Lie group, the conformal group CO(1,3). We thus have to do with a *homogeneous* contact bundle [23], wherein the Lie group G in (4) acts transitively and effectively on $\mathscr{R}$ and θ. The infinitesimal transformation of such a Lie group is known [24] to be fully determined by its action on a function, a vector field, and a differential form. In the present application we therefore invoke the concept of a Lie derivative "dragging" these fundamental structures along the orbits of the Lie group G.

DEFINITION 5.1. A function f, a vector field Y, or a differential form θ is said to be *Lie transported* or *dragged along the flow* of a vector field X that defines a Lie derivative $\mathscr{L}_X$ provided that

$$\mathscr{L}_X f = 0, \quad \mathscr{L}_X Y = 0, \text{ or } \mathscr{L}_X \theta, \qquad \text{respectively.} \tag{6}$$

DEFINITION 5.2. An infinitesimal contact structure $\mathscr{I}$ is one for which

$$\mathscr{L}_X \mathscr{I} = \mathscr{I}, \tag{7}$$

where $\mathscr{I}$ is the *contact ideal* consisting of all absolute and relative invariants of the group of $\mathscr{L}_X$, given by the relations

$$\mathscr{L}_X u = 0, \qquad \mathscr{L}_X u = f(x)u, \qquad \mathscr{L}_X \theta = 0, \quad \text{or} \quad \mathscr{L}_X \theta = g(x)\theta. \qquad (8)$$

The conditions (8) determine the *infinitesimal symmetries* of the Lie group.

All elements of such a differential ideal $\mathscr{I}$ pull back to zero on an integral submanifold. Any given contact-element field thus determines an ideal that consists of all 1-forms θ such that $\theta \cdot X = 0$ for all X belonging to the contact element. These 1-forms generate the differential ideal.

Integrability of the contact-element field is of prime importance in the present application if visual contours are to somehow arise out of microscopic ORFs. Frobenius' theorem asserts that such an integral submanifold will exist only if the contact element is *completely integrable*. This means that for any two vector fields X and Y in the contact field, their commutator, or Lie bracket, is also a member of the contact field.

A *Lie group connection* is a connection that, though not linear, is compatible with the Lie group structure on V [25]. Lie transport of a vector field X belonging to the Lie algebra admits *two* natural transport operations along a curve $\gamma(v)$ in V, one for left translations, the other for right translations, as shown in Figure 14. Let $x(v)$ in the figure denote any curve in V that has X as tangent vector and let $x(0) = g$. Then the group

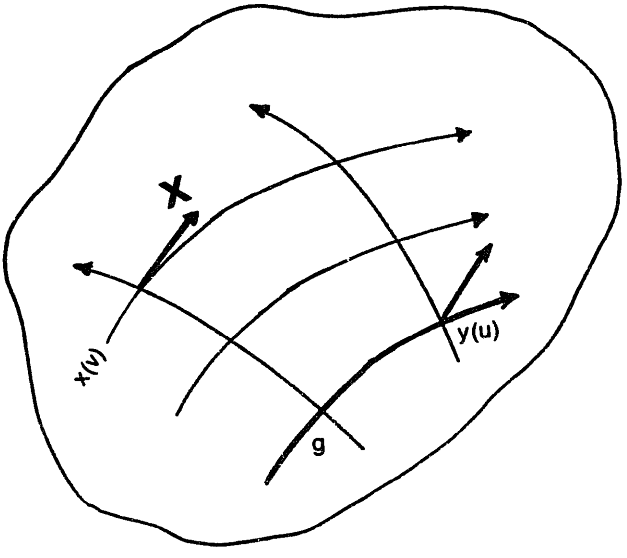


FIG. 14. One of the two natural connections on a Lie transformation group $G \times V \to V$, with action $\tau_g x = x'$. See the text for details. (After Burke [21].)

multiplication

$$v \to (u) g^{-1} x(v) \tag{9}$$

determines a tangent vector at $\gamma(u)$ that corresponds to left translation of X. Similarly the orbit

$$v \to x(v) g^{-1} \gamma(u) \tag{9'}$$

determines a tangent vector at $\gamma(u)$ that corresponds to right translation of X. Either transport operation can be used to provide a connection. For present purposes the important aspect is that the head and tail of a vector X will be carried along neighboring integral curves, whenever such exist, and they will if the Lie group vector field is completely integrable, i.e., the conditions on the Lie product of Frobenius' theorem do hold.

