

TOPOLOGIES OF RANDOM GEOMETRIC COMPLEXES ON RIEMANNIAN MANIFOLDS IN THE THERMODYNAMIC LIMIT

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ABSTRACT. We investigate the topologies of random geometric complexes built over random points sampled on Riemannian manifolds in the so-called “thermodynamic” regime. We prove the existence of universal limit laws for the topologies; namely, the random normalized counting measure of connected components (counted according to homotopy type) is shown to converge in probability to a deterministic probability measure. Moreover, we show that the support of the deterministic limiting measure equals the set of all homotopy types for Euclidean geometric complexes of the same dimension as the manifold.

1. INTRODUCTION

Sarnak and Wigman [SW16] recently established, utilizing methods developed by Nazarov and Sodin [NS16], the existence of universal limit laws for the topologies of nodal sets of random band-limited functions on Riemannian manifolds. In the current paper, we adapt these methods to the setting of random geometric complexes, that is, simplicial complexes with vertices arising from a random point process and faces determined by distances between vertices.

Kahle [Kah11] made the first extensive investigation into the topology of random geometric complexes generated by a point process in Euclidean space (zero-dimensional homology of random geometric graphs were also investigated earlier in [Pen03]). The expectation of each Betti number is studied within three main phases or regimes based on the relation between density of points and radius of the neighborhoods determining the complex: the subcritical regime (or “dust phase”) where there are many connected components with little topology, the critical regime (or “thermodynamic regime”) where topology is the richest (and where the percolation threshold appears), and the supercritical regime where the connectivity threshold appears. The thermodynamic regime is seen to have the most intricate topology. Many cycles of various dimensions begin to form as we enter this regime and many cycles become boundaries as we leave this regime.

Random geometric complexes on Riemannian manifolds were studied earlier in the influential work [NSW08] of Niyogi, Smale, and Weinberger, where the manifold is embedded in Euclidean space and the distance between vertices is given by the ambient Euclidean distance. In the current paper, we use geodesic distance to build the complexes when working in the Riemannian manifold setting. The main question in [NSW08] is motivated by applications in “manifold learning” and concerns the recovery of the topology of a manifold via a random sample of points on the manifold. Consequently, the authors only consider a certain window within the supercritical regime. The subsequent study [BM15] includes the thermodynamic regime where they provide upper and lower bounds of the same order of growth for each Betti number.

Yogeshwaran, Subag, and Adler [YSA17] have established limit laws in the thermodynamic regime for Betti numbers of random geometric complexes built over Poisson point processes in Euclidean space. Their results include limit theorems for expectations as well as concentration inequalities and central limit theorems.

A survey of other results on random geometric complexes is provided in [BK18]. Most progress in this area has been made only recently, but the problem of studying the topology of a random geometric complex (or equivalently the ε -neighborhood of a random point cloud) can be traced back to one of Arnold's problems (see the historical note at the end of the introduction).

A novelty of the current paper is that, whereas previous studies of random geometric complexes have focused on Betti numbers, we consider enumeration of connected components according to homotopy type, a count that provides more refined topological information. We also note that our results provide the first limit law addressing the thermodynamic regime for random geometric complexes in the Riemannian manifold setting, revealing universality (and reduction to the Euclidean setting) for these limits, see Theorem 1.1 below.

1.1. The Riemannian case. Let (M, g) be a compact Riemannian manifold of dimension $\dim(M) = d$, with normalized volume form $\text{Vol}(M) = 1$. Let $U_n = \{p_1, \dots, p_n\}$ be a set of points independently sampled from the uniform distribution on M . We denote by $\hat{B}(x, r)$ the Riemannian ball¹ centered at $x \in M$ of radius $r > 0$. We fix a positive number $\alpha > 0$ and build the random set:

$$(1.1) \quad \mathcal{U}_n = \bigcup_{k=1}^n \hat{B}(p_k, \alpha n^{-1/d}).$$

We denote by $\check{C}(\mathcal{U}_n)$ the corresponding Čech complex (which for $n > 0$ large enough, is homotopy equivalent to \mathcal{U}_n itself, see Lemma 6.1 below).

Let now $\hat{\mathcal{G}}$ be the set of equivalence classes of M -geometric, connected simplicial complexes, up to homotopy equivalence (observe that this is a countable set). In other words, $\hat{\mathcal{G}}$ consists of all the simplicial complexes that arise as Čech complexes of some finite family of balls in M . Note that different manifolds give rise to different sets $\hat{\mathcal{G}}$. For example, among all \mathbb{R}^d -geometric complexes we cannot find complexes with nonzero d -th Betti number; but if $M = S^d$, such complexes belong to $\hat{\mathcal{G}}$. When $M = \mathbb{R}^d$ we simply denote this set by \mathcal{G} .

Given \mathcal{U}_n as above, we define the random probability measure $\hat{\mu}_n$ on $\hat{\mathcal{G}}$:

$$\hat{\mu}_n = \frac{1}{b_0(\check{C}(\mathcal{U}_n))} \sum \delta_{[s]},$$

where the sum is over all connected components s of \mathcal{U}_n , $[s]$ denotes the type of s (i.e., the equivalence class of all connected complexes homotopy equivalent to s), and b_0 denotes the number of connected components.

Remark 1. Next theorem deals with the convergence of the random measure $\hat{\mu}_n$ in the limit $n \rightarrow \infty$. We endow the set \mathcal{P} of probability measures on the countable set $\hat{\mathcal{G}}$ with the *total variation distance*:

$$d(\mu_1, \mu_2) = \sup_{A \subset \hat{\mathcal{G}}} |\mu_1(A) - \mu_2(A)|.$$

In this way $\hat{\mu}_n$ is a random variable with values in the metric space (\mathcal{P}, d) . Convergence *in probability* (which is used in Theorem 1.1 and Theorem 1.3) of a sequence of random variables $\{\mu_n\}_{n \in \mathbb{N}}$ to a limit μ means that for every $\epsilon > 0$ we have $\lim_n \mathbb{P}\{d(\mu_n, \mu) > \epsilon\} = 0$.

Theorem 1.1. *The random measure $\hat{\mu}_n$ converges in probability to a universal deterministic measure $\mu \in \mathcal{P}$ supported on the set \mathcal{G} of \mathbb{R}^d -geometric complexes.*

¹In this paper we adopt the convention that when an object is denoted with a “hat” sign, then it is related to M . Analogous objects related to Euclidean space will have no “hat”. For example a ball in M is denoted by $\hat{B}(x, r)$ and a ball in \mathbb{R}^d by $B(x, r)$.

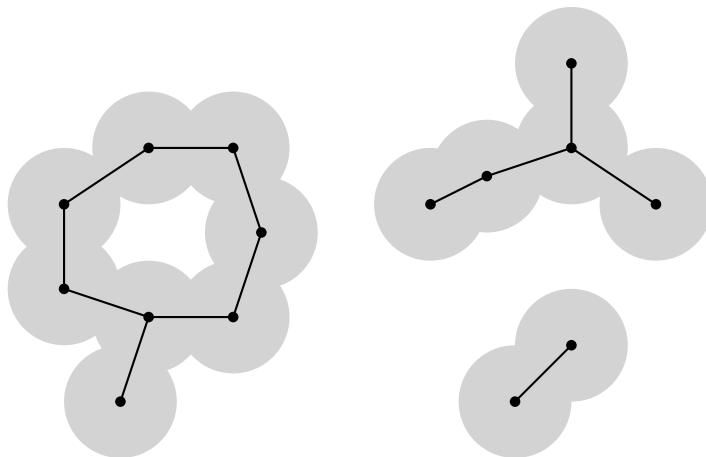


FIGURE 1. A geometric complex in the plane which is homotopy equivalent to $S^1 \cup \{p_1\} \cup \{p_2\}$. The corresponding measure on the set of homotopy classes of connected, simplicial complexes is: $\hat{\mu} = \frac{1}{3} (\delta_{[S^1]} + 2\delta_{[\text{pt}]})$. Theorem 1.1 says that as $n \rightarrow \infty$ the random measure $\hat{\mu}_n$ converges to a deterministic measure.

The “universal” in the previous statement means that μ does not depend on M (but it depends on d and on α).

Remark 1. Since \mathcal{G} is a proper subset of $\hat{\mathcal{G}}$, the measure μ does not charge some points in $\hat{\mathcal{G}}$. This is consistent with the findings of [BW17] where it was shown that an additional factor of $\log n$ is needed in the radii of the balls defining \mathcal{U}_n in order to see the so-called “connectivity threshold” where nontrivial d -dimensional homology appears.

Remark 2. We can write the limiting measure μ as:

$$\mu = \sum_{\gamma \in \mathcal{G}} a_\gamma \delta_\gamma$$

for some non-negative constants a_γ , $\gamma \in \mathcal{G}$, which depend on the $\alpha > 0$ appearing in (1.1), and are defined by Proposition 2.1; Proposition 1.2 below implies they are all strictly positive.

