NOTES ON TAMENESS IN MODEL-THEORETIC STRUCTURES

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We consider some notions from model theory, a branch of mathematical logic. At its root, there are quite a few important—but somewhat tedious—technical definitions that create pitfalls for the unwary outsider. But there is a small fragment of model theory that often suffices for applications to other subjects, especially for explaining and applying model-theoretic results: what can be called (first-order) definability theory. We present here a brief introduction to the subject. There are two equivalent approaches—informally, the top-down and bottom-up—each more useful than the other at times. *Neither history nor a comprehensive treatment of basic results is given here, and for ease of exposition, some minor technicalities are glossed over.*

Some conventions. $\mathbb{N}$ denotes the set of non-negative integers.

Given a set X and $n \in \mathbb{N}$, X^n denotes the n-fold cartesian power of X, where $X^0 := \{\emptyset\}$. We identify $X^m \times X^n$ with X^{m+n} whenever convenient, in particular, $X^n \times X^0 \cong X^0 \times X^n \cong X^n$. Identify functions $f\colon X^0 \to X$ with the constant $f(\emptyset)$.

We regard functions as set-theoretic objects, that is, there is no difference between a function and its graph provided that the codomain is understood.

1. Structures: The Top-Down Approach

In this scenario, we are interested in some particular class of sets that are (or we hope are) **closed under first-order definability**; we make this notion precise.

Let M be a nonempty set. A **structure on M** is a sequence $\mathfrak{M} = (\mathfrak{M}_n)_{n \in \mathbb{N}}$ such that for each $n \in \mathbb{N}$:

- (S1) $\mathfrak{M}_n$ is a collection of subsets of M^n that is closed under taking complements and finite unions.
- (S2) $\mathfrak{M}_n$ contains the diagonals $\{ (x_1, \dots, x_n) \in M^n : x_i = x_j \}$, $1 \le i \le j \le n$.
- (S3) If $A \in \mathfrak{M}_n$, then $A \times M,\ M \times A \in \mathfrak{M}_{n+1}$.
- (S4) If $A \in \mathfrak{M}_{n+1}$, then the projection of A on the first n coordinates is in $\mathfrak{M}_n$.

We say that a set $A \subseteq M^n$ is **definable in $\mathfrak{M}$**, or that **$\mathfrak{M}$ defines A**, if $A \in \mathfrak{M}_n$. If no ambient space M^n is mentioned, then "definable set" (in $\mathfrak{M}$) means "definable subset of M^n, for some $n \in \mathbb{N}$". Whenever a particular structure $\mathfrak{M}$ is under consideration, we just say "definable".

It is crucial to understand that definability is always taken with respect to some particular structure. Whenever we have more than one structure under consideration, we must take care to avoid ambiguities.

The use of the word "definable" comes from a connection to first-order logic. Note the correspondence between the set-theoretic operations of complementation, union, intersection and projection, and the logical operations of negation, disjunction, conjunction, and existential quantification. Closure under these logical operations is a very strong condition; see Appendices A and B of [6] for some examples of how this can be exploited.

There is a natural partial order on the class of all structures on M. Given structures $\mathfrak{M} = (\mathfrak{M}_n)$ and $\mathfrak{M}' = (\mathfrak{M}'_n)$ on M we put $\mathfrak{M} \subseteq \mathfrak{M}'$ if $\mathfrak{M}_n \subseteq \mathfrak{M}'_n$ for all $n \in \mathbb{N}$. If $\mathfrak{M} \subseteq \mathfrak{M}'$, then we say any of the following (depending on convenience):

- $\mathfrak{M}$ is a **reduct** of $\mathfrak{M}'$;
- $\mathfrak{M}'$ is an **expansion** of $\mathfrak{M}$;
- $\mathfrak{M}'$ **expands** $\mathfrak{M}$.

Clearly, M has a largest structure on it: For each $n \in \mathbb{N}$, just let $\mathfrak{M}_n$ be the collection of all subsets of M^n. This is not a very interesting structure, but its existence is occasionally useful for theoretical purposes. There is also a smallest structure on M (also not very interesting). Usually, we are interested in structures that come equipped with some extra basic information.