PRINCIPLE 5.3. *Transport of perceptual contour elements within the cortical contact bundle takes place via a Lie group connection.*

ARGUMENT. The base vector field on V is generated by the Lie group G_v of the psychological constancies [4, 14, 26–29]—size, shape, motion, color, pitch, loudness constancies, etc. The Lie group G_v may be readily identified by its actions on perceptual contours, in particular by those orbital structures that the group action leaves invariant. For the projection of the field of view on the frontal plane, these are listed in Table 1.

In general, $G_v = \mathrm{CO}(1,3)$, the group of conformal transformations of space-time [30, 31, 29, 32]. Here the group CO(1,3) acts on subjective space-time $S = (\mathbb{R} \times V, g)$ with signature $(+, -, -, -)$. One of the group actions is discrete: reflections in the vertical meridian. The remaining connected components of the identity constitute a 15-parameter Lie group [33]:

(i) the affine group: $v' = Av + b,\ A \in \mathrm{SO}(1,3), b \in S, v \in V$;
(ii) the group of dilations: $v' = \lambda v,\ \lambda \in \mathbb{R}^{+}$;
(iii) special conformal transformations, so-called "accelerations":

$$v' = \frac{v + av^2}{q(a, v)}.$$

TABLE 1

CONSTANCIES AND LIE TRANSFORMATION GROUPS

Constancy	Lie transformation group	Lie derivative(s)[a]
Shape constancy:	Affine (unimodular) group SL(2)	$\left(\partial_x = \frac{\partial}{\partial x}, \partial_y = \frac{\partial}{\partial y}, \partial_t = \frac{\partial}{\partial t}\right)$
location within the field of view	translation groups—horizontal, vertical	$L_x = \partial_x, L_y = \partial_y$
(form memory)	time translations	$L_t = \partial_t$
orientation	rotation group SO(2)	$L_O = -y\partial_x + x\partial_y$
Afferent binocular perception	Pseudo-Euclidean (hyperbolic) rotations	$L_b = y\partial_x + x\partial_y$
(Efferent binocular function)	Pseudo-Euclidean rotations in space[b]-time	$L_B = x\partial_x - y\partial_y$ $L_{B1} = t\partial_t - x\partial_x$ $L_{B2} = t\partial_t - y\partial_y$
Size constancy	Dilation group	$L_s = x\partial_x + y\partial_y$ $L_{s1} = x\partial_x + t\partial_t$ $L_{s2} = y\partial_y + t\partial_t$
Perception of moving objects	Generalized Lorentz group of order 2	$L_M = -L_O, L_{M1} = x\partial_\tau - \tau\partial_x$ $L_{M2} = y\partial_\tau - \tau\partial_y$ $L_m = L_M, L_{m1} = \tau\partial_x + x\partial_\tau$ $L_{m2} = \tau\partial_y + y\partial_\tau$

[a] $\tau = \hat{c}t$, $\hat{c}$ = peak cortical signal velocity.

[b] Space here is 2-dimensional, viz, the x, y-frontal plane, the projection of physical 3-space.