Proposition 1.2 (Existence of all topologies). *Let $\mathcal{P} \subset \mathbb{R}^d$ be a finite geometric complex and $\alpha > 0$. There exist $R, a > 0$ (depending on \mathcal{P} and α but not on M) such that for every $p \in M$:*

$$\mathbb{P} \left\{ \mathcal{U}_n \cap \hat{B}(p, Rn^{-1/d}) \simeq \mathcal{P} \right\} > a.$$

Example 1. An interesting consequence of the previous Proposition 1.2 is the following fact: given a compact, embedded manifold $P \hookrightarrow \mathbb{R}^d$, then for $R > 0$ large enough with positive probability the pair (\mathbb{R}^d, P) is homotopy equivalent to the pair $(\hat{B}(p, Rn^{-1/d}), \mathcal{U}_n \cap \hat{B}(p, Rn^{-1/d}))$. This follows from the fact that, by [NSW08, Proposition 3.1], one can cover P with (possibly many) small Euclidean balls $P \subset \bigcup_{k=1}^{\ell} B(p_k, \epsilon) = \mathcal{U}$ with the inclusion $P \hookrightarrow \mathcal{U}$ a homotopy equivalence – hence the pair (\mathbb{R}^d, P) is homotopy equivalent to a pair $(\mathbb{R}^d, \mathcal{P})$ with \mathcal{P} a \mathbb{R}^d -geometric complex.

1.2. The local model. The proof of Theorem 1.1 for the Riemannian case involves a study of a rescaled version of the problem in a small neighborhood of a given point. Specifically, one can fix $R > 0$ and a point $p \in M$ and study the asymptotic structure of our random complex only in the ball $\hat{B}(p, Rn^{-1/d})$. The random geometric complex that we obtain in the $n \rightarrow \infty$ limit can be described as follows.

For $R > 0$ let $P_R = \{p_1, p_2, \dots\}$ be a set of points sampled from the standard spatial Poisson distribution on $B(0, R) \subset \mathbb{R}^d$ and for $\alpha > 0$ consider the random set:

$$\mathcal{P}_R = \bigcup_{p \in P_R} B(p, \alpha).$$

Note that each $B(p, \alpha)$ is now convex and, by the Nerve Lemma, \mathcal{P}_R is homotopy equivalent to the simplicial complex $\check{C}(\mathcal{P}_R)$. The relation between $\mathcal{U}_n \cap \hat{B}(p, Rn^{-1/d})$ and \mathcal{P}_R is described in Theorem 3.1.

Similarly to what we have done above, we define the random probability measure μ_R on the set \mathcal{G} of homotopy types of finite and connected \mathbb{R}^d -geometric complexes:

$$\mu_R = \frac{1}{b_0(\check{C}(\mathcal{P}_R))} \sum \delta_{[s]},$$

where the sum is over all connected components s of \mathcal{P}_R . The following result proves a limit law for μ_R .

Theorem 1.3. *The family of random measures μ_R converges in probability to a deterministic universal measure $\mu \in \mathcal{P}$ whose support is all of \mathcal{G} .*

We conclude by observing that the limiting measure μ appearing in the previous Theorem 1.3 is the same one appearing in Theorem 1.1 (this fact implies the statement on the support of the limiting measure in Theorem 1.1).

Outline of the paper. We prove Theorem 1.3 addressing the Euclidean setting in Section 2. In Section 3, we establish the “semi-local” result involving a double-scaling limit within a neighborhood on the manifold, and in Section 4 we collect the semi-local information in order to prove the global result Theorem 1.1 for the manifold setting. We prove Proposition 1.2 in Section 5. The last Section 6 is an appendix that contains some basic tools used throughout the paper, including the integral geometry sandwiches that play an essential role.

Historical Note. The study of the topology of random simplicial complexes has taken shape only recently with intense activity in the past few years, but it is worth mentioning (as it seems to have been forgotten) that this theme was proposed by V.I. Arnold in the early 1970s, with specific attention given to random geometric complexes in the thermodynamic regime. In the collection [Arn04] of Arnold’s problems, the 28th problem from 1973 states (notice that the set considered is homotopy equivalent to a geometric complex by the nerve lemma):

Consider a random set of points in \mathbb{R}^d with density λ . Let $V(\alpha)$ be the α -neighborhood of this set. Consider the averaged Betti numbers

$$\beta_i(\alpha, \lambda) := \lim_{R \rightarrow \infty} \frac{b_i(V(\alpha) \cap B(0, R))}{R^d}.$$

Investigate these numbers.

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2. LIMIT LAW FOR THE EUCLIDEAN CASE

Definition 1 (Component counting function). *Let $Y_1, Y_2 \subset X$ and Z be topological spaces (in the case of our interest they will be homotopy equivalent to finite simplicial complexes). We denote by $\mathcal{N}(Y_1, Y_2; Z)$ the number of connected components of Y_1 entirely contained in the interior of Y_2 and which have the same homotopy type as Z . Similarly, we denote by $\mathcal{N}^*(Y_1, Y_2; Z)$ the number of connected components of Y_1 which intersect Y_2 and which have the same homotopy type as Z .*

Proposition 2.1. *For every $\gamma \in \mathcal{G}$ there exists a constant c_γ such that the random variable*

$$c_{R,\gamma} = \frac{\mathcal{N}(\mathcal{P}_R, B(0, R); \gamma)}{\text{Vol}(B(0, R))}$$

converges to c_γ in L^1 and almost surely as $R \rightarrow \infty$. Moreover the same is true for the random variable

$$c_R = \frac{\mathcal{N}(\mathcal{P}_R, B(0, R))}{\text{Vol}(B(0, R))},$$

(i.e. when we consider all components, with no restriction on their types): as $R \rightarrow \infty$, it converges to a nonzero constant $c > 0$ in L^1 and almost surely.

Proof. The proof follows closely the argument from [SW16, Theorem 3.3], with some needed modifications.

We will use the shortened notation $\mathcal{N}_R = \mathcal{N}(\mathcal{P}_R, B(0, R); \gamma)$, $\mathcal{N}_R(x, r) = \mathcal{N}(\mathcal{P}_R, B(x, r); \gamma)$ and $\mathcal{N}_R^*(x, r) = \mathcal{N}^*(\mathcal{P}_R, B(x, r); \gamma)$ (γ will be fixed for the rest of the proof and we omit dependence on it in the notation). Using Theorem 6.6 we can write, for $0 < \alpha < r < R$:

$$(2.1) \quad \left(1 - \frac{r}{R}\right)^d \frac{1}{\text{Vol}(B_{R-r})} \int_{B_{R-r}} \frac{\mathcal{N}_R(x, r)}{\text{Vol}(B_r)} dx \leq \frac{\mathcal{N}_R}{\text{Vol}(B_R)} \leq \left(1 + \frac{r}{R}\right)^d \frac{1}{\text{Vol}(B_{R+r})} \int_{B_{R+r}} \frac{\mathcal{N}_R^*(x, r)}{\text{Vol}(B_r)} dx.$$

Denoting by $A(x, r, \alpha)$ the annulus $\{r - \alpha \leq \|x - z\| \leq r\}$, we can estimate the integral on the r.h.s. of the previous equation with:

$$\int_{B_{R+r}} \frac{\mathcal{N}_R^*(x, r)}{\text{Vol}(B_r)} dx \leq \int_{B_{R+r}} \frac{\mathcal{N}_R(x, r)}{\text{Vol}(B_r)} + \frac{\#P_R \cap A(x, r, \alpha)}{\text{Vol}(B_r)} dx.$$

In fact, if a component of \mathcal{P}_R is not entirely contained in the interior of $B(x, r)$, then it touches the boundary of $B(x, r)$ and hence this component must contain a point $p \in U_n \cap A(x, r, \alpha)$.

We apply now the Ergodic theorem to the random variables:

$$\lambda_1(R-r) = \frac{1}{\text{Vol}(B_{R-r})} \int_{B_{R-r}} \frac{\mathcal{N}_R(x, r)}{\text{Vol}(B_r)} dx, \quad \lambda_2(R+r) = \frac{1}{\text{Vol}(B_{R+r})} \int_{B_{R+r}} \frac{\mathcal{N}_R^*(x, r)}{\text{Vol}(B_r)} dx,$$

where $\lambda_2(R)$ itself can be written as:

$$\lambda_2(R-r) = \lambda_1(R-r) + \underbrace{\frac{1}{\text{Vol}(B_{R+r})} \int_{B_{R+r}} \frac{\#P_R \cap A(x, r, \alpha)}{\text{Vol}(B_r)} dx}_{a(R+r)}.$$

As $R \rightarrow \infty$ all these random variables converge to constants in L^1 and almost surely:

$$\lambda_1(R-r) \rightarrow \bar{\lambda}_1(r) \quad \text{and} \quad \lambda_2(R+r) \rightarrow \bar{\lambda}_1(r) + \bar{a}(r).$$

On the other hand $\bar{a}(r) = O(r^{-1})$ and consequently, when taking the further limit $r \rightarrow \infty$, equation (2.1) guarantees that the middle term converges in L^1 and almost surely to a constant. \square

2.1. **Proof of Theorem 1.3.** We write the measure μ_R as:

$$\begin{aligned}\mu_R &= \frac{1}{\mathcal{N}(\mathcal{P}_R, B(0, R))} \sum_{\gamma \in \mathcal{G}} \mathcal{N}(\mathcal{P}_R, B(0, R); \gamma) \delta_\gamma \\ &= \frac{\text{Vol}(B(0, R))}{\mathcal{N}(\mathcal{P}_R, B(0, R))} \sum_{\gamma \in \mathcal{G}} \frac{\mathcal{N}(\mathcal{P}_R, B(0, R); \gamma)}{\text{Vol}(B(0, R))} \delta_\gamma \\ &= \sum_{\gamma \in \mathcal{G}} \frac{c_{R, \gamma}}{c_R} \delta_\gamma.\end{aligned}$$

Denoting by $a_{R, \gamma} = \frac{c_{R, \gamma}}{c_R}$, from the convergence in L^1 and almost surely of $c_{R, \gamma}$ and of c_R (the last one to a positive constant), it follows that $a_{R, \gamma}$ also converges in L^1 and almost surely to a constant. Convergence in L^1 implies convergence in probability and the result follows now from Lemma 6.3.

