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Spectral action gravity and cosmological models

Action spectrale, gravitation et modèles cosmologiques

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ABSTRACT

This paper surveys recent work of the author and collaborators on cosmological models based on the spectral action functional of gravity. A more detailed presentation of the topics surveyed here will be available in a forthcoming book [1].

RÉSUMÉ

Cet article passe en revue les travaux récents de l'auteure et de ses collaborateurs sur les modèles cosmologiques basés sur la fonctionnelle d'action spectrale de la gravitation. Une présentation plus détaillée des sujets abordés ici sera proposée dans un livre à venir [1].

1. Spectral action as a modified gravity model

In recent years, considerable interest has grown around various modified gravity models and their cosmological implications (for a comprehensive survey, see, for instance, [2]). These involve a range of different possibilities, including scalar-tensor theories, bigravity, MOND, conformal gravity, $f(R)$ theories, Hořava–Lifschitz gravity, and various braneworld scenarios. Such models have gained importance as a possible source of explanations for dark matter and dark energy phenomena. Observational data can severely constrain modified gravity models (see, for example, [3]). In this paper, we focus on another possible model of modified gravity, which arises naturally in the context of noncommutative geometry, in which gravity is described by the spectral action functional. We review the main aspects of the spectral action model of gravity and the recent development of cosmological applications, outlining where the link to observational constraints can be most significant.

The spectral action functional was introduced in [4] as a model of gravity (and gravity coupled with matter) on noncommutative spaces. The generalization to the noncommutative world of a compact Riemannian smooth spin manifold is provided by the notion of spectral triple $(\mathcal{A}, \mathcal{H}, D)$, which axiomatizes the relations between the algebra of smooth functions $\mathcal{A} = \mathcal{C}^\infty(X)$ and the metric on a manifold X, where the metric is encoded in the Dirac operator $\not{\partial}$ acting on the Hilbert space $\mathcal{H} = L^2(X, S)$ of square-integrable spinors. The main relation between the algebra and the Dirac operator is

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expressed by the condition that the commutators $[D, a]$ are bounded operators acting on $\mathcal{H}$. The operator D is required to be self-adjoint and with compact resolvent to encode the analytic properties of the usual Dirac operator on a compact manifold. Under the assumption that the operator D satisfies $\mathrm{Tr}(|D|^{-s}) < \infty$ for sufficiently large $Re(s)$, that is, that the spectral triple is finitely summable, the spectral action functional is defined as

$$\mathcal{S}_{\Lambda,f}(D) = \mathrm{Tr}(f(D/\Lambda) = \sum_{\lambda\in\mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda) \tag{1}$$

where $f \in \mathcal{S}(\mathbb{R})$ is an even rapidly decaying function (a smooth approximation to a cutoff function) and $\Lambda \in \mathbb{R}^+$ is an energy scale parameter that makes D/Λ dimensionless. Thus, one can think of the spectral action as a suitably regularized trace of the Dirac operator. It can be related to the heat kernel of D^2 and to the zeta function $\zeta_D(s) = \mathrm{Tr}(|D|^{-s})$ via Mellin transform. This provides, in the case where the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is an actual compact Riemannian spin manifold, an asymptotic expansion for the spectral action (see [4] and Chapter 1 of CoMa-book)

$$\mathcal{S}_{\Lambda,f}(D) \sim_{\Lambda\to\infty} \sum_{\beta\in\Sigma^+_{ST}} f_\beta\, \Lambda^\beta \, ⨍ |D|^{-\beta} \; + \; f(0)\, \zeta_D(0) \tag{2}$$

where $f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, \mathrm{d}v$ are the momenta of f. The summation is over the points of the non-negative dimension spectrum (poles of the zeta function on the non-negative real line), and the coefficients are residues of the zeta function,

$$⨍ |D|^{-\beta} = \frac{1}{2}\mathrm{Res}_{s=\beta}\; \zeta_D(s) \tag{3}$$

representing the noncommutative integration in dimension β.

The spectral action was proposed in [4] as a possible action functional for gravity coupled with matter, when computed for an almost commutative spectral triple (a product of a manifold and a finite noncommutative space). It was successfully applied to the construction of particle physics models, where its asymptotic expansion reconstructs the Lagrangian of the Standard Model with right-handed neutrinos and Majorana masses [5]. It was also shown in [5] that, in the gravity sector, the asymptotic expansion of the spectral action gives rise to a modified gravity model that includes, in addition to the Einstein–Hilbert action and the cosmological term of General Relativity, also a conformal gravity term (Weyl curvature) and a Gauss–Bonnet gravity term (which is non-dynamical and topological in dimension four). The particle physics models based on the spectral action have recently been shown to accommodate the correct Higgs mass, an additional scalar field (about which more later), supersymmetric models, and Pati–Salam grand unified theories, see [6–9].

