Bianchi IX Cosmologies, Spectral Action, and Modular Forms

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Topics in Mathematical Physics

References:

• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Spectral Action for Bianchi type-IX cosmological models*, arXiv:1506.06779, J. High Energy Phys. (2015) 85, 28 pp.

• Wentao Fan, Farzad Fathizadeh, Matilde Marcolli, *Modular forms in the spectral action of Bianchi IX gravitational instantons*, arXiv:1511.05321

Based on Wentao's SURF project with Farzad

$SU(2)$-Bianchi IX cosmologies (Euclidean, compactified)

• another version of Bianchi IX mixmaster cosmologies, with $SU(2)$ symmetry (Euclidean version)

$$g = w_1 w_2 w_3 \, dt^2 + \frac{w_2 w_3}{w_1} \sigma_1^2 + \frac{w_1 w_3}{w_2} \sigma_2^2 + \frac{w_1 w_2}{w_3} \sigma_3^2$$

with $w_i = w_i(t)$, or more generally

$$g = F \, (d\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with a conformal factor $F \sim w_1 w_2 w_3$

• $SU(2)$–invariant 1-forms $\{\sigma_i\}$ satisfying relations

$$d\sigma_i = \sigma_j \wedge \sigma_k$$

for all cyclic permutations (i, j, k) of $(1, 2, 3)$

- more explicitly ds^2 is

$$w_1 w_2 w_3\, dt\, dt + \frac{w_1 w_2 \cos(\eta)}{w_3} d\phi\, d\psi + \frac{w_1 w_2 \cos(\eta)}{w_3} d\psi\, d\phi$$
$$+ \left(\frac{w_2 w_3 \sin^2(\eta)\cos^2(\psi)}{w_1} + w_1 \left(\frac{w_3 \sin^2(\eta)\sin^2(\psi)}{w_2} + \frac{w_2 \cos^2(\eta)}{w_3} \right) \right) d\phi\, d\phi$$
$$+ \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta)\sin(\psi)\cos(\psi)}{w_1 w_2} d\eta\, d\phi + \frac{\left(w_1^2 - w_2^2\right) w_3 \sin(\eta)\sin(\psi)\cos(\psi)}{w_1 w_2} d\phi\, d\eta$$
$$+ \left(\frac{w_2 w_3 \sin^2(\psi)}{w_1} + \frac{w_1 w_3 \cos^2(\psi)}{w_2} \right) d\eta\, d\eta + \frac{w_1 w_2}{w_3} d\psi\, d\psi$$

- identifying S^3 with unit quaternions $SU(2)$
- The metrics on S^3

$$\frac{w_2 w_3}{w_1}\, \sigma_1^2 + \frac{w_1 w_3}{w_2}\, \sigma_2^2 + \frac{w_1 w_2}{w_3}\, \sigma_3^2$$

are left-invariants under the action of $SU(2)$ but *not* right-invariant (unlike the round metric on S^3)

Spectral action models of gravity (modified gravity)

- Spectral triple: $(\mathcal{A}, \mathcal{H}, D)$

1. unital associative algebra $\mathcal{A}$
2. represented as bounded operators on a Hilbert space $\mathcal{H}$
3. Dirac operator: self-adjoint $D^* = D$ with compact resolvent, with bounded commutators $[D, a]$

- prototype: $(C^\infty(M), L^2(M, S), \not{D}_M)$

- extends to non smooth objects (fractals) and noncommutative (NC tori, quantum groups, NC deformations, etc.)

Action functional

- Suppose *finitely summable* $ST = (\mathcal{A}, \mathcal{H}, D)$

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) < \infty, \quad \Re(s) >> 0$$

- Spectral action (Chamseddine–Connes)

$$\mathcal{S}_{ST}(\Lambda) = \mathrm{Tr}(f(D/\Lambda)) = \sum_{\lambda \in \mathrm{Spec}(D)} \mathrm{Mult}(\lambda) f(\lambda/\Lambda)$$

f = smooth approximation to (even) cutoff

Asymptotic expansion (Chamseddine–Connes) for (almost) commutative geometries:

$$\mathrm{Tr}(f(D/\Lambda)) \sim \sum_{\beta\in\Sigma^+_{ST}} f_\beta \, \Lambda^\beta \, \int\!\!\!\!\!\!- |D|^{-\beta} \; + \; f(0)\,\zeta_D(0)$$