3. SEMI-LOCAL COUNTS IN THE RIEMANNIAN CASE

Theorem 3.1. *Let $p \in M$ For every $\delta > 0$ and for $R > 0$ sufficiently big there exists n_0 such that for every $\gamma \in \mathcal{G}$ and for $n \geq n_0$:*

$$(3.1) \quad \mathbb{P} \left\{ \mathcal{N}(\mathcal{P}_R, B(0, R); \gamma) = \mathcal{N}(U_n, \hat{B}(p, Rn^{-1/d}); \gamma) \right\} \geq 1 - \delta.$$

Proof. Given $p \in M$, we introduce the following map (see also Proposition 6.2):

$$\psi_n : \hat{B}(p, Rn^{-1/d}) \xrightarrow{\exp_p^{-1}} B_{T_p M}(0, Rn^{-1/d}) \xrightarrow{n^{1/d}} B_{T_p M}(0, R) \simeq B(0, R).$$

Note that this map is a diffeomorphism, whose inverse we denote by φ_n . Moreover, through ψ_n , the stochastic point process $U_n \cap \hat{B}(p, Rn^{-1/d})$ induces a stochastic point process on $B(0, R)$ which converges in distribution to the uniform Poisson process on $B(0, R)$. By Skorokhod's representation theorem, we can assume that the convergence of these stochastic processes is almost surely.

For the proof of (3.1) we will need to establish the following three facts:

- (1) there exists $\ell_0 > 0$ and $n_1 > 0$ such that with probability at least $1 - \delta/3$ we have:

$$(3.2) \quad \#(U_n \cap \hat{B}(p, Rn^{-1/d})) = \#(P_R \cap B(0, R)) \leq \ell_0$$

(i.e. with positive probability for large n , depending on δ , both point processes have the same number of points and this number is bounded by some constant ℓ_0 , which also depends on δ).

- (2) There exists $W \subset \coprod_{\ell \leq \ell_0} B(0, R)^\ell$, $r > 0$ and $n_2 > 0$ such that $\mathbb{P}(W) \geq 1 - \delta/3$ and for every $x = (y_1, \dots, y_\ell) \in W$ if $\tilde{x} = (\tilde{y}_1, \dots, \tilde{y}_\ell)$ is such that $\|x - \tilde{x}\| < r$ and $n \geq n_2$ then:

$$\bigcup_{k=1}^{\ell} B(y_k, \alpha) \simeq \bigcup_{k=1}^{\ell} \hat{B}(\varphi_n(\tilde{y}_k), \alpha n^{-1/d}),$$

(i.e. the two spaces are homotopy equivalent), and for every connected component of $\bigcup_{k=1}^{\ell} B(y_k, \alpha)$ this component intersects $\partial B(0, R)$ if and only if the corresponding component of $\bigcup_{k=1}^{\ell} \hat{B}(\varphi_n(\tilde{y}_k), \alpha n^{-1/d})$ intersects $\partial \hat{B}(p, Rn^{-1/d})$.

- (3) assuming point (1), denoting by $\{x_1, \dots, x_\ell\} = P_R \cap B(0, R)$, and by $\{\tilde{x}_1, \dots, \tilde{x}_\ell\} = \psi_n(U_n \cap \hat{B}(p, Rn^{-1/d}))$, there exists $n_3 > 0$ such that for every $n \geq n_3$:

$$\mathbb{P} \{ \forall \ell \leq \ell_0, \forall k = 1, \dots, \ell, \|x_k - \tilde{x}_k\| \leq r \} \geq 1 - \delta/3.$$

Assuming these three facts, (3.1) follows arguing as follows. With probability at least $1 - \delta$ for $n \geq n_0 = \max\{n_1, n_2, n_3\}$ all the conditions from (1), (2) and (3) verify and the two random sets

$$\bigcup_{p \in \mathcal{P}_R} B(p, \alpha) \quad \text{and} \quad \bigcup_{p_k \in U_n \cap \hat{B}(p, Rn^{-1/d})} \hat{B}(p_k, \alpha n^{-1/d})$$

are homotopy equivalent and by the second part of point (3) also the unions of all the components entirely contained in $B(0, R)$ (respectively $\hat{B}(p, Rn^{-1/d})$) are homotopy equivalent. In particular the number of components of a given homotopy type $[S]$ is the same for both sets with probability at least $1 - \delta$.

It remains to prove (1), (2) and (3).

Point (1) follows from the fact that we have assumed the point process $\psi_n(U_n \cap \hat{B}(p, Rn^{-1/d}))$ converges almost surely to the Poisson point process on $B(0, R)$. In particular the sequence of random variables $\{\#(U_n \cap \hat{B}(p, Rn^{-1/d}))\}$ converges almost surely to $\#(P_R \cap B(0, R))$ and (3.2) follows from the fact that almost sure convergence implies convergence in probability.

For point (2) we argue as follows. Given ℓ_0 we consider the compact semialgebraic set:

$$X = \prod_{\ell \leq \ell_0} B(0, R)^\ell.$$

This set is endowed with the measure $d\rho$:

$$d\rho = \sum_{\ell \leq \ell_0} \frac{\text{vol}(B(0, R)^\ell)}{\ell!} \chi_{B(0, R)^\ell} d\lambda_{B(0, R)^\ell}$$

where $d\lambda$ denotes the Lebesgue measure (this is the measure induced from the Poisson distribution). Let now $Z \subset X$ be the set of points $x = (y_1, \dots, y_\ell)$ such that either the intersection $\bigcap_{j \in J_1} \partial B(y_j, \alpha)$ or the intersection $\partial B(0, R) \bigcap_{j \in J_2} \partial B(y_j, \alpha)$ is non-transversal for some index sets $J_1, J_2 \in \binom{[\ell]}{d}$ (note that the generic intersection of more than d spheres will be empty).

Let $U(Z)$ be an open neighborhood of Z such that $\rho(U(Z)^c) \geq 1 - \delta/3$ (for example one can take $U(Z) = \prod_{\ell \leq \ell_0} \{d(\cdot, Z) < \epsilon\}$ for $\epsilon > 0$ small enough). We set $W = U(Z)^c$ (note that $\mathbb{P}(W) \geq 1 - \delta/3$).

The property of transversal intersection implies that for every index sets $J_1, J_2 \in \bigcup_{\ell \leq \ell_0} \binom{[\ell]}{d}$ such that the intersection $\bigcap_{j \in J_1} B(y_j, \alpha)$ is nonempty, this intersection contains a nonempty open set, and there exists a point $\sigma_{J_1}(x)$ such that for every $j \in J_1$ we have $\|y_j - \sigma_{J_1}(x)\| < \alpha$. Similarly whenever an intersection $\partial B(0, R) \bigcap_{j \in J_2} \partial B(y_j, \alpha)$ is transversal and nonempty, there exists a point $\sigma_{J_2}(x)$ such that $\|\sigma_{J_2}(x)\| > R$ and for every $j \in J_2$ we have $\|y_j - \sigma_{J_2}(x)\| < \alpha$. Because these are open properties, there exists $r_1(x), r_2(x) > 0$ such that for every $w = (w_1, \dots, w_\ell)$ and $z = (z_1, \dots, z_\ell)$ with $\|w_j - x\| \leq r_1(x)$ and $\|z_j - w\| < r_2(x)$ for all $j = 1, \dots, \ell$, we have:

$$\forall j \in J_1 : \|z_j - \sigma_{J_1}(x)\| < \alpha \quad \text{and} \quad \forall j \in J_2 : \|z_j - \sigma_{J_2}(x)\| < \alpha.$$

Moreover since the property of having non-empty intersection is also stable under small perturbation, we can assume that $r_1(x), r_2(x)$ are small enough to guarantee also that:

$$\bigcap_{j \in J_3} B(z_j, \alpha) = \emptyset \iff \bigcap_{j \in J_3} B(x_j, \alpha) = \emptyset.$$

Observe now that the sequence of functions $d_n : B(0, R) \times B(0, R) \rightarrow \mathbb{R}$ defined by:

$$d_n(x_1, x_2) = d_M(\varphi_n(x_1), \varphi_n(x_2))n^{1/d}$$

converges uniformly to the Euclidean distance in \mathbb{R}^d . In particular there exists $n(x) > 0$ such that for every $n \geq n(x)$, for every $w = (w_1, \dots, w_\ell)$ and $z = (z_1, \dots, z_\ell)$ with $\|w_j - x\| \leq r_1(x)$ and $\|z_j - w\| < r_2(x)$, for all $j = 1, \dots, \ell$ and for $i = 1, 2$ we have:

$$d_M(\varphi_n(z_j), \varphi_n(\sigma_{J_i}(x))) < \alpha n^{-1/d}.$$

Moreover, for a possibly larger $n(x)$, we also have that

$$\bigcap_{j \in J_3} \hat{B}(\varphi_n(z_j), \alpha n^{-1/d}) = \emptyset \iff \bigcap_{j \in J_3} B(x_j, \alpha) = \emptyset.$$