Let $S_\beta \subseteq M^{n(\beta)}$ (β in some index set J) and $f_\alpha\colon M^{n(\alpha)} \to M$ be functions (α in some index set I). A **structure on** $(\, M, (S_\beta), (f_\alpha)\,)$ is a structure $\mathfrak{M}$ on M such that each f_α and each S_β is definable in $\mathfrak{M}$. As before, we may also say that $\mathfrak{M}$ expands, or is an expansion of, $(\, M, (S_\beta), (f_\alpha)\,)$.

Given $S \subseteq M$, we say that A is ***S*-definable** (in $\mathfrak{M}$), or **definable with parameters from *S*,** if A is definable in $(\mathfrak{M}, (c)_{c \in S})$, that is, in the expansion of $\mathfrak{M}$ by constants for each $c \in S$. In the case $S = M$, we say that A is **definable with parameters** or **parametrically definable**. Note that "definable" and "$\emptyset$-definable" mean the same thing. The distinction between "definable" and "parametrically definable" is often extremely important in model-theoretic statements and arguments.

Important. In some branches of contemporary model theory, the default has become that "definable" means "parametrically definable". We shall *not* follow this convention here. *When consulting the literature, one must determine in which sense "definable" is being used; when using the notion, one must take care to use it consistently.*

Examples.

Semilinear sets. Let K be a subfield of $\mathbb{R}$ and V be a K-linear subspace of $\mathbb{R}$. For each $n \in \mathbb{N}$, let $\mathfrak{M}_n$ be the collection of all finite unions of sets of the form

$$\{\, x \in V^n : f_1(x) = \cdots = f_k(x) = 0,\ g_1(x) < 0, \ldots, g_l(x) < 0 \,\}$$

where each f_i and each g_j are affine K-linear maps $V^n \to V$. (If $K = \mathbb{Q}$, then we can take the coefficients of the maps to be integers.) It is routine to check that each $\mathfrak{M}_n$ satisfies (S1)–(S3). Verifying (S4) takes a bit more work, but it is not difficult; see *e.g.* pages 25–27 of [4].

Constructible sets. Let K be an algebraically closed field ($\mathbb{C}$, for example). For each $n \in \mathbb{N}$, let $\mathfrak{M}_n$ be the collection of all finite unions of sets of the form

$$\{\, x \in K^n : f(x) \neq 0,\ g_1(x) = \cdots = g_l(x) = 0 \,\}$$

where f and each g_j are n-variable polynomial functions with coefficients from K. (If K is the field of complex algebraic numbers, then we can take the coefficients to be integers.) It is routine to check that each collection satisfies (S1)–(S3). That (S4) holds is a special case of Chevalley's constructibility theorem. For an interesting proof of the case $K = \mathbb{C}$, due to S. Łojasiewicz, see [4, Ch. 2, (2.25)].

Semialgebraic sets. Let R be a real-closed ordered field[1] ($\mathbb{R}$, for example). For each $n \in \mathbb{N}$, let $\mathfrak{M}_n$ be the collection of all finite unions of sets

$$\{\, x \in R^n : f(x) = 0,\ g_1(x) < 0, \dots, g_l(x) < 0 \,\}$$

where f and each g_j are n-variable polynomial functions with coefficients from R. (If R is the field of real algebraic numbers, then we can take the coefficients to be integers.) That (S4) holds is due to A. Tarski. Again, see [4, Ch. 2] for an interesting alternate proof.

Real subexponential sets. For each $n \in \mathbb{N}$, let $\mathfrak{M}_n$ be the collection of all projections on the first n variables of sets $\{\, (x, y) \in \mathbb{R}^{n+k} : F(x, y) = 0 \,\}$, where $k \in \mathbb{N}$ and $F : \mathbb{R}^{n+k} \to \mathbb{R}$ is a function from the ring

$$\mathbb{Z}[x_1, \dots, x_n, y_1, \dots, y_k, e^{x_1}, \dots, e^{x_n}, e^{y_1}, \dots, e^{y_k}].$$

In this case, verifying (S2)–(S4), along with showing that these collections are closed under finite unions, is the routine part. The (rather hard) work of showing that they are closed under complementation is due to A. Wilkie [27]. The result holds with $\mathbb{Z}$ replaced by $\mathbb{R}$, or even any real-closed subfield of $\mathbb{R}$.

Exercises.