- Residues

$$\int\!\!\!\!\!\!- |D|^{-\beta} = \frac{1}{2}\mathrm{Res}_{s=\beta} \; \zeta_D(s)$$

- Momenta $f_\beta = \int_0^\infty f(v)\, v^{\beta-1}\, dv$
- Dimension Spectrum Σ_{ST} poles of zeta functions $\zeta_{a,D}(s) = \mathrm{Tr}(a|D|^{-s})$
- positive dimension spectrum $\Sigma^+_{ST} = \Sigma_{ST} \cap \mathbb{R}^*_+$

Zeta function and heat kernel (manifolds)

- Mellin transform

$$|D|^{-s} = \frac{1}{\Gamma(s/2)} \int_0^\infty e^{-tD^2} t^{\frac{s}{2}-1} dt$$

- heat kernel expansion

$$\mathrm{Tr}(e^{-tD^2}) = \sum_\alpha t^\alpha c_\alpha \quad \text{for} \quad t \to 0$$

- zeta function expansion

$$\zeta_D(s) = \mathrm{Tr}(|D|^{-s}) = \sum_\alpha \frac{c_\alpha}{\Gamma(s/2)(\alpha + s/2)} + \text{holomorphic}$$

- taking residues

$$\mathrm{Res}_{s=-2\alpha}\zeta_D(s) = \frac{2c_\alpha}{\Gamma(-\alpha)}$$

Pseudo-differential Calculus: (manifold case)
to obtain *full* asymptotic expansion of the Spectral Action

- Dirac operator D and pseudodifferential symbol of D^2

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

each p_k homogeneous of order k in ξ

- Cauchy integral formula

$$e^{-tD^2} = \frac{1}{2\pi i} \int_\gamma e^{-t\lambda} \, (D^2 - \lambda)^{-1} \, d\lambda$$

- Seeley de-Witt coefficients ($m = \dim M$)

$$\mathrm{Tr}(e^{-tD^2}) \sim_{t\to 0+} \; t^{-m/2} \sum_{n=0}^{\infty} a_{2n}(D^2) t^n$$

Parametrix Method

- D^2 order 2 elliptic differential operator: exists a parametrix R_λ with

$$\sigma(R_\lambda) \sim \sum_{j=0}^{\infty} r_j(x,\xi,\lambda)$$

- $r_j(x,\xi,\lambda)$ pseudodifferential symbol order $-2-j$

$$r_j(x,t\xi,t^2\lambda) = t^{-2-j} r_j(x,\xi,\lambda)$$

- R_λ approximates $(D^2-\lambda)^{-1}$ with $\sigma((D^2-\lambda)R_\lambda) \sim 1$
- recursive equation:

$$\sigma((D^2-\lambda)R_\lambda) \sim ((p_2(x,\xi)-\lambda)+p_1(x,\xi)+p_0(x,\xi))\circ\left(\sum_{j=0}^{\infty} r_j(x,\xi,\lambda)\right) \sim 1$$

- solution for R_λ constructed recursively:

$$r_0(x,\xi,\lambda) = (p_2(x,\xi)-\lambda)^{-1}$$

$$r_n(x,\xi,\lambda) = -\sum \frac{1}{\alpha!}\partial_\xi^\alpha r_j(x,\xi,\lambda)\, D_x^\alpha p_k(x,\xi)\, r_0(x,\xi,\lambda),$$

summation over all $\alpha \in \mathbb{Z}^4_{\geq 0}, j \in \{0,1,\dots,n-1\}, k \in \{0,1,2\}$, with $|\alpha| + j + 2 - k = n$

Seeley-deWitt coefficients and Parametrix Method

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i}\int\int_\gamma e^{-\lambda}\,\mathrm{tr}\,(r_{2n}(x,\xi,\lambda))\, d\lambda\, d^m\xi$$

- odd j coefficients vanish: $r_j(x,\xi,\lambda)$ odd function of ξ

Dirac operator

• orthonormal coframe $\{\theta^a\}$

$$D = \sum_a \theta^a \nabla^S_{\theta_a}$$

• spin connection ∇^S with matrix of 1-forms $\omega = (\omega^a_b)$ with

$$\nabla \theta^a = \sum_b \omega^a_b \otimes \theta^b$$

• metric-compatibility and torsion-freeness (Levi–Civita connection)

$$\omega^a_b = -\omega^b_a, \qquad d\theta^a = \sum_b \omega^a_b \wedge \theta^b$$