Choosing $n(x)$ to be even larger, so that balls of radius smaller than $\alpha n^{-1/d}$ in M are geodesically convex, these conditions implies that the combinatorics of the cover $\{B(x_j, \alpha)\}_{j=1}^\ell$ and $\{\hat{B}(\varphi_n(z_j), \alpha n^{-1/d})\}_{j=1}^\ell$ are the same and, by Lemma 6.1, the two sets $\bigcup_{j \leq \ell} B(x_j, \alpha)$ and $\bigcup_{j \leq \ell} \hat{B}(\varphi_n(z_j), \alpha n^{-1/d})$ are homotopy equivalent. Also, the above condition on $\sigma_{J_2}(x)$ implies that a component of $\bigcup_{j \leq \ell} B(x_j, \alpha)$ intersects $\partial B(0, R)$ if and only if the corresponding component of $\bigcup_{j \leq \ell} \hat{B}(\varphi_n(z_j), \alpha n^{-1/d})$ intersects $\hat{B}(p, Rn^{-1/d})$.

Finally, we cover now $W = X \setminus U(Z)$ with the family of open sets $\bigcup_{x \in W} B(x, r_1(x))$ and find, by compactness of W , finitely many points x_1, \dots, x_L such that the union of the balls $B(x_k, r_1(x_k))$ with $k = 1, \dots, L$ covers W . With the choice $n_2 = \max\{n(x_k), k = 1, \dots, L\}$ and $r = \min\{r_2(x_k), k = 1, \dots, L\}$ property (2) is true.

Concerning point (3), we observe that again this follows from the fact that the point process $\psi_n(U_n \cap \hat{B}(p, Rn^{-1/d}))$ converges almost surely (hence in probability) to the Poisson point process on $B(0, R)$. \square

Corollary 3.2. *For each $\gamma \in \mathcal{G}$, $\alpha > 0$, $x \in M$, and $\varepsilon > 0$, we have*

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \frac{\mathcal{N}(\mathcal{U}_n, B(x, Rn^{-1/d}); \gamma)}{\text{Vol}(B_R)} - c_\gamma \right| > \varepsilon \right\} = 0.$$

Proof. This follows from Theorem 3.1 combined with Proposition 2.1. Indeed, let $\varepsilon > 0$ and $\delta > 0$ be arbitrary. By Proposition 2.1 there exists R_0 such that for $R > R_0$ we have

$$\mathbb{P} \left\{ \left| \frac{\mathcal{N}(\mathcal{P}_R, B(x, R); \gamma)}{\text{Vol}(B_R)} - c_\gamma \right| > \varepsilon \right\} < \delta.$$

Fix any such $R > R_0$ and apply Proposition 2.1; there exists n_0 such that for all $n \geq n_0$ the event

$$\left| \frac{\mathcal{N}(\mathcal{U}_n, B(x, Rn^{-1/d}); \gamma)}{\text{Vol}(B_R)} - c_\gamma \right| > \varepsilon$$

is contained in the union of the event

$$\left| \frac{\mathcal{N}(\mathcal{P}_R, B(x, R); \gamma)}{\text{Vol}(B_R)} - c_\gamma \right| > \varepsilon$$

and another event E_δ with $\mathbb{P}\{E_\delta\} \leq \delta$. Thus,

$$\limsup_{n \rightarrow \infty} \mathbb{P} \left\{ \left| \frac{\mathcal{N}(\mathcal{U}_n, B(x, Rn^{-1/d}); \gamma)}{\text{Vol}(B_R)} - c_\gamma \right| > \varepsilon \right\} < 2\delta.$$

Since $\delta > 0$ was arbitrary, this completes the proof of Corollary 3.2. \square

4. THE GLOBAL COUNT FOR THE RIEMMANIAN CASE: PROOF OF THEOREM 1.1

Theorem 4.1. *For every $\gamma \in \mathcal{G}$, the random variable*

$$c_{n,\gamma} = \frac{\mathcal{N}(\mathcal{U}_n, M; \gamma)}{n}$$

converges in L^1 to the constant $c_\gamma = c_\gamma(\alpha)$ (the same constant as in Proposition 2.1). The same statement is true for the random variable

$$c_n = \frac{\mathcal{N}(\mathcal{U}_n, M)}{n}$$

(i.e. when we consider all components, with no restriction on their type): as $n \rightarrow \infty$, it converges in L^1 to a constant $c = \sum_{\gamma \in \mathcal{G}} c_\gamma > 0$.

4.1. Proof of Theorem 1.1 assuming Theorem 4.1. Since convergence in L^1 implies convergence in probability, Theorem 4.1 ensures that the random variable $c_{n,\gamma} = \frac{\mathcal{N}(\mathcal{U}_n, M; \gamma)}{n}$ converges in probability to the constant c_γ ; similarly the the random variable $c_n = \frac{\mathcal{N}(\mathcal{U}_n, M)}{n}$ converges in L^1 (hence in probability) to $c > 0$. The proof now proceeds similarly to the proof of Theorem 1.3. We write the measure $\hat{\mu}_n$ as:

$$\begin{aligned} \hat{\mu}_n &= \frac{1}{b_0(\check{C}(\mathcal{U}_n))} \sum_{\gamma \in \hat{\mathcal{G}}} \mathcal{N}(\mathcal{U}_n, M; \gamma) \delta_\gamma \\ &= \frac{1}{b_0(\check{C}(\mathcal{U}_n))} \left(\sum_{\gamma \in \mathcal{G}} \mathcal{N}(\mathcal{U}_n, M; \gamma) \delta_\gamma + \sum_{\gamma \in \hat{\mathcal{G}} \setminus \mathcal{G}} \mathcal{N}(\mathcal{U}_n, M; \gamma) \delta_\gamma \right) \\ &= \frac{1}{\mathcal{N}(\mathcal{U}_n, M)} \sum_{\gamma \in \mathcal{G}} \mathcal{N}(\mathcal{U}_n, M; \gamma) \delta_\gamma + \frac{1}{\mathcal{N}(\mathcal{U}_n, M)} \sum_{\gamma \in \hat{\mathcal{G}} \setminus \mathcal{G}} \mathcal{N}(\mathcal{U}_n, M; \gamma) \delta_\gamma \\ &= \frac{n}{\mathcal{N}(\mathcal{U}_n, M)} \sum_{\gamma \in \mathcal{G}} \frac{\mathcal{N}(\mathcal{U}_n, M; \gamma)}{n} \delta_\gamma + \frac{n}{\mathcal{N}(\mathcal{U}_n, M)} \sum_{\gamma \in \hat{\mathcal{G}} \setminus \mathcal{G}} \frac{\mathcal{N}(\mathcal{U}_n, M; \gamma)}{n} \delta_\gamma \\ (4.1) \quad &= \sum_{\gamma \in \mathcal{G}} \frac{c_{n,\gamma}}{c_n} \delta_\gamma + \sum_{\gamma \in \hat{\mathcal{G}} \setminus \mathcal{G}} \frac{c_{n,\gamma}}{c_n} \delta_\gamma. \end{aligned}$$

Applying now Lemma 6.3, and using again the fact that L^1 convergence implies convergence in probability, we get that the measure on the left in (4.1) converges in probability to μ . Since μ is a probability measure, this implies that

$$\sum_{\gamma \in \mathcal{G}} \frac{c_{n,\gamma}}{c_n}$$

converges to 1 in probability. For any $\gamma_0 \in \hat{\mathcal{G}} \setminus \mathcal{G}$ this implies that $\frac{c_{n,\gamma_0}}{c_n}$ converges to zero in probability, since

$$0 \leq \frac{c_{n,\gamma_0}}{c_n} \leq 1 - \sum_{\gamma \in \mathcal{G}} \frac{c_{n,\gamma}}{c_n}.$$

Thus, the measure on the right in (4.1) converges in probability to zero, and the measure $\hat{\mu}_n$ converges in probability to μ .

4.2. Proof of Theorem 4.1. Note: Since $\alpha > 0$ and $\gamma \in \mathcal{G}$ are fixed, we will simply use \mathcal{N}_n below to denote $\mathcal{N}(\mathcal{U}_n, M; \gamma)$, the number of components of \mathcal{U}_n in M of type γ . We will use

$$\mathcal{N}_n^*(x, r) := \mathcal{N}^*(\mathcal{U}_n, \hat{B}(x, r); \gamma)$$

to denote the number of such components intersecting the geodesic ball $\hat{B}(x, r)$ of radius r centered at x and

$$\mathcal{N}_n(x, r) := \mathcal{N}(\mathcal{U}_n, \hat{B}(x, r); \gamma)$$

to denote the number of components completely contained in $\hat{B}(x, r)$. We will also write $c := c_\gamma$.