- In the definition of semilinear set, replace $\mathbb{R}$ by $\mathbb{C}$ and $<$ by $\neq$. Show that (S1)–(S3) holds. Can you show that (S4) holds?
- Verify (S1)–(S3) for semilinear, constructible and semialgebraic sets. (*Hint.* Since R is real closed, we have $p(x) = q(x) = 0$ iff $p(x)^2 + q(x)^2 = 0$.)
- Show that if A is a semialgebraic subset of R^n then there exists $k \in \mathbb{N}$ and an $(n+k)$-variable polynomial P with coefficients in R such that A is the projection on the first n coordinates of the zeroset of P. (*Hint.* Since R is real closed, we have $g(x) > 0$ iff $g(x)$ is the square of a nonzero element of R.) Does the analog of this hold for constructible sets?
- Show that the subexponential subsets of $\mathbb{R}^n$ are closed under taking finite unions and intersections.
- In the definition of subexponential sets, replace $\mathbb{R}$ by $\mathbb{C}$. Show that $\mathbb{Q}$ is a complex subexponential set, but $\mathbb{C} \setminus \mathbb{Q}$ is not. (*Hint.* The irrational numbers are a nonmeager subset of $\mathbb{R}$.) Hence, the complex analog of Wilkie's theorem fails.

2. Structures: The Bottom-Up Approach

In this case, we are given the set M together with some functions on, and subsets of, various cartesian products, and we "close off under definability".

For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a (possibly empty) collection of subsets of M^n and $\mathcal{F}_n$ be a (possibly empty) collection of functions $M^n \to M$. Elements of $\mathcal{P}_n$ and $\mathcal{F}_n$ are sometimes called **primitive** relations and functions, or just **primitives**.[2] We now regard

[1] That is, every positive element has a square root and every odd-degree polynomial over R has a root. See *e.g.* Chapter XI of Lang's Algebra for basic information.

[2] For technical reasons, the elements of $\mathcal{F}_n$ must have domain M^n. Functions from proper subsets of M^n into M must be regarded as relations.

M as being equipped with these relations and functions, that is, we consider the **structure** $(\,M, (\mathcal{P}_n), (\mathcal{F}_n)\,)$ as an algebraic object. We shall see that our use of the word "structure" for two formally different objects causes no trouble when we are concerned with definability.

For each $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the smallest set of functions on M^n such that:

- $\mathcal{T}_n$ contains the coordinate projections $x \mapsto x_i \colon M^n \to M$ for $i = 1, \dots, n$;
- $F \circ (f_1, \dots, f_m) \in \mathcal{T}_n$ for all $m \in \mathbb{N}$, $F \in \mathcal{F}_m$ and $f_1, \dots, f_m \in \mathcal{T}_n$.

We construct collections $\mathfrak{M}_{n,k}$ of subsets of M^n by induction on $k \in \mathbb{N}$. For $k = 0$ and $n \in \mathbb{N}$, let $\mathfrak{M}_{n,0}$ be the boolean algebra of M^n generated by the collection of all sets of the following forms:

- $\{\, x \in M^n : f(x) = g(x)\,\}$, where $f, g \in \mathcal{T}_n$
- $\{\, x \in M^n : (f_1(x), \dots, f_m(x)) \in P\,\}$, where $f_1, \dots, f_m \in \mathcal{T}_n$, $P \in \mathcal{P}_m$, and $m \in \mathbb{N}$.

Let $k \in \mathbb{N}$ and assume that we have constructed $\mathfrak{M}_{n,k}$ for all $n \in \mathbb{N}$. Let $\mathfrak{M}_{n,k+1}$ be the boolean algebra of subsets of M^n generated over $\mathfrak{M}_{n,k}$ by the collection of all projections on the first n variables of sets in $\mathfrak{M}_{n+1,k}$.

For $n \in \mathbb{N}$, put $\mathfrak{M}_n := \bigcup_{k\in\mathbb{N}} \mathfrak{M}_{n,k}$. It is easy to see that $(\mathfrak{M}_n)_{n\in\mathbb{N}}$ is the smallest structure in the top-down sense on $\big(M, (\mathcal{P}_n), (\mathcal{F}_n)\big)$, so we just denote it by $\big(M, (\mathcal{P}_n), (\mathcal{F}_n)\big)$ and call it the **structure on M generated by** the $\mathcal{P}_n$ and $\mathcal{F}_n$. Often, for convenience, we just list the primitives, *e.g.*, $(\mathbb{R}, +, \cdot)$. And, of course, we say that $A \subseteq M^n$ is definable in $\big(M, (\mathcal{P}_n), (\mathcal{F}_n)\big)$ if $A \in \mathfrak{M}_n$.