We will discuss here only the case where the spectral triple $(\mathcal{A}, \mathcal{H}, D)$ is the commutative spectral triple $(\mathcal{C}^\infty(X), L^2(X, S), \not{D})$ associated with a compact spin Riemannian manifold X. In this case, the spectral action provides a model of (modified) Euclidean gravity on X (see [10]) that includes, in addition to the usual Einstein–Hilbert action with cosmological constant, an additional modified gravity term that includes conformal gravity and Gauss–Bonnet gravity. The reason why the spectral action requires an Euclidean signature lies in the property of the Dirac operator: on a compact Riemannian spin manifold, the Dirac operator is self-adjoint with compact resolvent, hence in particular the spectrum is discrete and with finite multiplicities, so that (1) is well defined, while these properties typically do not hold in the Lorentzian setting. However, though the spectral action itself is defined only in Euclidean signature, it is often possible to make sense of a Wick rotation to Lorentzian signature for the individual terms of its asymptotic expansion.

In the case of a 4-dimensional compact Riemannian spin manifold M, the leading terms of the asymptotic expansion of [11] for large Λ correspond to the points $\beta = 0, 2, 4$, respectively with contributions

$$\mathrm{Tr}(f(D/\Lambda)) \sim 2\Lambda^4 f_4 a_0 + 2\Lambda^2 f_2 a_2 + f_0 a_4$$

The coefficients a_0, a_2 and a_4 correspond, respectively, to the cosmological term, the Einstein–Hilbert term, and the Weyl curvature and Gauss–Bonnet modified gravity terms. We will discuss here some of the cosmological implications of this model of gravity.

The cosmological implications of conformal gravity models (Weyl curvature) are analyzed for instance in [12] and specific astrophysical and cosmological effects of the presence of the Weyl curvature terms in the spectral action were analyzed in [13,14], for example with respect to the effects on gravitational wave equations.

2. RGE flows and early universe scenarios

In the spectral action functional for models of gravity coupled with matter, based on an almost commutative geometry $X \times F$, the choice of the finite noncommutative space F determines the particle physics sector of the model. Indeed, the finite space $F = (A_F, H_F, D_F)$ consists of a spectral triple where the algebra A_F and the Hilbert space H_F are finite dimensional. The Hilbert space specifies the fermion content of the model, with the representation of A_F determining the hypercharges, and the unitaries $U(A_F)$ determine the gauge symmetries. The Dirac operator D_F contains the information on the Yukawa parameters (masses and mixing angles) of the particle sector. The gauge boson and the Higgs sector arise from the Dirac operator on the product $D = \not{D}_X \otimes 1 + \gamma_5 \otimes D_F$, by considering, respectively fluctuations in the manifold direction (gauge bosons) and fluctuations in the finite noncommutative direction (Higgs boson), see Chapter 1 of [15] for a detailed account of this method. In particular, $A_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C})$, with $\mathbb{H}$ the quaternions, is the finite algebra that gives rise to an extension of the Minimal Stantard Model with right-handed neutrinos and Majorana mass terms, [5]. As shown in [5] (see also Chapter 1 of [15]) in the asymptotic expansion of the spectral action for this almost-commutative geometry, one obtains coefficients of the gravitational term that depend on the Yukawa parameters of the particle sector. This determines certain relations between these parameters at unification energy where the initial conditions for this model are set.

More precisely, the asymptotic expansion of the spectral action for the almost commutative geometry is of the form [5]

$$\begin{aligned}
\mathcal{S}_{\Lambda,f}(D) \sim\ & \frac{1}{2\kappa_0^2(\Lambda)} \int R \sqrt{g}\, \mathrm{d}^4x + \gamma_0(\Lambda) \int \sqrt{g}\, \mathrm{d}^4x \\
&+ \alpha_0(\Lambda) \int C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} \sqrt{g}\, \mathrm{d}^4x + \tau_0(\Lambda) \int R^* R^* \sqrt{g}\, \mathrm{d}^4x \\
&+ \frac{1}{2} \int |DH|^2 \sqrt{g}\, \mathrm{d}^4x - \mu_0^2(\Lambda) \int |H|^2 \sqrt{g}\, \mathrm{d}^4x \\
&- \xi_0(\Lambda) \int R\, |H|^2 \sqrt{g}\, \mathrm{d}^4x + \lambda_0(\Lambda) \int |H|^4 \sqrt{g}\, \mathrm{d}^4x \\
&+ \frac{1}{4} \int (G^i_{\mu\nu} G^{\mu\nu i} + F^\alpha_{\mu\nu} F^{\mu\nu\alpha} + B_{\mu\nu} B^{\mu\nu}) \sqrt{g}\, \mathrm{d}^4x
\end{aligned}$$