Thus, our goal, stated in this notation, is to prove

$$(4.2) \quad \mathbb{E} \left[\left| \frac{\mathcal{N}_n}{n} - c \right| \right] \rightarrow 0.$$

Using the integral geometry sandwich from Theorem 6.7 we have

$$(4.3) \quad (1 - \varepsilon) \int_M \frac{\mathcal{N}_n(x, Rn^{-1/d})}{\text{Vol}(B_R)} dx \leq \frac{\mathcal{N}_n}{n} \leq (1 + \varepsilon) \int_M \frac{\mathcal{N}_n^*(x, Rn^{-1/d})}{\text{Vol}(B_R)} dx.$$

Letting I_1 denote the integral on the left side and I_2 the one on the right side, we subtract I_1 from each part of (4.3) and write

$$-\varepsilon I_1 \leq \frac{\mathcal{N}_n}{n} - I_1 \leq \varepsilon I_1 + (1 + \varepsilon)(I_2 - I_1).$$

In order to estimate $I_2 - I_1$ we note that the number of connected components of \mathcal{U}_n that intersect, but are not completely contained in, the geodesic ball $\hat{B}(x, Rn^{-1/d})$ is bounded above by the number of points that fall within distance $\alpha n^{-1/d}$ to the boundary $\partial \hat{B}(x, Rn^{-1/d})$. This $\alpha n^{-1/d}$ -neighborhood of $\partial \hat{B}(x, Rn^{-1/d})$ is the same as the geodesic annulus centered at x with inner radius $(R - \alpha)n^{-1/d}$ and outer radius $(R + \alpha)n^{-1/d}$. The average number of points in this annulus equals its volume which can be estimated (uniformly over $x \in M$) by that of the Euclidean annulus, and this gives $I_2 - I_1 = O(R^{-1})$ on average. This implies

$$\mathbb{E} \left| \frac{\mathcal{N}_n}{n} - I_1 \right| = O(\varepsilon) + O(R^{-1}).$$

Next we apply this to the expectation appearing in the statement of the theorem.

$$\begin{aligned} \mathbb{E} \left[\left| \frac{\mathcal{N}_n}{n} - c \right| \right] &= \mathbb{E} \left[\left| \frac{\mathcal{N}_n}{n} - I_1 + I_1 - c \right| \right] \\ &\leq \mathbb{E} [|I_1 - c|] + O(\varepsilon) + O(R^{-1}). \\ &= \mathbb{E} \left[\left| \int_M \frac{\mathcal{N}_n(x, Rn^{-1/d})}{\text{Vol}(B_R)} - c dx \right| \right] + O(\varepsilon) + O(R^{-1}). \end{aligned}$$

Thus, in order to prove the theorem it suffices to show that the above term $\mathbb{E} [|I_1 - c|]$ can be made arbitrarily small for all sufficiently large n .

Define the “bad” event

$$\Omega_{x,R,n} := \left\{ \left| \frac{\mathcal{N}_n(x, Rn^{-1/d})}{\text{Vol}(B_R)} - c \right| > \varepsilon \right\}.$$

Claim: There exists a sequence $R_j \rightarrow \infty$ such that for every $\delta > 0$ there exists $M_\delta \subset M$ with $\text{Vol}(M_\delta) > 1 - \delta$ such that

$$(4.4) \quad \lim_{R_j \rightarrow \infty} \limsup_{n \rightarrow \infty} \sup_{x \in M_\delta} \mathbb{P}(\Omega_{x, R_j, n}) = 0.$$

The proof of this claim closely follows [SW16] and uses Egorov's theorem as well as the idea from the proof of Egorov's theorem. We start by recalling the point-wise limit stated in Corollary 3.2. For each $x \in M$, we have

$$\lim_{R \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P}\{\Omega_{x, R, n}\} = 0.$$

Apply Egorov's theorem to obtain $M'_\delta \subset M$ with $\text{Vol}(M'_\delta) > 1 - \frac{\delta}{2}$ such that

$$(4.5) \quad \lim_{R \rightarrow \infty} \sup_{x \in M'_\delta} \limsup_{n \rightarrow \infty} \mathbb{P}\{\Omega_{x, R, n}\} = 0.$$

Next we use an additional Egorov-type argument in order to obtain the statement in the claim (where we will obtain the set M_δ by slightly shrinking M'_δ). For each fixed integer $j > 0$, we can find by (4.5) an R_j sufficiently large so that

$$(4.6) \quad \sup_{x \in M'_\delta} \limsup_{n \rightarrow \infty} \mathbb{P}\{\Omega_{x, R_j, n}\} < \frac{1}{j}.$$

Letting $F_m(j)$ denote the monotone decreasing (with m) sequence of sets

$$F_m(j) = \bigcup_{k \geq m} \left\{ x \in M'_\delta : \mathbb{P}(\Omega_{x, R_j, k}) > \frac{2}{j} \right\},$$

we see from (4.6) that

$$\bigcap_{m \geq 1} F_m(j) = \emptyset.$$

Thus, there exists $m = m(j)$ such that $\text{Vol}(F_m(j)) < \frac{\delta}{2^{j+1}}$. We take

$$M_\delta = M'_\delta \setminus \left(\bigcup_{j \geq 1} F_{m(j)}(j) \right),$$

which satisfies $\text{Vol}(M_\delta) > \text{Vol}(M'_\delta) - \frac{\delta}{2} > 1 - \delta$. It follows from the definition of $F_m(j)$ that

$$\limsup_{n \rightarrow \infty} \sup_{x \in M_\delta} \mathbb{P}\{\Omega_{x, R_j, n}\} \leq \frac{2}{j},$$

and we see that (4.4) is satisfied.

Denoting the whole probability space as Ω , we separate the integration (defining the expectation) over the two sets $\Omega_{x, R_j, n}$ and $\Omega \setminus \Omega_{x, R_j, n}$.

$$\mathbb{E}[|I_1 - c|] = \int_{\Omega \setminus \Omega_{x, R_j, n}} |I_1 - c| d\omega + \int_{\Omega_{x, R_j, n}} |I_1 - c| d\omega.$$

We use the definition of $\Omega_{x, R_j, n}$ to estimate the first integral:

$$\int_{\Omega \setminus \Omega_{x, R_j, n}} |I_1 - c| d\omega \leq \int_{\Omega \setminus \Omega_{x, R_j, n}} \int_M \left| \frac{\mathcal{N}_n(x, Rn^{-1/d})}{\text{Vol}(B_R)} - c \right| dx d\omega \leq \varepsilon.$$

For the second integral, we use the estimate

$$(4.7) \quad \frac{\mathcal{N}_n(x, Rn^{-1/d})}{B_R} \leq (1 + \varepsilon)n\xi^{-1} = O(1),$$

where $\xi > 0$ is the minimum (over $x \in M$) volume of a ball of radius $\alpha n^{-1/d}$, which is uniformly (over $x \in M$) comparable to the volume of the Euclidean ball of the same radius. Since $\xi = \Theta(n^{-1})$, the estimate (4.7) is based on the fact that the minimal volume of a component times the number of components cannot exceed the volume of the region where they are contained (while fixing attention on components of type γ as we are throughout the proof).

$$\begin{aligned} \int_{\Omega_{x,R_j,n}} |I_1 - c| d\omega &\leq \int_M \int_{\Omega_{x,R_j,n}} \left| \frac{\mathcal{N}_n(x, Rn^{-1/d})}{\text{Vol}(B_R)} - c \right| d\omega dx \\ &\leq (O(1) + c) \cdot \int_M \mathbb{P}(\Omega_{x,R_j,n}) dx. \end{aligned}$$

Next, we split this last integration over M_δ and $M \setminus M_\delta$:

$$\begin{aligned} \int_M \mathbb{P}(\Omega_{x,R_j,n}) dx &= \int_{M_\delta} \mathbb{P}(\Omega_{x,R_j,n}) dx + \int_{M \setminus M_\delta} \mathbb{P}(\Omega_{x,R_j,n}) dx \\ &\leq \sup_{x \in M_\delta} \mathbb{P}(\Omega_{x,R_j,n}) + \delta. \end{aligned}$$

Bringing these estimates together, we have

$$\mathbb{E}[|I_1 - c|] \leq \varepsilon + O(1) \left(\delta + \sup_{x \in M_\delta} \mathbb{P}(\Omega_{x,R_j,n}) \right),$$

which can be made arbitrarily small using (4.4). This establishes (4.2) and completes the proof of the first part of Theorem 4.1.

5. QUANTITATIVE ESTIMATES

Remark 3. We will say that a \mathbb{R}^d -geometric complex $\bigcup_{k=1}^\ell B(y_k, r)$ is *nondegenerate* if for every $1 \leq k \leq d$ and $J = \{j_1, \dots, j_k\} \in \binom{[\ell]}{k}$ the intersection $\bigcap_{j \in J} \partial B(y_j, r)$ is transversal. The set of homotopy types of \mathbb{R}^d -geometric, connected, nondegenerate complexes coincides with \mathcal{G} (where we did not assume the nondegeneracy condition). In fact given a possibly degenerate $\mathcal{P} = \bigcup_{k=1}^\ell B(y_k, r)$, let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be the semialgebraic and continuous function defined by

$$f(x) = d(x, \{y_1, \dots, y_k\}) = \min_k \|y_k - x\|,$$

and observe that:

$$\bigcup_{k=1}^\ell B(y_k, r) = \{f \leq r\}.$$

We consider now the semialgebraic, monotone family $\{X(r + \varepsilon) = \{f \leq r + \varepsilon\}\}_{\varepsilon \geq 0}$. By [BPR06, Lemma 16.17] for $\varepsilon > 0$ the inclusion $X(r) \hookrightarrow X(r + \varepsilon)$ is a homotopy equivalence. It suffices therefore to show that for $\varepsilon > 0$ small enough $X(r + \varepsilon)$ is nondegenerate; this follows from the fact that given points $y_1, \dots, y_\ell \in \mathbb{R}^d$, for every $1 \leq k \leq d$ and $J = \{j_1, \dots, j_k\} \in \binom{[\ell]}{k}$ there

are only finitely many $r > 0$ such that the intersection $\bigcap_{j \in J} \partial B(y_j, r)$ is nontransversal (and the number of possible multi-indices to consider is also finite).