Let $\mathfrak{M} = (\mathfrak{M}_n)$ be a structure on M in the top-down sense. Clearly, a set A is definable in $\mathfrak{M}$ in the top-down sense if and only if A is definable in $\big(M, (\mathfrak{M}_n)\big)$ in the bottom-up sense.

Two bottom-up structures on M are **interdefinable** if they have the same top-down form, *i.e.*, if they have exactly the same definable sets.

The sets in $\mathfrak{M}_{n,0}$ are the **quantifier-free definable** sets (in M^n). The structure $\big(M, (\mathcal{P}_n), (\mathcal{F}_n)\big)$ **admits quantifier elimination**—or **has QE** for short—if every definable set is quantifier-free definable, and is **model complete** if every definable set is a coordinate projection of a quantifier-free definable set.

Examples. (*Suggestion:* First, review the examples in §1.)

The sets definable in $(\mathbb{Q}, <, +, 1)$ are exactly the $\mathbb{Q}$-semilinear sets. To see this, note that the function $t \mapsto -t \colon \mathbb{Q} \to \mathbb{Q}$ and the constant 0 are definable in $(\mathbb{Q}, <, +, 1)$, so we might as well work over $(\mathbb{Q}, <, +, -, 0, 1)$. Each $\mathcal{T}_n$ is then exactly the collection of all affine linear maps $\mathbb{Q}^n \to \mathbb{Q}$ with integer coefficients. If $f, g \in \mathcal{T}_n$ and $x \in \mathbb{Q}^n$, then

$$f(x) = g(x) \Leftrightarrow (f - g)(x) = 0$$
$$f(x) < g(x) \Leftrightarrow (f - g)(x) < 0,$$

so the quantifier-free definable sets are exactly the semilinear sets. Now note that coordinate projections of semilinear sets are semilinear, *i.e.*, $(\mathbb{Q}, <, +, -, 0, 1)$ has QE, so every definable set is quantifier-free definable, hence semilinear.

(On the other hand, 1 is not definable in $(\mathbb{Q}, <, +)$, nor is $<$ definable in $(\mathbb{Q}, +, 1)$.)

The sets definable in $\big(\mathbb{R}, +, \cdot, (c)_{c\in\mathbb{R}}\big)$ are the (real) semialgebraic sets. Again, the reason is that $(\mathbb{R}, <, +, -, \cdot, (c)_{c\in\mathbb{R}}\big)$ has QE (what are the $\mathcal{T}_n$?) and $<$ is definable:

$$x < y \Leftrightarrow \exists z \in \mathbb{R},\ (x - y)z^2 + 1 = 0.$$

On the other hand, $(\mathbb{R}, +, \cdot, (c)_{c\in\mathbb{R}})$ is only model complete: $<$ is not quantifier-free definable (exercise).

The sets definable in $(\mathbb{C}, +, \cdot, (c)_{c\in\mathbb{C}})$ are the $\mathbb{C}$-constructible sets.

The sets definable in $(\mathbb{R}, +, e^x)$ are the subexponential sets (see the exercises below). $(\mathbb{R}, +, \cdot, e^x)$ is model complete, but $(\mathbb{R}, <, +, -, \cdot, 0, 1, e^x)$ does not have QE.

There are scads of other interesting examples, but let us be content with these for now.

Exercises.

- Let $\mathfrak{M}$ be a structure in the bottom-up sense that has QE. Show that, for any $S \subseteq M$, $(\mathfrak{M}, (c)_{c\in S})$ has QE. Show that this holds with "has QE" replaced by "is model complete".
- Let $(G, *, e)$ be a group, written multiplicatively, with identity e. Show that both e and $g \mapsto g^{-1} \colon G \to G$ are definable in $(G, *)$.
- Show that every set quantifier-free definable in $(\mathbb{R}, +, -, \cdot, 0, 1, e^x)$ is subexponential. Conclude that every set definable in $(\mathbb{R}, +, e^x)$ is subexponential, and (from Wilkie's theorem) that $(\mathbb{R}, +, -, \cdot, 0, 1, e^x)$ is model complete.