• Dirac operator

$$D = \sum_{a,\mu} \gamma^a dx^\mu(\theta_a) \frac{\partial}{\partial x^\mu} + \frac{1}{4} \sum_{a,b,c} \gamma^c \omega^b_{ac} \gamma^a \gamma^b$$

with $\omega^b_a = \sum_c \omega^b_{ac} \theta^c$

• matrices γ^a Clifford action of θ^a on spin bundle:
$(\gamma^a)^2 = -I$
$\gamma^a\gamma^b + \gamma^b\gamma^a = 0$ for $a \neq b$

Dirac operator on Bianchi IX metrics

- local coordinates $(x^\mu) = (t, \eta, \phi, \psi)$ with $\mathbb{S}^3$ parametrized by

$$(\eta, \phi, \psi) \mapsto \left(\cos(\eta/2) e^{i(\phi+\psi)/2}, \sin(\eta/2) e^{i(\phi-\psi)/2}\right)$$

with $0 \leq \eta \leq \pi, 0 \leq \phi < 2\pi, 0 \leq \psi < 4\pi$.

- orthonormal frame

$$\begin{aligned}
\theta^0 &= \sqrt{w_1 w_2 w_3}\, dt, \\
\theta^1 &= \sin(\eta)\cos(\psi)\sqrt{\frac{w_2 w_3}{w_1}}\, d\phi - \sin(\psi)\sqrt{\frac{w_2 w_3}{w_1}}\, d\eta, \\
\theta^2 &= \sin(\eta)\sin(\psi)\sqrt{\frac{w_1 w_3}{w_2}}\, d\phi + \cos(\psi)\sqrt{\frac{w_1 w_3}{w_2}}\, d\eta, \\
\theta^3 &= \cos(\eta)\sqrt{\frac{w_1 w_2}{w_3}}\, d\phi + \sqrt{\frac{w_1 w_2}{w_3}}\, d\psi.
\end{aligned}$$

- non-vanishing ω^b_{ac}

$$\omega^0_{11} = -\frac{w_2\left(w_1 w_3' - w_3 w_1'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega^0_{22} = -\frac{w_2\left(w_3 w_1' + w_1 w_3'\right) - w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}},$$

$$\omega^0_{33} = -\frac{w_2\left(w_3 w_1' - w_1 w_3'\right) + w_1 w_3 w_2'}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega^1_{23} = -\frac{w_1^2 w_2^2 - w_3^2\left(w_1^2 + w_2^2\right)}{2(w_1 w_2 w_3)^{3/2}},$$

$$\omega^1_{32} = -\frac{w_1^2\left(w_2^2 - w_3^2\right) + w_2^2 w_3^2}{2(w_1 w_2 w_3)^{3/2}}, \quad \omega^2_{31} = -\frac{w_2^2 w_3^2 - w_1^2\left(w_2^2 + w_3^2\right)}{2(w_1 w_2 w_3)^{3/2}}.$$

pseudo-differential symbol of Dirac

$$
\begin{aligned}
\sigma(D)(x,\xi) &= \sum_{a,\mu} i\gamma^a e_a^\mu \xi_{\mu+1} + \frac{1}{4\sqrt{w_1 w_2 w_3}}\left(\frac{w_1'}{w_1}+\frac{w_2'}{w_2}+\frac{w_3'}{w_3}\right)\gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4 \\
&= -\frac{i\gamma^2\sqrt{w_1}\left(\csc(\eta)\cos(\psi)\left(\xi_4\cos(\eta)-\xi_3\right)+\xi_2\sin(\psi)\right)}{\sqrt{w_2}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^3\sqrt{w_2}\left(\sin(\psi)\left(\xi_3\csc(\eta)-\xi_4\cot(\eta)\right)+\xi_2\cos(\psi)\right)}{\sqrt{w_1}\sqrt{w_3}} \\
&\quad + \frac{i\gamma^1\xi_1}{\sqrt{w_1}\sqrt{w_2}\sqrt{w_3}} + \frac{i\gamma^4\xi_4\sqrt{w_3}}{\sqrt{w_1}\sqrt{w_2}} \\
&\quad + \frac{1}{4\sqrt{w_1 w_2 w_3}}\left(\frac{w_1'}{w_1}+\frac{w_2'}{w_2}+\frac{w_3'}{w_3}\right)\gamma^1 \\
&\quad - \frac{\sqrt{w_1 w_2 w_3}}{4}\left(\frac{1}{w_1^2}+\frac{1}{w_2^2}+\frac{1}{w_3^2}\right)\gamma^2\gamma^3\gamma^4
\end{aligned}
$$