where G, F, B are the gauge bosons, H is the Higgs field, $C_{\mu\nu\rho\sigma}$ is the Weyl curvature, and R^*R* is the (topological) Gauss–Bonnet term. The coefficients in this expansion represent an "effective cosmological constant" $\gamma_0(\Lambda)$ and an "effective gravitational constant" $8\pi G_{\mathrm{eff}}(\Lambda) = \kappa_0^2(\Lambda)$. They are given by the expressions

$$\begin{aligned}
&\frac{1}{2\kappa_0^2(\Lambda)} = \frac{96 f_2 \Lambda^2 - f_0 \mathfrak{c}(\Lambda)}{24\pi^2} && \alpha_0 = -\frac{3 f_0}{10\pi^2} \\
&\lambda_0(\Lambda) = \frac{\pi^2 \mathfrak{b}(\Lambda)}{2 f_0 \mathfrak{a}^2(\Lambda)} && \tau_0 = \frac{11 f_0}{60\pi^2} \\
&\mu_0^2(\Lambda) = 2\frac{f_2 \Lambda^2}{f_0} - \frac{\mathfrak{e}(\Lambda)}{\mathfrak{a}(\Lambda)} && \xi_0 = \tfrac{1}{12} \\
&\gamma_0(\Lambda) = \frac{1}{\pi^2}(48 f_4 \Lambda^4 - f_2 \Lambda^2 \mathfrak{c}(\Lambda) + \frac{f_0}{4}\mathfrak{d}(\Lambda))
\end{aligned}$$

where the terms $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}, \mathfrak{d}, \mathfrak{e}$ are functions of the Yukawa parameters Y and the Majorana mass matrix M, which in turn run with the energy scale Λ,

$$\begin{aligned}
\mathfrak{a} &= \mathrm{Tr}(Y_\nu^\dagger Y_\nu + Y_e^\dagger Y_e + 3(Y_u^\dagger Y_u + Y_d^\dagger Y_d)) \\
\mathfrak{b} &= \mathrm{Tr}((Y_\nu^\dagger Y_\nu)^2 + (Y_e^\dagger Y_e)^2 + 3(Y_u^\dagger Y_u)^2 + 3(Y_d^\dagger Y_d)^2) \\
\mathfrak{c} &= \mathrm{Tr}(MM^\dagger) \\
\mathfrak{d} &= \mathrm{Tr}((MM^\dagger)^2) \\
\mathfrak{e} &= \mathrm{Tr}(MM^\dagger Y_\nu^\dagger Y_\nu)
\end{aligned}$$

There are different ways of interpreting this expression in the model. As a boundary condition at unification energy, this determines some constraints on the initial conditions of the renormalization group equations (RGE) given by the relation between the gravitational and Yukawa terms expressed above. As treated in [5], at lower energies the Yukawa terms run according to the RGE flow of the Minimal Standard Model [16], and the running of the gravitational terms (with compatible initial conditions) is derived independently (see [17–19]). In [20], this approach is revisited by considering the effect of replacing the RGE for the Minimal Standard Model with those of the extension with right-handed neutrinos and Majorana mass terms derived in [16,21]. These consist of different effective field theories between the see-saw scales of the Majorana masses. A compatible set of initial conditions at unification based is also identified in [20]. In particular, it is also shown that the RGE equations exhibit a sensitive dependence on the initial condition, which causes a fine-tuning problem in the model.

In [22], a version of this model is analyzed, where one allows the relation described above between gravitational and Yukawa terms to persist for some range of energies sufficiently close to unification energy, and not only as an initial condition, compatibly with the fact that the relation above expresses the form of the asymptotic expansion of the spectral action for sufficiently large Λ. Thus, one looks at the effect of the running of the Yukawa couplings on the effective gravitational and cosmological constant for large Λ. One finds that this model reproduces several scenarios that have been previously studied in cosmology, which include primordial black holes with varying gravitational constant (gravitational memory) [23–25] and with a modified evaporation law for primordial black hole, with implications on their possible relation to gamma-ray bursts considered in [26]; Linde's antigravity in the early universe hypothesis [27] and "gravity balls" [28]; emergent Hoyle–Narlikar cosmologies [29] near the see-saw scales where the Einstein–Hilbert terms may become subdominant with respect to the conformal gravity terms; a variable cosmological constant as in the model of [30]; an inflation scenario based on the conformal coupling of the Higgs bosons with gravity as in [31], which is however severely constrained by the results of [32,33].