3. Tameness

The word "tameness" in this context is more cultural than precise, but I shall try to give some feeling for it.

All of the examples given so far have the property that they have some explicit, fairly simple, top-down form arising visibly from some nice collection of primitives. It is probably fair to say that this is the exception rather than the rule. For example, the structure $(\mathbb{R}, +, \cdot, \mathbb{N})$ is extremely complicated, involving even set-theoretic independence issues; see *e.g.* [13, Ch. V]. Thus, though the generating primitives may be familiar and fairly simple (at least, seemingly so), the definable sets that they generate can become more and more complicated.

The central issue in definability theory is to understand the definable sets generated from given sets of primitives, dually, to find useful sets of primitives that generate a given structure.

Let $(M, (\mathcal{P}_n), (\mathcal{F}_n))$ be a structure in the bottom-up sense. The first approximation of a definition of tameness is that there is some fixed $J \in \mathbb{N}$ such that for all $n \geq 1$ and $k \geq J$, we have $\mathfrak{M}_{n,k} = \mathfrak{M}_{n,J}$ (notation as in the previous section). Roughly speaking, this says that there is absolute bound on the complexity of the definable sets *relative to the given primitives*. Now, there is nothing wrong with this definition technically, but morally, it definitely has some problems. For example, suppose that each $\mathcal{P}_n$ is the power set of M^n; then we can take $J = 0$, but it does not do us any good, because the primitives are already as complicated as possible. Even if we do understand the primitives as well as we would like, then we still may have little hope of understanding the definable sets if we have to take J large; even dealing with $J \geq 2$ can be quite challenging.

The notion of tameness carries with it some judgement by the user as to what constitutes acceptable behavior of the definable sets, as well as what constitutes acceptable knowledge of the behavior.

All of the examples mentioned in §1 are generally regarded as tame, though understanding the subexponential sets in detail takes a fair amount of work (*e.g.*, it is known that no nondegenerate arc of $\sin x$ is subexponential, but the only known proof is fairly involved). The structure $(\mathbb{R}, +, \cdot, \mathbb{N})$ is generally regarded as "wild" (*i.e.*, not tame).

Here is something a bit more esoteric: If G is a divisible subgroup of $\mathbb{R}$ containing 1, then every subset of $\mathbb{R}^n$ definable in $(\mathbb{R}, +, G, 1)$ is a boolean combination of projections on the first n coordinates of sets of the form $A \cap (\mathbb{R}^n \times G^k)$ where $k \in \mathbb{N}$ and $A \subseteq \mathbb{R}^{n+k}$ is semilinear (and recall our convention about what G^k means). The result is just a special case of much more general theorem [3], in particular, if G is also a real-closed field, then the result holds for $(\mathbb{R}, +, \cdot, G)$ with "semialgebraic" in place of "semilinear".

O-minimality. Let $(M, <)$ be a dense linearly ordered set without endpoints, and let $\mathfrak{M}$ be an expansion of $(M, <)$. Note that every finite union of points and open intervals (with endpoints from $M \cup \{\pm\infty\}$) contained in M is parametrically definable in $\mathfrak{M}$. The structure $\mathfrak{M}$ is **o-minimal** (short for "order-minimal") if every parametrically definable subset of M—*i.e.*, definable in $(\mathfrak{M}, (c)_{c\in M})$—is a finite union of points and open intervals; for $M = \mathbb{R}$, this is the same as saying that every parametrically definable subset of $\mathbb{R}$ has finitely many connected components. For expansions of $(\mathbb{R}, <, +, 1)$, this is the same as requiring only that every definable subset of $\mathbb{R}$ have finitely many connected components [18].

O-minimal structures—especially o-minimal structures on $(\mathbb{R}, +, \cdot\,)$—have so many nice properties that it takes many pages to describe (let alone prove) them; for starters, see [4–6]. They are generally regarded as tame, but it would take us too far afield here to go into details.

Examples.