- with non-vanishing e_a^μ:

$$e_0^0 = \frac{1}{\sqrt{w_1 w_2 w_3}}, \qquad e_1^1 = -\frac{\sqrt{w_1}\sin(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^1 = \frac{\sqrt{w_2}\cos(\psi)}{\sqrt{w_1 w_3}}, \qquad e_1^2 = \frac{\sqrt{w_1}\csc(\eta)\cos(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^2 = \frac{\sqrt{w_2}\csc(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, \qquad e_1^3 = -\frac{\sqrt{w_1}\cot(\eta)\cos(\psi)}{\sqrt{w_2 w_3}},$$

$$e_2^3 = -\frac{\sqrt{w_2}\cot(\eta)\sin(\psi)}{\sqrt{w_1 w_3}}, \qquad e_3^3 = \frac{\sqrt{w_3}}{\sqrt{w_1 w_2}}$$

- get from the symbol the homogeneous components $p_k(x,\xi)$ with

$$\sigma(D^2)(x,\xi) = p_2(x,\xi) + p_1(x,\xi) + p_0(x,\xi)$$

- Example: for $p_0(x,\xi)$ get

$$\begin{aligned}
&\Big(-\frac{w_1'}{8w_1w_2^2} - \frac{w_1'}{8w_1w_3^2} + \frac{3w_1'}{8w_1^3} - \frac{w_2'}{8w_1^2w_2} - \frac{w_3'}{8w_1^2w_3} - \frac{w_2'}{8w_2w_3^2} \\
&\qquad + \frac{3w_2'}{8w_2^3} - \frac{w_3'}{8w_2^2w_3} + \frac{3w_3'}{8w_3^3}\Big)\gamma^1\gamma^2\gamma^3\gamma^4 + \\
&\Big(-\frac{w_1''}{4w_1^2w_2w_3} + \frac{w_1'w_2'}{8w_1^2w_2^2w_3} + \frac{w_1'w_3'}{8w_1^2w_2w_3^2} + \frac{5w_1'^2}{16w_1^3w_2w_3} - \frac{w_2''}{4w_1w_2^2w_3} \\
&\quad + \frac{w_2'w_3'}{8w_1w_2^2w_3^2} + \frac{5w_2'^2}{16w_1w_2^3w_3} - \frac{w_3''}{4w_1w_2w_3^2} + \frac{5w_3'^2}{16w_1w_2w_3^3} + \frac{w_2w_3}{16w_1^3} \\
&\qquad + \frac{w_3}{8w_1w_2} + \frac{w_1w_3}{16w_2^3} + \frac{w_2}{8w_1w_3} + \frac{w_1}{8w_2w_3} + \frac{w_1w_2}{16w_3^3}\Big)I.
\end{aligned}$$

- also manageable expression for $p_2(x,\xi)$, longer one for $p_1(x,\xi)$

Applying Parametrix Method to this Dirac operator

$$a_{2n}(x, D^2) = \frac{(2\pi)^{-m}}{2\pi i} \int \int_\gamma e^{-\lambda} \operatorname{tr}\left(r_{2n}(x, \xi, \lambda)\right) \, d\lambda \, d^m \xi$$

- Find a_0, a_2, a_4 explicitly

$$a_0(D^2) = 4 w_1 w_2 w_3$$

$$\begin{aligned} a_2(D^2) \quad = \quad & -\frac{w_1^2}{3} - \frac{w_2^2}{3} - \frac{w_3^2}{3} + \frac{w_1^2 w_2^2}{6 w_3^2} + \frac{w_1^2 w_3^2}{6 w_2^2} + \frac{w_2^2 w_3^2}{6 w_1^2} - \frac{(w_1')^2}{6 w_1^2} - \frac{(w_2')^2}{6 w_2^2} \\ & - \frac{(w_3')^2}{6 w_3^2} - \frac{w_1' w_2'}{3 w_1 w_2} - \frac{w_1' w_3'}{3 w_1 w_3} - \frac{w_2' w_3'}{3 w_2 w_3} + \frac{w_1''}{3 w_1} + \frac{w_2''}{3 w_2} + \frac{w_3''}{3 w_3}. \end{aligned}$$

and a much longer and more complicated expression for $a_4(D^2)$

Observation: all coefficients in these expressions (also for a_4) are rational numbers ... what about other terms in expansion?