3. Cosmic topology

The question of "cosmic topology" has been variously investigated in theoretical cosmology over the past two decades. The 3-dimensional spatial sections of a 4-dimensional spacetime are (compact or non-compact) smooth 3-dimensional manifolds. Under the hypothesis of compactness, and under standard cosmological assumptions of homogeneity and isotropy, such a 3-manifold should be either a spherical space form (either the 3-sphere S^3 or a quotient of S^3 by a group of isometries) or a flat torus or Bieberbach manifold (a quotient of a 3-dimensional torus by a group of isometries), or else a compact hyperbolic 3-manifold, depending on whether the curvature is positive, flat, or negative. From the mathematical viewpoint, the richest and most interesting among these classes of manifolds is the hyperbolic one, but cosmological data seem to indicate that the negative curvature case is ruled out and the curvature is either flat or slightly positively curved. So the remaining candidate topologies are the spherical space forms and the tori and Bieberbach manifolds. There is a complete classification of all of these cases, and the cosmic topology question investigates whether it is possible to identify from cosmological and astrophysical data the most likely topology among these choices. Note that, while geometry (curvature) is a local information that is encoded in the Einstein equations of gravity, topology is a global phenomenon that is not visible at the level of General Relativity. So far the main approach to investigating cosmic topology has been the search for signatures in the background radiation of the existence of a periodic structure that indicates a nontrivial topology (the "matching circles in the sky" method), see [34–42]. While it has been proposed that the dodecahedral space (that is, the Poincaré homology sphere) may be a plausible candidate for cosmic topology, the results so far have been inconclusive in detecting the presence of non-trivial topology.

In [43–45], the problem of cosmic topology is considered from a new viewpoint. Assuming that we model gravity using the spectral action, can this model of gravity provide information on the cosmic topology? Unlike the usual Einstein–Hilbert action of General Relativity, which is sensitive to the local information of curvature but to on the global information of topology, the spectrum of the Dirac operator, on which the spectral action is based, is sensitive to global properties. This approach relies on computing the spectral action directly through the Dirac spectrum, which is generally not explicitly known. However, in the special case of highly symmetric spaces like spherical space forms and Bieberbach manifolds, explicit computations based on the Dirac spectrum can be performed via a Poisson summation technique. This was observed for the case of S^3 in [11] and can be generalized to the other spaces in the list of candidate cosmic topologies, see [43,44, 46], using explicit information on the Dirac spectra [47–49]. The basic idea (see [11]) is that, if X is a manifold for which the eigenvalues of the Dirac operator $\not{D}_X$ form an arithmetic progression (or a union of finitely many arithmetic progressions), $\mathrm{Spec}(\not{D}_X) = \cup_i A_i$, and the multiplicities of the eigenvalues in each arithmetic progression are interpolated by a polynomial $\mathrm{Mult}(\lambda) = P_i(\lambda)$ for all $\lambda \in A_i \subset \mathrm{Spec}(\not{D}_X)$, then the product $h(\lambda) = P_i(\lambda) f(\lambda/\Lambda)$ is a rapidly decaying function and the spectral action can be computed via a Poisson summation formula

$$\sum_{n\in\mathbb{Z}} h(x+\alpha n) = \frac{1}{\alpha} \sum_{m\in\mathbb{Z}} \exp(\frac{2\pi i m x}{\alpha})\, \hat{h}(\frac{m}{\alpha})$$

where $\hat{h}$ is the Fourier transform. One then shows that all the $m \neq 0$ terms in the dual series are smaller than Λ^{-k} for any $k \in \mathbb{N}$, hence the leading contribution to the spectral action comes only from the $m = 0$ term. For example, in the case of a sphere S^3_a of radius a, one finds

$$\mathcal{S}_{\Lambda,f}(\not{D}_{S^3_a}) \sim (\Lambda a)^3 \hat{f}^{(2)}(0) - \frac{1}{4}\Lambda a \hat{f}(0)$$

with $\hat{f}^{(2)}$ the Fourier transform of $x^2 f(x)$. In the case of the spherical space forms, although the Dirac spectrum and multiplicities depend on the choice of the spin structure, the spectral action itself does not and it turns out to be simply a multiple of the spectral action of the 3-sphere in the case of the spherical space forms, or of a 3-torus in the case of the Bieberbach manifolds, where the overall factor divides by the order of the group of isometries. So, for instance, for s spherical space form $Y = S^3/\Gamma$ one finds $\mathcal{S}_{\Lambda,f}(\not{D}_Y) = \mathcal{S}_{\Lambda,f}(\not{D}_{S^3})/\#\Gamma$. A simple derivation of this fact via the heat kernel expansion is given in [45].

It seems from this that the spectral action depends only very mildly on the different topologies (only through the overall factor $\#\Gamma$) in order to get any useful information regarding the signatures of nontrivial topology. However, it is possible to generate from the spectral action a slow-roll inflation potential, obtained from a scalar perturbation $D^2 + \phi^2$ of the Dirac operator D [43], and this can be used to derive observable quantities that can be used to distinguish between the candidate cosmic topologies [43,44].