- $(\mathbb{Q}, <, +)$ is o-minimal; this is immediate from QE. On the other hand, $(\mathbb{Q}, <, +, \cdot\,)$ is not o-minimal: The definable set $\{\, x \in \mathbb{Q} : x^2 < 2 \,\}$ is not a finite union of points and open intervals of $(\mathbb{Q}, <)$. In fact, $(\mathbb{Q}, <, +, \cdot\,)$ defines $\mathbb{N}$, and is regarded as wild.
- If R is a real-closed field, then $(R, +, \cdot\,)$ is o-minimal (again by QE).
- If $\mathfrak{R}$ is an o-minimal expansion of $(\mathbb{R}, <, +)$, then the expansion of $\mathfrak{R}$ by exponentiation is o-minimal (hence so is the expansion of $\mathfrak{R}$ by multiplication). This result is difficult; see [22] and [24]. In particular, $(\mathbb{R}, +, e^x)$ is o-minimal (but this was first proved via Wilkie's theorem on subexponential sets).

As a counterpoint to the last item above, o-minimal expansions of $(\mathbb{R}, <, +)$ that do not define multiplication are exceptional (in the sense that they have rather special properties) as are o-minimal expansions of $(\mathbb{R}, +, \cdot)$ that do not define exponentiation; see [14, 20] for information.

For an exposition of an application of o-minimality to diophantine geometry, see [23].

For further general discussion of tameness in the real context, see [15], but some of the material there is now quite out of date. For some more recent works in the subject by me and some of my collaborators, see [1, 2, 7, 10, 12, 16, 17, 19, 21]. *It is worth noting that there is a good record of significant contributions being made by grad students* (*e.g.*, [25, 26]) *and postdocs* (*e.g.*, [8, 9]). (Indeed, I did [14] while a grad student and [6, 18, 20] while a postdoc.)

4. Dimensional coincidence

I now describe what is arguably the current state of the art in tameness over the real field. We first need three different notions of dimension for $E \subseteq \mathbb{R}^n$.

Inductive dimension. If $E = \emptyset$, put $\operatorname{ind} E = -\infty$. If $E \neq \emptyset$, then $\operatorname{ind} E$ is the least $k \in \mathbb{N}$ such that, for every $x \in E$ and open $V \ni x$, there is an open $U \ni x$ such that the closure of U is contained in V and $\operatorname{ind}(E \cap \operatorname{bd} U) < k$.

Note that ind is a purely topological notion.

Assouad dimension. For $S \subseteq \mathbb{R}^n$ and $r > 0$, let $N(S, r)$ be the number (allowing $+\infty$) of open balls of radius r needed to cover S. Let $\operatorname{Dim} E$ be the infimum of $\alpha \in \mathbb{R}$ such that the set

$$\{ (r/R)^\alpha N(E \cap B(x, R), r) : x \in E, 0 < r < R \}$$

is bounded. ($B(x, r)$ is the open ball in $\mathbb{R}^n$centered at x and having radius r.)

Note that Dim is a purely metric notion.

An empirical observation: All dimensions commonly encountered in geometric measure theory, fractal geometry and analysis on metric spaces are bounded below by ind and above by Dim. Thus, sets on which ind and Dim are equal can be regarded as tame from the point of view of topological- and metric-dimension theory; indeed, some authors call such sets "antifractal", as they are not fractal under any of the various definitions one finds in the literature.

"Naive" dimension. $\dim E$ is the maximal $k \in \mathbb{N}$ such that some coordinate projection of E on $\mathbb{R}^k$ has interior.

Now, dim is neither a topological nor a metric notion per se, but all *metric* dimensions commonly encountered in geometric measure theory, fractal geometry and analysis on metric spaces are bounded below by dim and above by Dim. On the other hand, the relation of dim to ind is complicated: Each of dim $<$ ind, dim $=$ ind and dim $>$ ind occur even among the G_δ sets in $\mathbb{R}^3$.

Theorem. *If E is closed, $f\colon E \to \mathbb{R}^p$ is continuous and $(\mathbb{R}, +, \cdot, f)$ does not define $\mathbb{N}$, then* $\operatorname{ind} f(E) = \dim f(E) = \operatorname{Dim} f(E)$.

It would take too much time and space to explain the consequences of this here. If you are interested, feel free to contact me, and we can talk about it. Or try tackling it on your own; see [11].

In addition to the below, see `https://www.asc.ohio-state.edu/miller.1987/` for corrections and upgrades to some of my papers, as well as some unpublished notes.

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