The infinitesimal transformations of these Lie group actions are generated by

$$\begin{aligned} &\text{translations} && p_{,i} = \frac{\partial}{\partial x_i}; \\ &\text{general Lorentz transformations} && L_{ij} = x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}; \\ &\text{dilations} && D = \sum_i x_i \frac{\partial}{\partial x_i}; \\ &\text{special conformal transformations} && k_i = 2x_i \sum_j x_j \frac{\partial}{\partial x_j} - x^2 \frac{\partial}{\partial x_i}. \end{aligned} \tag{10}$$

On the visual manifold V, translations correspond to invariant recognition of any given perceived form wherever it may be in the field of view—right or left, up or down. The L_{ij} generate motion constancy as well as other components of shape constancy (rotation, obliquity, afferent and efferent binocular function) and color constancy. Dilations are of course the transformations that generate size constancy—the invariance of perceived forms whether close up or distant. The special conformal transformations are also involved in size constancy.

In visual processing the constancy transformations are the first to act on a perceived form stimulus, thus generating equivariant perception and so a considerable economy in memory storage. A given form does not have to be remembered in every one of its possible distortions under the conditions of viewing. Hence we take the generating vector fields of the constancy transformations as the vector(s) X in (9) or (9′) and "drag" them along the visual contour [the $\gamma(u)$ in (9) or (9′)]. A visual stimulus contour is therefore traced out within the perceptual cortex as a lift of the visual path on the retina. Examination of Figure 10 makes it clear how this comes about. Such a connection is known to induce parallel displacement of fibres within the associated contact bundle [34], thus generating configurations like that of Figure 10.

Contact bundles are preludes to jet bundles [35], and the latter correspond to prolongations of a Lie transformation group [36]. Prolongation provides the answer to the natural question: integral curves (the orbits) of CO(1,3) may suffice for such basic visual patterns as those of the constancies (rectangular grids—the so-called "spatial frequency response," circles, rays, and hyperbolas), but what about higher visual forms, such as those we see all about us at this very moment?

PRINCIPLE 5.4 [7, 27–29]. *Higher visual perception represents invariance of higher perceived forms through extension of the contact bundle* $\mathscr{C}\mathscr{R}$ *to higher jet bundles through prolongation of the Lie derivatives of* CO(1,3).

ARGUMENT. The ideal of Equations (7) and (8) is extended to higher differential invariants u_k by such relations as

$$\mathscr{L}_X^{(k)} u_k = 0, \qquad (k = 1,2,\ldots). \tag{11}$$

In the frontal plane, u_k is a function of x and y and the first k derivatives

$y', y'', \ldots, y^{(k)}$. The prolonged Lie derivative takes the form

$$\mathscr{L}_X^{(k)} = \mathscr{L}_X + \sum_{j=1}^{k} X_{(j)}(x, y, y', y'', \ldots, y^{(j)}) \frac{\partial}{\partial y^{(j)}}. \tag{12}$$

The $X_{(j)}$ are thus determined recursively from the equations involving higher and higher order differential forms according to the rule

$$\mathscr{L}_X^{(j)}\left[dy^{(j-1)} - y^{(j)}dx\right] = 0, \qquad (j = 1, 2, \ldots, k).$$

These last expressions correspond to the extended contact structure and have their cortical correlate in the proliferation of the neuronal arborescence, regarding the latter as the local phase portrait for the Pfaffian system associated with (11) and (12). Compare Figure 5.

COROLLARY 5.5. *The jet bundles in Principle* 5.4 *are contact bundles of dimension one order lower than the corresponding tangent bundles.*

ARGUMENT. Let $\alpha(x, y, y', y'', \ldots, y^{(k)})$ denote an nth order differential form that is invariant under the group of $\mathscr{L}_X$. Then substitution of the canonical variables u and u_1 defined by $\mathscr{L}_X u = 0$ and $\mathscr{L}_X^{(1)} u_1(x, y, y') = 0$ reduces the differential form to one of order $n - 1$ that is determined by a $(n - 1)$st order differential equation in the canonical variables themselves:

$$\frac{d^{n-1}u_1}{du^{n-1}} = \beta\left(u, u_1, \frac{du_1}{du}, \ldots, \frac{d^{n-2}u_1}{du^{n-2}}\right). \tag{13}$$

Of course it is well known that any contact bundle is one dimension lower than the corresponding tangent bundle [37]; but the important point in the present context is the—