In slow-roll inflation models, the slow-roll potential $V(\phi)$ of the scalar field ϕ determines the slow-roll parameters

$$\epsilon = \frac{m_{\rm Pl}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2, \quad \eta = \frac{m_{\rm Pl}^2}{8\pi}\frac{V''(\phi)}{V(\phi)}, \quad \xi = \frac{m_{\rm Pl}^4}{64\pi^2}\frac{V'(\phi)V'''(\phi)}{V^2(\phi)}$$

where $m_{\rm Pl}$ is the Planck mass. These, in turn, determine very constrained measurable quantities in cosmology, such as the spectral index $n_{\rm s}$ and the tensor-to-scalar ratio r

$$n_{\rm s} \simeq 1 - 6\epsilon + 2\eta, \quad n_{\rm t} \simeq -2\epsilon, \quad r = 16\epsilon$$
$$\alpha_s \simeq 16\epsilon\eta - 24\epsilon^2 - 2\xi, \quad \alpha_t \simeq 4\epsilon\eta - 8\epsilon^2$$

The power spectra for the scalar and tensor fluctuations of a Friedmann cosmology depend on these quantities as

$$\mathcal{P}_{\rm s}(k) \sim \mathcal{P}_{\rm s}(k_0)\left(\frac{k}{k_0}\right)^{1-n_{\rm s}+\frac{\alpha_s}{2}\log(k/k_0)}$$
$$\mathcal{P}_{\rm t}(k) \sim \mathcal{P}_{\rm t}(k_0)\left(\frac{k}{k_0}\right)^{n_{\rm t}+\frac{\alpha_t}{2}\log(k/k_0)}$$

where the expressions $n_{\rm s}$, $n_{\rm t}$, $\alpha_{\rm s}$, $\alpha_{\rm t}$ in the exponent are as above, and the amplitude depends on the slow-roll potential as

$$\mathcal{P}_{\rm s}(k_0) \sim \frac{V^3}{(V')^2}, \quad \mathcal{P}_{\rm t}(k_0) \sim V$$

with a proportionality constant that contains a power of the Planck mass $m_{\rm Pl}$, see [50–52]. The quantities ϵ, η, ξ (hence the $n_{\rm s}$, $n_{\rm t}$, $\alpha_{\rm s}$, $\alpha_{\rm t}$ and the exponents of the power law) detect the difference between spherical and flat case, but do not distinguish between different spherical space forms or between different Bieberbach manifolds, while the amplitudes $\mathcal{P}_{\rm s}(k_0)$ and $\mathcal{P}_{\rm t}(k_0)$ distinguish between almost all the different cases of spherical space forms and between the different Bieberbach manifolds. Thus, in a model of gravity based on the spectral action functional, the different cosmic topologies leave a measurable signature on the shape of the inflation potential and on the corresponding power laws for scalar and tensor fluctuations. The latter are in turn constrained by observational data on the cosmic microwave background (CMB).

It is further shown in [45] that, in the case where the spectral action is computed on an almost commutative geometry, to incorporate the matter sector coupled with gravity, the inflation potential $V(\phi)$ and the amplitudes $\mathcal{P}_{\rm s}(k_0)$ and $\mathcal{P}_{\rm t}(k_0)$ also acquire a multiplicative factor that depends on the number of fermionic elementary particles in the matter sector of the model. This is based on results of [48] for spectra of twisted Dirac operators.

Using a general technique of [53,54] to construct Dirac spectra, it is also possible to engineer different kinds of inflation potentials $V(\phi)$ using the spectral action. Note that the scalar field ϕ that plays the role of inflaton field is not a Higgs field, although it also coupled conformally with gravity. It is more closely related to the scalar field introduced in [6] and derived in [55] from the inner fluctuations of a "fused algebra" that combines all the spectral triple data into a single object. Note that, because the spectral action model admits both a Higgs field and an additional inflaton scalar field, it is also possible to develop spectral-action-based multifield inflation models. In such models, isocurvature perturbations can arise, which are dependent on the inflaton–Higgs coupling and their nonminimal couplings with gravity. Multifield inflation models are severely constrained by CMB data (see for instance [56]). Such multifield inflationary models within a spectral action scenario are currently being investigated.