5.1. Proof of Proposition 1.2.

Proof. Let $y_1, \dots, y_\ell \in \mathbb{R}^d$ and $r > 0$ such that

$$\bigcup_{k=1}^{\ell} B(y_k, r) = \mathcal{P}$$

with $\bigcup_{k=1}^{\ell} B(y_k, r)$ a nondegenerate complex (it is not restrictive to consider nondegenerate complexes by Remark 3 above). Let now $R' > 0$ such that $B(0, R')$ contains $\bigcup_{k=1}^{\ell} B(y_k, r)$ and set $R = \frac{R'\alpha}{r}$. Consider also the sequence of maps:

$$\psi_n : \hat{B}(p, Rn^{-1/d}) \xrightarrow{\exp_p^{-1}} B_{T_p M}(0, Rn^{-1/d}) \xrightarrow{\frac{r}{\alpha} n^{1/d}} B_{T_p M}(0, R') \simeq B(0, R').$$

Proposition 6.2 implies that there exists $\epsilon_0 > 0$ and n_0 such that if $\|\tilde{y}_k - y_k\| \leq \epsilon_0$ then for $n \geq n_0$ the two complexes $\bigcup_{k=1}^{\ell} B(y_k, r)$ and $\bigcup_{k=1}^{\ell} \hat{B}(\varphi_n(\tilde{y}_k), \alpha n^{-1/d})$ are homotopy equivalent.

We are interested in the the event:

$$E_n = \left\{ \exists I_\ell \in \binom{[n]}{\ell} \mid \forall j \in I_\ell : p_j \in \psi_n^{-1}(B(y_j, \epsilon)), \quad \forall j \notin I_\ell : p_j \in \hat{B}(p, (R + \alpha)n^{-1/d})^c \right\}.$$

Observe that if E_n verifies, then $\mathcal{U}_n \cap \hat{B}(p, Rn^{-1/d}) \simeq \mathcal{P}$: in fact, since there is no other point in $\hat{B}(p, (R + \alpha)n^{-1/d})$ other than $\{p_j\}_j \in J$, then the complex \mathcal{U}_n is the disjoint union of the two complexes $\mathcal{U}_n \cap \hat{B}(p, \alpha n^{-1/d})$ and $\mathcal{U}_n \cap \hat{B}(p, (R + \alpha)n^{-1/d})^c$; the complex $\mathcal{U}_n \cap \hat{B}(p, \alpha n^{-1/d}) \simeq \mathcal{P}$ by Proposition 6.2.

It is therefore enough to estimate from below the probability of E_n . Note that for every measurable subset $B \subset B(0, R')$ there exists a constant $c_B > 0$ such that $\text{Vol}(\psi_n^{-1}(B)) \geq \frac{c_B}{n}$. In particular, using the independence of the points in U_n , we can estimate:

$$\begin{aligned} \mathbb{P}(E_n) &= \binom{n}{\ell} \mathbb{P} \left\{ \forall j \leq \ell : p_j \in \psi_n^{-1}(B(p_j, \epsilon)) \quad \text{and} \quad \forall j \geq \ell + 1 : p_j \in \hat{B}(p, (R + \alpha)n^{-1/d})^c \right\} \\ &= \binom{n}{\ell} \left(\prod_{j=1}^{\ell} \text{vol}(\psi_n^{-1}(B(y_j, \epsilon))) \right) \left(\text{vol}(\hat{B}(p, (R + \alpha)n^{-1/d})^c) \right)^{n-\ell} \\ &\geq \binom{n}{\ell} \left(\frac{c_1}{n} \right)^\ell \left(1 - \frac{c_2}{n} \right)^{n-\ell} \xrightarrow{n \rightarrow \infty} \frac{c_1^\ell}{\ell!} (1 - c_2)^{-\ell} e^{-c_2}. \end{aligned}$$

In particular there exists $c > 0$ such that:

$$\mathbb{P} \left\{ \mathcal{U}_n \cap \hat{B}(p, Rn^{-1/d}) \simeq \mathcal{P} \right\} \geq \mathbb{P}(E_n) > c,$$

and this concludes the proof. \square

5.2. Proof Proposition 1.2 implies positivity of all coefficients. Next we prove that for every $\gamma \in \mathcal{G}$ we have $c_\gamma > 0$.

Recall that, by Theorem 4.1, for every $\gamma \in \mathcal{G}$ we have:

$$c_\gamma = \lim_{n \rightarrow \infty} \mathbb{E} \left(\frac{\mathcal{N}(\mathcal{U}_n, M; \gamma)}{n} \right).$$

Let now R, a be given by Proposition 1.2 for the choice of $[\mathcal{P}] = \gamma$. Then, there exists $\beta > 0$ such that in M , for $n > 0$ large enough, we can fit in $k \geq \beta n$ many disjoint Riemannian balls

$$B_1 = \hat{B}(p_1, Rn^{-1/d}), \dots, B_k = \hat{B}(p_k, Rn^{-1/d}).$$

Observe now that:

$$\mathcal{N}(U_n, M; \gamma) \geq \sum_{j=1}^k \mathcal{N}(U_n, B_j; \gamma)$$

and consequently:

$$\mathbb{E} \left(\frac{\mathcal{N}(U_n, M; \gamma)}{n} \right) \geq \sum_{j=1}^k \mathbb{E} \left(\frac{\mathcal{N}(U_n, B_j; \gamma)}{n} \right) \geq k \frac{a}{n} \geq \beta a > 0.$$

6. SOME ADDITIONAL TOOLS

In this section we collect some additional tools used throughout the paper.

6.1. Geometry. A subset A of a Riemannian manifold (M, g) is called *strongly convex* if for any pair of points $y_1, y_2 \in \text{clos}(A)$ there exists a unique minimizing geodesic joining these two points such that its interior is entirely contained in A (see [CE08, dC92]).

Lemma 6.1. *Let (M, g) be a compact Riemannian manifold. There exists $r_0 > 0$ such that for every point $x \in M$ and every $r < r_0$ the ball $\hat{B}(x, r)$ is strongly convex and contractible. Moreover for every $x_1, \dots, x_k \in M$ and $0 < r_1, \dots, r_k < r_0$ the set $\bigcap_{j=1}^k \hat{B}(x_j, r_j)$ is also strongly convex and contractible. In particular, by the Nerve Lemma, the set $\bigcup_{j=1}^k \hat{B}(x_j, r_j)$ is homotopy equivalent to its associated Cech complex.*

Proof. By [CE08, Theorem 5.14] there exists a positive and continuous function $r : M \rightarrow (0, \infty)$ such that if $r < r(x)$, then $\hat{B}(x, r)$ is strictly convex (this is in fact due to Whitehead). Since M is compact, then $r_0 = \min r > 0$. Any strongly convex set in a Riemannian manifold is contractible with respect to any of its point (star-shaped in exponential coordinates), hence it follows that for $r < r_0$ the ball $\hat{B}(x, r)$ is also contractible. To finish the proof, we simply observe that the intersection of strongly convex sets A_1, A_2 is still strongly convex: in fact given two points $y_1, y_2 \in A_1 \cap A_2$, by strong convexity of the sets, the unique minimizing geodesic joining the two points is contained in both sets. \square

The following Proposition plays an important role in all asymptotic stability arguments.

Proposition 6.2. *Let (M, g) be a compact Riemannian manifold of dimension d and $p \in M$. Let $\mathcal{P} \subset \mathbb{R}^d$ be a nondegenerate complex such that:*

$$\mathcal{P} = \bigcup_{j=1}^{\ell} B(y_j, r) \subset B(0, R')$$

for some points $y_1, \dots, y_\ell \in \mathbb{R}^d$ and $r, R' > 0$. Given $\alpha > 0$ set $R = \frac{\alpha R'}{r}$ and consider the sequence of maps:

$$\psi_n : \hat{B}(p, Rn^{-1/d}) \xrightarrow{\exp_p^{-1}} B_{T_p M}(0, Rn^{-1/d}) \xrightarrow{\frac{r}{\alpha n^{1/d}}} B_{T_p M}(0, R') \simeq B(0, R').$$

Denoting by φ_n the inverse of ψ_n , there exist $\epsilon_0 > 0$ and $n_0 > 0$ such that if $\|\tilde{y}_k - y_k\| \leq \epsilon_0$ for every $k = 1, \dots, \ell$ then for $n \geq n_0$ we have:

$$\bigcup_{k=1}^{\ell} \hat{B}(\varphi_n(\tilde{y}_k), \alpha n^{-1/d}) \simeq \bigcup_{k=1}^{\ell} B(y_k, r).$$

Proof. For $k \leq d$ and for every $J = \{j_1, \dots, j_k\} \in \{\ell\}$ either one of these possibilities can verify:

- (1) $\bigcap_{j \in J} B(y_j, r) \neq \emptyset$, in which case, by nondegeneracy, there exists ϵ_J and y_J such that $\|y_J - y_j\| < r - \epsilon_J$ for all $j \in J$;
- (2) $\bigcap_{j \in J} B(y_j, r) = \emptyset$, in which case there is no y solving $\|y - y_j\| \leq r$ for all $j \in J$.