APPLICATION. The local phase portrait of a dynamical system—or the corresponding Pfaffian system or the Lie-group-generated flow—represents the orbit structure of the Lie transformation group and its prolongations. Flows through the neuronal arborescence thus *simulate* the computations involved in the group action. The absolute invariant u and the first differential invariant u_1 are generated not by actually solving a Pfaffian system but rather by afferent flows of nerve impulses through an appropriate configuration of neuronal arborescences. (Compare Figure 5.) Incoming nerve signals from the optic radiation are known to terminate at several thousand synaptic junctions on both the soma and dendrites of a neuron, but mainly on the latter. The complicated morphology of the neuronal arborescence is thereafter replaced by a single path, that of the emergent axon, along which discharges of nerve impulses constitute translations in the arborescence-generated canonical variables u and u_1, in the style of Equation (13). For planar

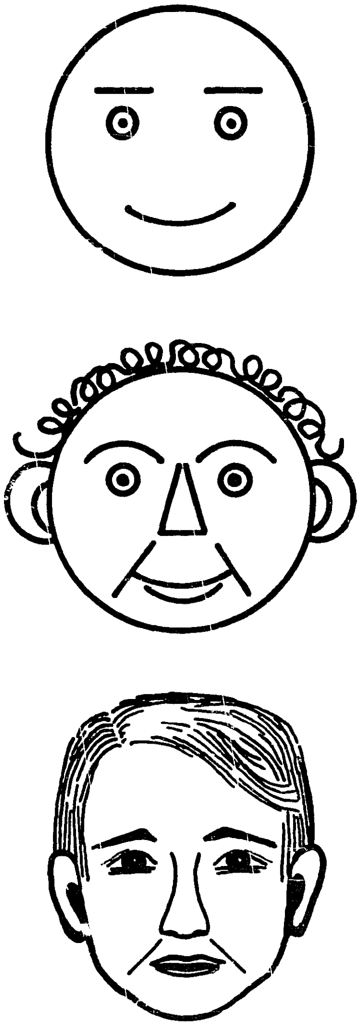


FIG. 15. Progressive modification via absolute and differential invariants of a picture face that leads eventually to generation of a more or less realistic face. In (a) and (b) the basic figures are generated as orbits of the translation group, the rotation group O(2), and linear combinations of these two actions.

visual stimulus patterns the u_1 invariant is a differential form in y', and du_1/du represents local curvature [17] in accord with the well-known theorem [37] that any plane curve is determined by its intrinsic equation in the curvature variable.

As a somewhat simplified instance of the perceptual use of differential invariants, consider Figure 15. Figure 15(a) depicts a picture face, familiar from countless posters and advertisements. All the contours of Figure 15(a) consist of orbits of either O(2) or the group of horizontal translations. In Figure 15(b) the orbits of O(2) are again apparent, but now we also see a triangular nose (an invariant of the projective conformal group [38]), curly hair (partial orbits of the combined rotation and dilation groups that generate spirals), and some mouth lines (orbits of the dilation group, centered at the triangular nose). The realistic face of Figure 15(c) superimposes upon these basic invariants—the group orbits—the higher differential invariants generated by prolongation of the Lie derivatives of the foregoing constancy groups. It is these curves that undergo Lie-group transport within the cortical jet bundle.

6. THE AREAL ASPECT—SYMPLECTIC AND CONTACT STRUCTURE

Noted above in connection with Figure 7 was the fact that any ORF has an areal as well as a directional aspect. The cortical retina is 3-dimensional, but subjective space-time $S = \mathbb{R} \times V$ is 4-dimensional. Such an even dimensional manifold carrying a closed 2-form Ω of maximal rank is a *symplectic manifold*. Any cotangent bundle T^*M is an instance of such, and there always exists on any cotangent bundle an invariant 1-form. This canonical 1-form was taken above to be the connection 1-form θ. In local coordinates $(q_1, \ldots, q_n; p_1, \ldots, p_n)$ on V, the invariant 1-form is

$$\theta = \sum_{i=1}^{n} p_i \, dq_i, \tag{14}$$

and the 2-form on the associated symplectic manifold is

$$\Omega = d\theta = \sum_i dp_i \wedge dq_i. \tag{15}$$

Since $d\Omega = dd\theta = 0$, Ω is in fact closed and of maximal rank (since $\Omega^n \neq 0$ in S).