4. Arithmetic structures in gravity

The residues of the zeta function $\zeta_D(s)$ that appear in the coefficients of the asymptotic expansion of the spectral action are related to the coefficients in the expansion of the heat kernel

$$\mathrm{Tr}(\mathrm{e}^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for } t \to 0$$

via the Mellin transform relation

$$|D|^{-s} = \frac{1}{\Gamma(s/2)}\int_0^\infty \mathrm{e}^{-tD^2} t^{\frac{s}{2}-1}\mathrm{d}t$$

so that one has

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

In the case of a manifold of dimension $m = \dim M$, the heat kernel expansion of $D^2 = \not{\partial}_X^2$ is given by the Seeley–DeWitt coefficients

$$\mathrm{Tr}(\mathrm{e}^{-tD^2}) \sim_{t\to 0+} \; t^{-m/2} \sum_{n=0}^{\infty} a_{2n}(D^2) t^n$$

These can be computed via a recursively constructed parametrix R_λ satisfying $\sigma((D^2-\lambda)R_\lambda) \sim 1$.

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi \mathrm{i}} \int \int_{\gamma} \mathrm{e}^{-\lambda} \, \mathrm{tr}\,(r_{2n}(x,\xi,\lambda)) \, \mathrm{d}\lambda \, \mathrm{d}^m \xi$$

In sufficiently regular and symmetric cases, like the (Euclidean) Robertson–Walker spacetimes, $\mathrm{d}s^2 = \mathrm{d}t^2 + a(t)^2 \mathrm{d}\sigma^2$, with the scaling factor $a(t)$ and the round metric $\mathrm{d}\sigma^2$ on S^3, the coefficients of the asymptotic expansion of the spectral action can be computed explicitly, see [57,58]. Surprisingly, all the terms a_{2n} in the asymptotic expansion of the spectral action of a Robertson–Walker spacetime are expressible as a rational function with $\mathbb{Q}$-coefficients of the scaling factor and its derivative up to $2n$. This was conjectured in [57] and proved in [58]. The occurrence of this kind of rationality result is indicative of the presence of underlying arithmetic structures in the spectral action model of gravity, at least for sufficiently regular spacetimes. The same rationality result was proved in [59] for $SU(1)$-Bianchi IX spacetimes. The method used in [59] relies on a faster computation of the Seeley–DeWitt coefficients in terms Wodzicki residues after taking products of the 4-dimensional spacetime with auxiliary flat tori, so that the Wodzicki residue extracts the coefficient of a given order from the expansion.

In [60], it is further shown that, after a simple change of coordinates, the integrals computing the Seeley–DeWitt coefficients in terms of Wodzicki residues in the case of the Robertson–Walker metrics can be written as periods of algebraic varieties. Periods are numbers that can be obtained by integrating an algebraic differential form on a cycle defined by algebraic equations inside an algebraic variety, see [61]. Thus, although periods themselves need not be algebraic numbers, they are in this sense numbers obtained by an algebraic procedure. The kind of numbers that can occur as periods of a given algebraic variety is closely related to the nature of the motive of the variety. The theory of motives was introduced by Grothendieck in the early 1960s as a universal cohomology theory for algebraic varieties. The case of mixed motives, which includes varieties that are not necessarily smooth and projective, was formulated by Voevodsky in terms of a triangulated category, [62]. In the case of the asymptotic expansion of the spectral action for Robertson–Walker metrics, the motives involved are complements in affine spaces of a union of a quadric hypersurface and two hyperplanes. This motive lies in the subcategory of mixed Tate motives. These are in a sense the "simplest" kind of motives, which heuristically correspond to varieties with filtrations whose graded pieces "look like" projective spaces. A way to check the nature of the motive is to compute its "universal Euler characteristic", namely the class in the Grothendieck ring of varieties. This is the ring generated by isomorphism classes of varieties with the inclusion–exclusion relations $[Y] + [X \setminus Y] = [X]$ for $Y \hookrightarrow X$ a closed embedding, and product $[X] \cdot [Y] = [X \times Y]$. The subring that corresponds to Tate motives is the polynomial ring $\mathbb{Z}[\mathbb{L}]$ generated by the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$, the class of the affine line. The motive underlying the a_{2n} term in the asymptotic expansion of the spectral action on a Robertson–Walker spacetime has Grothendieck class of the form

$$\mathbb{L}^{2n+3} - 3\mathbb{L}^{2n+2} + 2\mathbb{L}^{2n+1} - \mathbb{L}^{n+2} + 3\mathbb{L}^{n+1} - 2\mathbb{L}^n$$

This result on motives and periods in the spectral action of Robertson–Walker metrics should be compared with the situation arising in quantum field theory [63], where the Feynman integrals in the perturbative expansion in Feynman graphs of a Euclidean scalar massless field theory can be written as periods of an algebraic variety that is also given by a complement of a hypersurface obtained from the combinatorics of the graph. In the quantum field theory case, however, the integrals are in general divergent and require renormalization, and the motives cease to be mixed Tate for some sufficiently large graphs. In contrast, in the models of gravity based on the spectral action, the integrals in the terms of the asymptotic expansion are all convergent and the motives are all mixed Tate.