Since the sequence of maps $d_n : B(0, R') \times B(0, R') \rightarrow \mathbb{R}$ defined by

$$\frac{rn^{1/d}}{\alpha} \cdot d_n(z_1, z_2) = d_M(\varphi_n(z_1), \varphi_n(z_2))$$

converges uniformly to $d_{\mathbb{R}^d}$, then for every $\delta > 0$ there exists $n_1 > 0$ such that for all pairs of points $z_1, z_2 \in B(0, R')$ and for all $n \geq n_1$ we have:

$$(6.1) \quad \left| \frac{rn^{1/d}}{\alpha} \cdot d_M(\varphi_n(z_1), \varphi_n(z_2)) - \|z_1 - z_2\| \right| \leq \delta.$$

For every index set J satisfying condition (1) above, choosing $\delta = \frac{\epsilon_J r}{3\alpha}$ and setting $\epsilon_J = \delta$, the previous inequality (6.1) implies that, if $\|\tilde{y}_k - y_k\| < \epsilon_J$ for every $k = 1, \dots, \ell$, then for $n \geq n_J$:

$$d_M(\varphi_n(\tilde{y}_j), \varphi_n(y_J)) < \alpha n^{-1/d}.$$

This means that the combinatorics of the covers $\{B(y_j, r)\}_{j \in J}$ and $\{\hat{B}(\varphi_n(\tilde{y}_j), \alpha n^{-1/d})\}_{j \in J}$ are the same if $\|y_j - \tilde{y}_j\| < \epsilon_J$ for $j \in J$ and $n \geq n_J$.

Let us consider now an index set J satisfying condition (2) above. We want to prove that there exists $\epsilon_J > 0$ and n_J such that if $\|\tilde{y}_j - y_j\| < \epsilon_J$ for all $j \in J$, then for $n \geq n_J$ the intersection $\bigcap_{j \in J} \hat{B}(\varphi_n(\tilde{y}_j), \alpha n^{-1/d})$ is still empty. We argue by contradiction and assume there exist a sequence of points $x_n \in \hat{B}(p, Rn^{-1/d})$ and for $j \in J$ points $y_{j,n} \in B(0, R')$ with $\|y_{i,n} - y_j\| \leq \frac{1}{n}$ such that for all $j \in J$ and all n large enough:

$$(6.2) \quad d_M(x_n, \varphi_n(y_{j,n})) < \alpha n^{-1/d}.$$

We call $y_n = \psi_n(x_n)$ and assume that (up to subsequences) it converges to some $\bar{y} \in B(0, R')$. Using again the uniform convergence of d_n to $d_{\mathbb{R}^d}$, the inequality (6.2) would give:

$$r > \lim_{n \rightarrow \infty} \frac{n^{1/d} r}{\alpha} \cdot d_M(x_n, \varphi_n(y_{j,n})) = \|\bar{y} - y_j\| \quad \forall j \in J$$

which gives the contradiction $\bar{y} \in \bigcap_{j \in J} B(y_j, r) = \emptyset$.

Set now $n_1 = \max_{J \in \{\ell\}, k \leq d} n_J$ and $\epsilon_0 = \min_{J \in \{\ell\}, k \leq d} \epsilon_J$. We have proved that, if $\|\tilde{y}_j - y_j\| < \epsilon_0$ for all $j = 1, \dots, \ell$, then for all $n \geq n_1$ the two open covers $\{B(y_j, r)\}_{j \in J}$ and $\{\hat{B}(\varphi_n(\tilde{y}_j), \alpha n^{-1/d})\}_{j \in J}$ have the same combinatorics. In particular their Čech complex is the same. Moreover, Lemma 6.1 implies that for a possibly larger $n_0 \geq n_1$ all the balls $\hat{B}(x, \alpha n^{-1/d})$ are strictly convex in M ; consequently, by the Nerve Lemma, for n larger than such n_0 these two open covers are each one homotopy equivalent to their Čech complexes, hence they are themselves homotopy equivalent. \square

6.2. Measure theory.

Lemma 6.3. *Let $\mu_\lambda = \sum a_{\lambda,k} \delta_k$ be a one-parameter family of random probability measures on \mathbb{N} , and let $\mu = \sum a_k \delta_k$ be a deterministic probability measure on \mathbb{N} . Assume that for every $k \in \mathbb{N}$ $a_{\lambda,k} \rightarrow a_k$ in probability as $\lambda \rightarrow \infty$. Then $\mu_\lambda \rightarrow \mu$ in probability, i.e., for every $\varepsilon > 0$ we have*

$$\lim_{\lambda \rightarrow \infty} \mathbb{P}\{d(\mu_\lambda, \mu) \geq \varepsilon\} = 0,$$

where d denotes the total variation distance.

Proof. Let $\delta > 0$ be arbitrary.

Since μ is a probability measure on \mathbb{N} , there exists K such that

$$(6.3) \quad \sum_{k \geq K} a_k < \frac{\varepsilon}{4}.$$

We have

$$\mathbb{P}\left\{|a_{\lambda,k} - a_k| > \frac{\varepsilon}{4K}\right\} < \frac{\delta}{2K},$$

which implies (by a union bound)

$$(6.4) \quad \mathbb{P}\left\{\sum_{k < K} |a_{\lambda,k} - a_k| > \frac{\varepsilon}{4}\right\} < \frac{\delta}{2},$$

and also (by the triangle inequality)

$$(6.5) \quad \mathbb{P}\left\{\left|\sum_{k < K} a_{\lambda,k} - \sum_{k < K} a_k\right| > \frac{\varepsilon}{4}\right\} < \frac{\delta}{2},$$

for $\lambda \geq \lambda_0$.

The estimate (6.5) implies an estimate for the tails:

$$(6.6) \quad \mathbb{P}\left\{\left|\sum_{k \geq K} a_{\lambda,k} - \sum_{k \geq K} a_k\right| > \frac{\varepsilon}{4}\right\} < \frac{\delta}{2},$$

since

$$\left|\sum_{k < K} a_{\lambda,k} - \sum_{k < K} a_k\right| = \left|\sum_{k \geq K} a_{\lambda,k} - \sum_{k \geq K} a_k\right|,$$

which follows from μ_λ and μ being probability measures.

For any $\lambda > \lambda_0$, we then have

$$(6.7) \quad \mathbb{P}\left\{\sum_{k \geq K} a_{\lambda,k} > \frac{\varepsilon}{2}\right\} < \frac{\delta}{2}.$$

Indeed, if

$$\sum_{k \geq K} a_{\lambda,k} > \frac{\varepsilon}{2}$$

then equation (6.3) gives

$$\left|\sum_{k < K} a_{\lambda,k} - \sum_{k < K} a_k\right| > \frac{\varepsilon}{4},$$

and (6.7) then follows from (6.6).

In order to estimate the total variation distance between μ_λ and μ , let $A \subset \mathbb{N}$ be arbitrary. We have:

$$\begin{aligned} \left| \sum_{k \in A} a_{\lambda,k} - \sum_{k \in A} a_k \right| &= \left| \sum_{k \in A, k < K} (a_{\lambda,k} - a_k) + \sum_{k \in A, k \geq K} a_{\lambda,k} - \sum_{k \in A, k \geq K} a_k \right| \\ &\leq \sum_{k \in A, k < K} |a_{\lambda,k} - a_k| + \sum_{k \in A, k \geq K} a_{\lambda,k} + \sum_{k \in A, k \geq K} a_k \\ &\leq \sum_{k < K} |a_{\lambda,k} - a_k| + \sum_{k \geq K} a_{\lambda,k} + \sum_{k \geq K} a_k \\ &\leq \sum_{k < K} |a_{\lambda,k} - a_k| + \sum_{k \geq K} a_{\lambda,k} + \frac{\varepsilon}{4}. \end{aligned}$$

Using a union bound, this implies

$$\mathbb{P} \left\{ \left| \sum_{k \in A} a_{\lambda,k} - \sum_{k \in A} a_k \right| > \varepsilon \right\} \leq \mathbb{P} \left\{ \sum_{k < K} |a_{\lambda,k} - a_k| > \frac{\varepsilon}{4} \right\} + \mathbb{P} \left\{ \sum_{k \geq K} a_{\lambda,k} > \frac{\varepsilon}{2} \right\},$$

which is less than δ by (6.4) and (6.7).