The corresponding canonical contact form on $\mathbb{R} \times T^*\mathscr{R} \approx \mathbb{R} \times \mathscr{C}\mathscr{R}$ is

$$\omega = dt - \theta = dt - \sum_i p_i\, dq_i. \tag{16}$$

It is well known [39] that there exists a symplectization functor that associates a symplectic manifold of dimension $2(n+1)$ with a contact manifold of dimension $2n+1$, and conversely, a contactization functor that takes a symplectic manifold of dimension $2n$ to a contact manifold of dimension $2n+1$. Here we invoke the latter and in the static configuration associated with the ORFs regard

$$\sum_{i=1}^{3} p_i\, dx_i \tag{17}$$

as the canonical 1-form determined by the local direction field, while

$$\sum_{i=1}^{3} dp_i \wedge dx_i \tag{18}$$

is the symplectic 2-form determining the areal nature of the ORF. Both directional and areal characteristics of ORFs are thus described in mutually consistent terms, terms that lead at once to a coherent means for representing visual forms of arbitrary degrees of complexity that are at the same time in accord with the essential nature of ORFs, as indicated in Figures 7 and 10.

7. AXIOMS FOR VISUAL PERCEPTION

Any mathematical model is an axiom system, whether stated verbally, or in terms of equations, or graph-theoretically, or whatever. Here we take this opportunity to update the visual system axioms laid down earlier [32]. There, although the contact structure was explicitly mentioned, visual contours were assumed to be integrals of the Lie-group-generated vector fields. As mentioned above, the ORFs do not line up in proper fashion to be a vector-field-generated integral curve.

Axiom 1. *The figure-ground relation and visual contours are equivalent to a path-connected visual manifold embedded in subjective space-time.*

Principle 7.1. *Visual contours lift along the visual pathway from retina $\mathcal{R}$ to cortex V into a contact bundle $\mathcal{C}\mathcal{R} = V$ possessing a parallel connection. The latter represents the directional nature of ORFs.*

Argument. This is the essential nature of Hubel and Wiesel's orientation response fields (ORFs) for individual cortical neurons: they are at the same time contact elements and symplectic elements that lift visual contours from retina to cortex. Visual contours are the "lifts" of retinal contours and are generated by contact differential forms.

Axiom 2. *The distortion-free invariant perceptions generated from perceived shapes by the action of the psychological constancies (shape, size, motion, color, etc.) correspond to the action of the conformal group* CO(1,3) *on subjective space-time* $S = \mathbb{R} \times V$.

The key word here is "invariance." Form memory and the constancies would be pointless phenomena in a completely chaotic environment. For the expression of invariance of integral curves on a differentiable manifold, Lie transformation groups governing these orbits are well known to be the natural mathematical structure. Which transformation groups it is that are acting can readily be determined by noting which shapes are left invariant by the distortion-generating transformations and then making comparison with the corresponding orbit structure. It follows that the parallel connection of Principle 7.1 is a Lie group connection (Principle 5.3).

Principle 7.2. *The modern version of the neuron doctrine, including what is known about the "synaptology" of cortical neurons, implies that the neuron is a Lie group germ and that the neuronal arborescences constitute local phase portraits for the corresponding Pfaffian systems.*

Argument. Principle 7.2 is similar to the cell theory of biology in that it unifies the "local" (microscopic) and "global" (psychological) levels of neuropsychological organization. The corresponding Lie group structures are infinitesimal transformations at the local level and orbit structures at the global. The latter part of the proposition follows as an immediate consequence of Axiom 2, given the well-known relations among the Lie group germ, the (prolonged) Lie derivative, and the associated Pfaffian system whose solution yields the invariants of the transformation group, and making comparison with the morphology of Golgi-Cox preparations of cortical neurons [4, 7, 28, 40, 2, 41]. See Figures 2 through 6 above.