Another occurrence of interesting arithmetic structures in spectral action models of gravity can be seen in the computation of the asymptotic expansion of the spectral action for Bianchi IX gravitational instantons, obtained in [59,64]. The $SU(2)$-Bianchi IX metrics are homogeneous but non-isotropic spacetimes of the form

$$g = F\,(\mathrm{d}\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with a conformal factor $F \sim w_1 w_2 w_3$, and with the scaling factors $w_i = w_i(t)$, and where σ_i are the $SU(2)$-invariant 1-forms on S^3 satisfying $\mathrm{d}\sigma_i = \sigma_j \wedge \sigma_k$ for all cyclic permutations (i, j, k) of $(1, 2, 3)$. The Bianchi IX gravitational instantons are metrics of this form that satisfy the Einstein equations (with or without cosmological constant) and are self-dual. It is known from results of [65,66] that because of the high degree of symmetry of these solutions, the Einstein and self-dual equations reduce to a system of singular ODEs that are a special case of Painlevé VI equations. The solutions can then be parameterized explicitly in terms of a two-parameter (p, q) family of theta functions. See also [67,68] for a discussion of Bianchi IX cosmologies in the context of algebro-geometric models for cosmology and for a discussion of the asymptotic behavior of the Bianchi IX theta function parameterizations. The parameterization in terms of theta functions is used in [64] to compute the coefficients of the asymptotic expansion of the spectral action. It is shown that the terms $a_{2n}(p,q)$ of the asymptotic expansion for the two-parameter family of solutions are vector-valued modular forms.

The Bianchi IX gravitational instantons play a crucial role in the theory of quantum gravity and quantum cosmology, as minisuperspace models, see [69]. They are also closely related to the mixmaster spacetimes, built out of Kasner metrics, which provide very interesting models of anisotropic cosmologies with chaotic dynamical behavior near the cosmological singularity, see [70–75].

5. Multifractal cosmologies

While investigating the spectral action model of gravity on homogeneous but unisotropic cosmologies like the Bianchi IX spacetimes gives rise to the interesting arithmetic structures described above, one can also investigate spectral action models of isotropic but non-homogeneous spacetimes like the Packed Swiss Cheese Cosmologies obtained by iterating the construction of [76] of an isotropic non-homogeneous spacetime over an Apollonian packing of 3-spheres in a 4-dimensional spacetime. The resulting type of multifractal cosmologies has been proposed as a model for fractal structures in the large scale distribution of galaxies, see [77,78].

A spectral action model of the Packed Swiss Cheese Cosmologies was studied in [79]. The first step in order to obtain a model of gravity based on the spectral action functional for this type of multifractal cosmologies consists in constructing a spectral triple associated with an Apollonian packing of 3-dimensional spheres. Following the construction of [80], developed for other kinds of fractal geometries, a spectral triple is constructed using a direct sum of the data $(\mathcal{H}, D)$ of each sphere and a dense subalgebra of the algebra of continuous functions on the sphere packing. For a round 3-sphere S^3_a of radius a the zeta function of the Dirac operator is of the form

$$\zeta_{\not{D}_{S^3_a}}(s) = a^s(2\zeta(s-2,\frac{3}{2}) - \frac{1}{2}\zeta(s,\frac{3}{2}))$$

where $\zeta(s,c)$ is the Hurwitz zeta function. Thus, for an Apollonian sphere packing of 3-spheres, with radii $a_{n,k}$, where $n \in \mathbb{N}$ is the level of the packing and $k = 1,\ldots,6\cdot 5^{n-1}$ (see [81]), we have

$$\zeta_D(s) = \zeta_{\mathcal{L}}(s)\cdot \zeta_{\not{D}_{S^3}}(s)$$

where $\zeta_{\mathcal{L}}(s) = \sum_n \sum_{k=1}^{6\cdot 5^{n-1}} a^s_{n,k}$ is the zeta function of the fractal string $\mathcal{L} = \{a_{n,k}\}$, in the sense of [82]. In addition to the poles of $\zeta_{\not{D}_{S^3}}(s)$ at $s=1$ and $s=3$, now the zeta function $\zeta_{\mathcal{L}}(s)$ also has a pole at the real number σ given by the packing constant of the Apollonian packing (which is an upper bound for the Hausdorff dimension of the residual set of the packing), as well as poles off the real line. If the Apollonian packing is regular enough for the fractal string $\mathcal{L}$ to be well approximated by fractal strings with exact self-similarity, then the zeta function $\zeta_{\mathcal{L}}(s)$ and consequently the leading terms in the expansion of the spectral action $\mathcal{S}_{\Lambda,f}(D)$ of the sphere packing can be computed in terms of the case with exact self-similarity.