This implies that for every $\delta > 0$ we have, for all λ sufficiently large,

$$\mathbb{P} \left\{ \sup_{A \subset \mathbb{N}} |\mu_\lambda(A) - \mu(A)| \geq \varepsilon \right\} \leq \delta,$$

i.e. we have shown

$$\lim_{\lambda \rightarrow \infty} \mathbb{P} \{d(\mu_\lambda, \mu) \geq \varepsilon\} = 0.$$

□

Lemma 6.4 (Topology does not leak to infinity). *For every $\delta > 0$ there exists a finite set $g \subset \mathcal{G}$ and $R_0 > 0$ such that for all $R \geq R_0$*

$$\mathbb{E} \sum_{\gamma \in g^c} c_{R,\gamma} < \frac{\delta}{4}.$$

Proof. First we observe that

$$(6.8) \quad \sum_{\gamma \in \mathcal{G}} \frac{\mathbb{E} N(\mathcal{P}_r, B(0, r); \gamma)}{\text{Vol}(B(0, r))} < a_0 < \infty,$$

where a_0 is independent of r . Indeed,

$$\sum_{\gamma \in \mathcal{G}} \frac{\mathbb{E} N(\mathcal{P}_r, B(0, r); \gamma)}{\text{Vol}(B(0, r))} = \frac{\mathbb{E} N(\mathcal{P}_r, B(0, r))}{\text{Vol}(B(0, r))} \leq \frac{\mathbb{E} |\{P_r \cap B(0, r)\}|}{\text{Vol}(B(0, r))},$$

which is a constant independent of r (the average number of points of a Poisson process in a given region is proportional to the volume of the region).

Let $A \subset \mathcal{G}$ be arbitrary. Then, using the Integral Geometry Sandwich, we obtain:

$$\begin{aligned} \int_{\Omega} \sum_{\gamma \in A} c_{R,\gamma}(\omega) d\omega &\leq \left(1 + \frac{r}{R}\right)^d \frac{1}{\text{Vol}(B_{R+r})} \mathbb{E} \left\{ \sum_{\gamma \in A} \int_{B_{R+r}} \frac{N^*(\mathcal{P}_r, B(x, r), \gamma)}{\text{Vol}(B_r)} dx \right\} \\ &\leq \left(1 + \frac{r}{R}\right)^d \left(\sum_{\gamma \in A} \mathbb{E} \frac{N(\mathcal{P}_r, B(x, r), \gamma)}{\text{Vol}(B_r)} + O(r^{-1}) \right). \end{aligned}$$

Let $\delta > 0$ be arbitrary, and choose r sufficiently large that the above $O(r^{-1})$ error term is smaller than $\delta/16$.

By the convergence (6.8) there exists a finite set $g \subset \mathcal{G}$ such that

$$\sum_{\gamma \in g^c} \frac{\mathbb{E}N(\mathcal{P}_r, B(0, r); \gamma)}{\text{Vol}(B(0, r))} < \frac{\delta}{16}.$$

Choosing R_0 large enough that $\left(1 + \frac{r}{R}\right)^d < 2$ we then have for all $R \geq R_0$

$$\mathbb{E} \sum_{\gamma \in g^c} c_{R, \gamma} < 2 \left(\frac{\delta}{16} + \frac{\delta}{16} \right) = \frac{\delta}{4},$$

as desired. □

Proposition 6.5. *μ is a probability measure.*

Proof. Let $\delta > 0$ and take $g \subset \mathcal{G}$ to be the set guaranteed by Lemma 6.4.

We want to show that

$$(6.9) \quad \left| \sum_{\gamma \in \mathcal{G}} c_\gamma - 1 \right| \leq \delta,$$

which will then immediately imply $\sum_{\gamma \in \mathcal{G}} c_\gamma = 1$, since $\delta > 0$ is arbitrary.

Observe that by Fatou's lemma

$$\sum_{\gamma \in g^c} c_\gamma \leq \liminf_{R \rightarrow \infty} \sum_{\gamma \in g^c} c_{R, \gamma},$$

and applying Fatou's lemma again followed by Tonelli's theorem, we have

$$\begin{aligned} \int_{\Omega} \sum_{\gamma \in g^c} c_\gamma d\omega &\leq \int_{\Omega} \liminf_{R \rightarrow \infty} \sum_{\gamma \in g^c} c_{R, \gamma}(\omega) d\omega \\ &\leq \liminf_{R \rightarrow \infty} \int_{\Omega} \sum_{\gamma \in g^c} c_{R, \gamma}(\omega) d\omega \\ &= \liminf_{R \rightarrow \infty} \sum_{\gamma \in g^c} \int_{\Omega} c_{R, \gamma}(\omega) d\omega. \end{aligned}$$

Combining this with Lemma 6.4 gives

$$(6.10) \quad \int_{\Omega} \sum_{\gamma \in g^c} c_\gamma d\omega \leq \frac{\delta}{4}.$$

We proceed to estimate $\left| \sum_{\gamma \in \mathcal{G}} c_\gamma - 1 \right|$:

$$\begin{aligned}
 \left| \sum_{\gamma \in \mathcal{G}} c_\gamma - 1 \right| &= \int_{\Omega} \left| \sum_{\gamma \in \mathcal{G}} c_\gamma - 1 \right| d\omega \\
 &\leq \int_{\Omega} \left| \sum_{\gamma \in \mathcal{G}} c_\gamma - \sum_{\gamma \in \mathcal{G}} c_{R,\gamma} \right| d\omega \\
 &\leq \int_{\Omega} \left| \sum_{\gamma \in g} c_\gamma - \sum_{\gamma \in g} c_{R,\gamma} \right| d\omega + \int_{\Omega} \left| \sum_{\gamma \in g^c} c_\gamma - \sum_{\gamma \in g^c} c_{R,\gamma} \right| d\omega \\
 &\leq \int_{\Omega} \left| \sum_{\gamma \in g} c_\gamma - \sum_{\gamma \in g} c_{R,\gamma} \right| d\omega + \int_{\Omega} \sum_{\gamma \in g^c} c_\gamma d\omega + \int_{\Omega} \sum_{\gamma \in g^c} c_{R,\gamma} d\omega \\
 &\leq \frac{\delta}{2} + \frac{\delta}{4} + \frac{\delta}{4}.
 \end{aligned}$$

In the last line, we have estimated the first term by $\delta/2$ by choosing R sufficiently large to apply Lemma 6.3, we have estimated the second term by $\delta/4$ using (6.10), and we have estimated the last term by $\delta/4$ by choosing R sufficiently large to apply Lemma 6.4. This establishes (6.9) and concludes the proof of the proposition. \square

6.3. The integral geometry sandwiches.

Theorem 6.6 (Integral Geometry Sandwich). *Let \mathcal{P} be a generic geometric complex in \mathbb{R}^d and fix $\gamma \in \mathcal{G}$. Then for $0 < r < R$*

$$\int_{B_{R-r}} \frac{\mathcal{N}(\mathcal{P}, B(x, r); \gamma)}{\text{Vol}(B_r)} dx \leq \mathcal{N}(\mathcal{P}, B_R; \gamma) \leq \int_{B_{R+r}} \frac{\mathcal{N}^*(\mathcal{P}, B(x, r); \gamma)}{\text{Vol}(B_r)} dx.$$

Theorem 6.7 (Integral Geometry Sandwich on a Riemannian manifold). *Let \mathcal{U} be a generic geometric complex on M and fix $\gamma \in \mathcal{G}$. Then for any $\varepsilon > 0$ there exists $\eta > 0$ such that for every $r < \eta$*

$$(1 - \varepsilon) \int_M \frac{\mathcal{N}(\mathcal{U}, \hat{B}(x, r); \gamma)}{\text{Vol}(B_r)} dx \leq \mathcal{N}(\mathcal{U}, M; \gamma) \leq (1 + \varepsilon) \int_M \frac{\mathcal{N}^*(\mathcal{U}, \hat{B}(x, r); \gamma)}{\text{Vol}(B_r)} dx,$$

where B_r still denotes the Euclidean ball of radius r .

Proofs of Theorems 6.6 and 6.7. These results follow from the same proof as in [SW16]. \square

Remark 2. We observe that similar statements hold true if we take the sum over *all* components, ignoring their type. More precisely, denoting by $\mathcal{N}(Y_1, Y_2)$ the number of components of Y_1 entirely contained in the interior of Y_2 and by $\mathcal{N}^*(Y_1, Y_2)$ the number of components of Y_1 that intersect Y_2 , we have the following inequality:

$$\int_{B_{R-r}} \frac{\mathcal{N}(\mathcal{P}, B(x, r))}{\text{Vol}(B_r)} dx \leq \mathcal{N}(\mathcal{P}, B_R) \leq \int_{B_{R+r}} \frac{\mathcal{N}^*(\mathcal{P}, B(x, r))}{\text{Vol}(B_r)} dx$$

and, in the Riemannian framework:

$$(1 - \varepsilon) \int_M \frac{\mathcal{N}(\mathcal{U}, \hat{B}(x, r))}{\text{Vol}(B_r)} dx \leq \mathcal{N}(\mathcal{U}, M) \leq (1 + \varepsilon) \int_M \frac{\mathcal{N}^*(\mathcal{U}, \hat{B}(x, r))}{\text{Vol}(B_r)} dx.$$

Since both \mathcal{P} and \mathcal{U} have only finitely many components, these inequalities follow by simply summing up the two inequalities from the previous theorems over all components type (the

sums are over finitely many elements). In fact the integral geometry sandwiches as proved in [SW16] are adaptations of the original construction from [NS16], where the case of all components was considered. We use this observation multiple times in the paper.

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