PRINCIPLE 7.3. *Invariant and programmable neuropsychological systems are fibrations: Serre or Hurwicz fibre bundles in the case of the posterior perceptual systems, and Kan fibrations for the frontal-inferotemporal cognitive systems.*

ARGUMENT. Throughout the central nervous system (CNS) the basic structure and function appear to be that of fibration: a total space (cortex) that projects (via efferent nerve pathways) onto a base space (the midbrain region). (Compare Figure 1.) Owing to the presence of the ORFs and the resultant implication of the polyhedral covering homotopy property (PHCP), the cortical contact bundle is a Serre fibration. The cross section map in this case consists of the afferent volleys of nerve impulses that transmit visual stimuli from retina to brain. Since there is a structure group, viz., CO(1,3), the fibration is in fact a Serre fibre bundle. The PHCP is known to imply the CHP for arbitrary base spaces, and it therefore follows that the cortical contact bundle is a fibre bundle in the sense of Hurwicz as well. Finally [42], "the image of a fibration in the sense of Serre under the singular complex functor is a fibration in the sense of Kan." The latter is general enough to handle any aspect of information-processing psychology, and the presence in cognitive phenomena of the simplicial functor (the category of simplicial objects) has been argued at length in [29] and [43].

Invariance in the Kan fibration case, which is embodied in such higher cognitive faculties as conscious thought, plans, long-term behavioral sequences, etc., apparently corresponds to Freyd's "The Theorem" [44]: *an elementary property on categories is invariant up to equivalence types of categories iff it is a diagrammatic property.*" Thus "chasing around the diagram" in "trains of thought" is more basic than "logical," conscious thought. Intuition precedes insight and inspiration and certainly the laborious processes of conscious thought itself.

According to dialectical psychology [45, 46], thought processes consist of continual transformation of contradictory experience into momentarily stable cognitive structures. The essential feature is sensed contradiction between thesis and antithesis that progresses toward synthesis of the commonalities present, perhaps at a higher level. It is not an "if...then..." kind of reasoning but rather, as Sherlock Holmes himself is supposed to have put it,

> … when you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.

An appropriate mathematical structure for expressing such dialectical thought processes appears to be the set-theoretic operation of *symmetric difference*. The symmetric difference[2] $A\$B$ of two sets A and B consists of those elements that are in A or B but not both:

$$A\$B = A\cup B-(A\cap B)=(A\cap\tilde{B})\cup(\tilde{A}\cap B). \tag{19}$$

Riegel laid down the following more or less traditional dialectical "laws" for dialectical psychology:

I. The unity and struggle of opposites.
II. The transformation of quantitative into qualitative change.
III. The negation of the negation.

Law II is well expressed by such a set operation as (19). For law III we require the complement of $A\$B$:

$$\sim(A\$B)=(A\cap B)\cup(\tilde{A}\cap\tilde{B}). \tag{20}$$

Law I thus consists of the two-stage combination of processes (19) and (20). In $A\cap B$ in (20) we have the commonality—the "synthesis"—of A and B. In $\tilde{A}\cap\tilde{B}$ we have everything else—that which is outside of A and B. The second stage (20) thus admits both "convergent thinking" via $A\cap B$ and "divergent thinking" in terms of $\tilde{A}\cap\tilde{B}$.

Introduction of the symmetric difference into a paper oriented about differential geometry and topology may seem somewhat far afield, but the isomorphism between the symmetric difference and mod-2 homology [47, 48] justifies such a model, once the category of simplicial objects has been brought in to represent cognitive objects and processes. We thus have—

PRINCIPLE 7.2. *The symmetric difference operation represents the fundamental operations of dialectical psychology.*

---

[2] The notation \$ is used here instead of the usual $\triangle$ because the latter is already preempted by the simplicial category. \$ may be regarded as the superposition of S (for "symmetric") and | , which is at least close to the standard symbol \ for set difference, and besides \$ is on the typewriter keyboard.