In the case of a fractal geometry with a single exact self-similarity, namely with a Dirac operator D such that the eigenvalues of $|D|$ grow exponentially like b^n for some $b>1$ while the spectral multiplicities also grow exponentially like a^n for some $a>1$, the leading terms in the spectral action expansion take the form

$$\mathcal{S}_{\Lambda,f}(D) \sim \Lambda^{\sigma} \sum_{m\in\mathbb{Z}} \Lambda^{\frac{2\pi \mathrm{i}}{\log b}} f_{s_m}$$

where $\sigma = \frac{\log b}{\log a}$ is the Hausdorff dimension and $s_m = \sigma + \frac{2\pi \mathrm{i} m}{\log b}$ are the poles of the zeta function, which in this case (a single exact self-similarity) are lined up periodically on the vertical line with real part σ. The log-oscillatory terms come from the contributions of these poles off the real line, with the heat kernel expansion given by (see [83])

$$\mathrm{Tr}(\mathrm{e}^{-tD^2}) \sim \frac{t^{-\frac{\log a}{2\log b}}}{2\log b} \sum_{m\in\mathbb{Z}} \Gamma(\frac{\log a}{2\log b} + \frac{\pi \mathrm{i} m}{\log b}) \exp(-\frac{\pi \mathrm{i} m}{\log b}\log t)$$

A simple example of a multifractal cosmology that has a single exact self-similarity (and non-trivial cosmic topology [39]) can be obtained by a fractal arrangement of dodecahedral spaces, see [79]. The case of the Packed Swiss Cheese Cosmologies based on Apollonian sphere packings are more complicated, because in Apollonian sphere packings do not have a single exact self-similarity and can at best be approximated by fractal strings with finitely many exact-self similarities. One can still, in such cases, use the result above as a template and obtain a form of the leading terms in the asymptotic expansion of the spectral action

$$\mathrm{Tr}(f(\mathcal{D}_{\mathcal{P}}/\Lambda)) \sim \Lambda^3 \zeta_{\mathcal{L}}(3) f_3 - \Lambda \frac{1}{4}\zeta_{\mathcal{L}}(1) f_1 + \Lambda^{\sigma} (\zeta(\sigma-2,\frac{3}{2}) - \frac{1}{4}\zeta(\sigma,\frac{3}{2}))\mathcal{R}_{\sigma} f_{\sigma} + \mathcal{S}^{\mathrm{osc}}_{\mathcal{P}}(\Lambda)$$

with σ the packing constant, $\mathcal{R}_\sigma = \mathrm{Res}_{s=\sigma}\zeta_{\mathcal{L}}(s)$ the residue of the zeta function of the fractal string $\mathcal{L}$, and $f_\beta = \int_0^\infty v^{\beta-1} f(v)\mathrm{d}v$ the momenta of the test function. The oscillatory term $\mathcal{S}^{\mathrm{osc}}_{\mathcal{P}}(\Lambda)$ is approximated by a sequence

$$\mathcal{S}^{\mathrm{osc}}_{\mathcal{P}}(\Lambda)_{\leq R} \sim \sum_{j=0}^{N_n} \Lambda^{\sigma_{n,j}} \mathfrak{f}_{\sigma_{n,j}}(\theta_n(\Lambda))$$

where $n \to \infty$ as $R \to \infty$ and with $\sigma_{n,j} = \Re(s_{n,j,m})$, for

$$\{s_{n,j,m} = \sigma_{n,j} + \mathrm{i}(\alpha_{n,j} + \frac{2\pi m}{\log b_n})\}_{j=0,\ldots,N_n, m\in\mathbb{Z}}$$

the set of non-real poles of the zeta functions $\zeta_{\mathcal{L}_n}(s)$ with exact self-similarity approximating $\zeta_{\mathcal{L}}(s)$. It is possible to obtain from the spectral action a slow-roll inflation potential $V(\phi)$, as discussed above in relation to the cosmic topology question. The shape of the slow-roll potential, which in the case of a single sphere S^3 as spatial sections is built out of two functions of the form

$$\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))\mathrm{d}u, \quad \mathcal{W}(x) = \int_0^x h(u)\mathrm{d}u$$

where h is the test function of the spectral action on the 4-dimensional spacetime, acquired an additional term in the case of the spectral action of the Packed Swiss Cheese Cosmology, which is built out of a function of the form

$$\mathcal{U}_\sigma(x) = \int_0^\infty u^{(\sigma-1)/2}(h(u+x) - h(u))\mathrm{d}u$$

with σ the packing constant, see [79]. This additional term changes the shape of the slow-roll potential and consequently changes the slow-roll parameters and the power law of the scalar and tensor fluctuation, as discussed above. Thus, one can conclude that, in a spectral action model of gravity, one will find in the slow-roll parameters detectable signatures of the presence of multifractality in the spacetime structure.

For further details on the cosmological models described in this paper, we refer the reader to [1].

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