ARGUMENT. The category of contact bundles is functorial to the category of simplicial objects, since the former is included in the category of fibre bundles. The category of simplicial objects describes information processing psychology nicely [29, 49]. Mod-2 homology is isomorphic with the symmetric difference. The modifications of information processing psychology required for the dialectical processes of dialectical psychology are, as indicated above, well represented by the symmetric difference and its complement.[3] The symmetric difference and its complement can, like the Sheffer stroke, be used to generate a Boolean algebra in its entirety. Hence all of "logical" thought is accessible via $ even though the simplicial objects and functors involved in cognitive processing are more basic.

REFERENCES

1 S. A. Talbot and W. H. Marshall, *Amer. J. Ophthalmol.* 24:1255 (1941).
2 D. A. Sholl, *The Organization of the Cerebral Cortex*, Hafner, New York, 1967.
3 M. Colonnier, *J. Anatomy (London)* 98:327 (1964).
4 W. C. Hoffman, *Kybernetik* 8:151 (1971).
5 P. O. Bishop, J. S. Coombs, and G. H. Henry, *J. Physiol.* 219:659 (1971).
6 G. M. Shepherd, *The Synaptic Organization of the Brain*, Oxford U.P., New York, 1974.
7 W. C. Hoffman, Figural synthesis by vectorfields: Geometric neuropsychology, in *Figural Synthesis* (P. C. Dodwell and T. M. Caelli, Eds.), Erlbaum, Hillsdale, N.J., 1984.
8 D. O. Hebb, *The Organization of Behavior*, Wiley, New York, 1949.
9 D. H. Hubel and T. N. Wiesel, *J. Physiol.* 160:106 (1962).
10 D. H. Hubel and T. N. Wiesel, *J. Neurophysiol.* 28:229 (1965).
11 D. H. Hubel and T. N. Wiesel, *Proc. Roy. Soc. London* 198B:1 (1977).
12 V. B. Mountcastle, An organizing principle for cerebral function: The unit module and the distributed system, in *The Mindful Brain* (G. M. Edelman and V. B. Mountcastle, Eds.), M.I.T. Press, Cambridge, Mass., 1978.
13 M. von Senden, *Space and Sight: The Perception of Space and Shape in the Congenitally Blind Before and After Operation*, Free Press, Glencoe, Ill., 1960.
14 W. C. Hoffman, *Cahiers de Psychologie* 20:135 (1977).
15 G. Werner, The topology of the body representation in the somatic afferent pathway, in *The Neurosciences: Second Study Program* (F. O. Schmitt et al., Eds.), Rockefeller U.P., New York, 1970.
16 J. Dugundji, *Topology*, Allyn & Bacon, Boston, 1966, p. 396.
17 E. H. Spanier, *Algebraic Topology*, McGraw-Hill, New York, 1966, p. 66ff.
18 D. W. Anderson, *Bull. Amer. Math. Soc.* 84:765 (1978).

---

[3] A full description of what is involved here is given in [50].

19 W. A. Poor, *Differential Geometric Structures*, McGraw-Hill, New York, 1981, pp. 44–46.

20 W. L. Burke, *Applied Differential Geometry*, Cambridge U.P., New York, 1985, p. 137.

21 W. L. Burke, *Applied Differential Geometry*, Cambridge U.P., New York, 1985.

22 J. Martinet, Formes de contact sur les variétés de dimension trois, in *Proceedings of the Liverpool Singularities Symposium II* (C. T. C. Wall, Ed.), Springer-Verlag, New York, 1971.

23 W. M. Boothby, *Bull. Inst. Math. Acad. Sinica* 8:341 (1980).

24 S. Sternberg, *Lectures on Differential Geometry*, Prentice-Hall, Englewood Cliffs, N.J., 1964, p. 92.

25 W. L. Burke, *Applied Differential Geometry*, Cambridge U.P., New York, 1985, p. 356.

26 W. C. Hoffman, *J. Math. Psychol.* 3:65 (1966), 4:348 (1967).

27 W. C. Hoffman, *Mathematical Biosci.* 6:437 (1970).

28 W. C. Hoffman, The Lie transformation group approach to visual neuropsychology, in *Formal Theories of Visual Perception* (E. L. J. Leeuwenberg and H. Buffart, Eds.), Halsted Press, Chichester, England, 1978.

29 W. C. Hoffman, *Internat. J. Math. Modelling* 1:349 (1980).

30 P. C. Dodwell, A model for adaptation to distortions of the visual field, in *Models for the Perception of Space and Visual Form* (W. Wathen-Dunn, Ed.), M.I.T. Press, Cambridge, Mass., 1967.

31 P. C. Dodwell, *Visual Pattern Recognition*, Holt, Rinehart & Winston, New York, 1970.

32 W. C. Hoffman and P. C. Dodwell, *Canad. J. Psychol.* 39:491 (1985).

33 F. Bayen, Conformal invariance in physics, in *Differential Geometry & Relativity* (M. Cahen and M. Flato, Eds.), D. Reidel, Dordrecht, 1976.

34 S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Wiley-Interscience, New York, 1963, p. 113.

35 W. L. Burke, *Applied Differential Geometry*, Cambridge U.P., New York, 1985, p. 106.

36 V. Guillemin and S. Sternberg, *Deformation Theory of Pseudogroup Structures*, AMS Memoir No. 64, Amer. Math. Soc., Providence, R.I., 1966.

37 H. W. Guggenheimer, *Differential Geometry*, Dover, New York, 1977, pp. 143–144.

38 G. W. Bluman and J. D. Cole, *Similarity Methods for Differential Equations*, Springer-Verlag, New York, 1974, p. 53.

39 V. I. Arnold, Critical points of smooth functions, in *Proceedings of the International Congress of Mathematicians, Vancouver, 1974,* Vol. 1, Canad. Math. Congress, 1974, p. 30.

40 W. C. Hoffman, *Quart. Appl. Math.* 25:423 (1968).

41 R. Lorente de No, Architectonics and structure of the cerebral cortex, in *Physiology of the Nervous System* (J. F. Fulton, Ed.), Oxford U.P., Oxford, 1949.

42 P. Gabriel and M. Zisman, *Calculus of Fractions and Homotopy Theory*, Springer-Verlag, Berlin, 1967, p. 133.

43 W. C. Hoffman, Mathematical models of Piagetian psychology, in *Towards a Theory of Psychological Development* (S. Modgil and C. Modgil, Eds.), N.F.E.R. Publishing Co., Windsor, England, 1980.
44 P. Freyd, Properties invariant within equivalence types of categories, in *Algebra, Topology, and Category Theory* (A. Heller and M. Tierney, Eds.), Academic, New York, 1976.
45 K. Riegel, *Human Develop.* 16:346 (1973).
46 J. F. Rychlak (Ed.), *Dialectic: Humanistic Rationale for Behavior and Development*, S. Karger, Basel, 1976.
47 M. Henle, *A Combinatorial Introduction to Topology*, Freeman, San Francisco, 1979, p. 149.
48 J. G. Hocking and G. S. Young, *Topology*, Addison-Wesley, Reading, Mass., 1961, p. 232ff.
49 W. C. Hoffman, *Internat. J. Man-Machine Stud.* 22:613 (1985).
50 W. C. Hoffman, A formal model for dialectical psychology, preprint, 1987.
51 K. H. Pribram, The New neurology: Memory, Novelty, Thought, and Choice, in *EEG and Behavior* (H. Glaser, Ed.), Basic Books, New York, 1963.