A different method: Wodzicki residue

• Wodzicki residue: unique trace functional on algebra of pseudodifferential operators on smooth sections of vector bundle over smooth manifold

• classical pseudodifferential operator P_σ of order $d \in \mathbb{Z}$ local symbol

$$\sigma(x,\xi) \sim \sum_{j=0}^{\infty} \sigma_{d-j}(x,\xi) \qquad (\xi \to \infty),$$

σ_{d-j} positively homogeneous order $d-j$ in ξ

• Residue:

$$\mathrm{Res}(P_\sigma) = \int_{S^*M} \mathrm{Tr}\left(\sigma_{-m}(x,\xi)\right) d^{m-1}\xi \, d^m x,$$

$S^*M = \{(x,\xi) \in T^*M; ||\xi||_g = 1\}$ cosphere bundle

• spectral formulation of residue: pseudodifferential operator P_σ, Laplacian Δ

$$P_\sigma \mapsto \mathrm{Res}_{s=0} \mathrm{Tr}(P_\sigma \Delta^{-s})$$

same up to a constant $c_m = 2^{m+1}\pi^m$

• Mellin transform (for simplicity $\mathrm{Ker}(\Delta) = 0$):

$$\mathrm{Tr}(\Delta^{-s}) = \frac{1}{\Gamma(s)} \int_0^\infty \mathrm{Tr}(e^{-t\Delta})\, t^s \, \frac{dt}{t}$$

• heat kernel expansion

$$\mathrm{Tr}\left(e^{-t\Delta}\right) = t^{-m/2} \sum_{n=0}^{N} a_{2n} t^n + O(t^{-m/2+N+1})$$

- find for any non-negative integer $n \leq m/2 - 1$:

$$\mathrm{Res}_{s=m/2-n} \mathrm{Tr}(\Delta^{-s}) = \frac{a_{2n}(\Delta)}{\Gamma(m/2-n)},$$

- in particular

$$\mathrm{Res}_{s=1} \mathrm{Tr}(\Delta^{-s}) = a_{m-2}(\Delta)$$

- in terms of Wodicki residue:

$$a_{m-2}(\Delta) = \frac{1}{c_m} \mathrm{Res}(\Delta^{-1}) = \frac{1}{2^{m+1}\pi^m} \mathrm{Res}(\Delta^{-1})$$

applied to $\Delta = D^2$

• coefficient $a_2(D^2)$

$$a_2(D^2) = \frac{1}{c_4}\mathrm{Res}(D^{-2}) = \frac{1}{32\pi^4}\int_{S^*M} \mathrm{Tr}\left(\sigma_{-4}(D^{-2})\right)\, d^3\xi\, d^4x$$

• for other coefficients, introduce an auxiliary product space for correct counting of dimensions: use flat r-dimensional torus $\mathbb{T}^r = (\mathbb{R}/\mathbb{Z})^r$

$$\Delta = D^2 \otimes 1 + 1 \otimes \Delta_{\mathbb{T}^r},$$

$\Delta_{\mathbb{T}^r}$ flat Laplacian on $\mathbb{T}^r$

$$a_{2+r}(D^2) = \frac{1}{2^5\,\pi^{4+r/2}}\mathrm{Res}(\Delta^{-1})$$

because Künneth formula gives

$$a_{2+r}((x,x'),\Delta) = a_{2+r}(x,D^2)a_0(x',\mathbb{T}^r) = 2^{-r}\pi^{-r/2}a_{2+r}(x,D^2)$$

with volume term only non-zero heat coefficient for flat metric

- obtain for all coefficients

$$a_{2+r}(D^2) = \frac{1}{2^5 \pi^{4+r/2}} \int \mathrm{Tr}\left(\sigma_{-4-r}(\Delta^{-1})\right) \, d^{3+r}\xi \, d^4x.$$

- writing $\sigma(\Delta^{-1}) \sim \sum_{j=-2}^{-\infty} \sigma_j(x,\xi)$ inductively

$$\begin{aligned} \sigma_{-2}(x,\xi) &= p_2'(x,\xi)^{-1}, \\ \sigma_{-2-n}(x,\xi) &= -\sum \frac{1}{\alpha!} \partial_\xi^\alpha \sigma_j(x,\xi) \, D_x^\alpha p_k(x,\xi) \, \sigma_{-2}(x,\xi) \qquad (n>0), \end{aligned}$$

summation over all multi-indices non-negative integers α, $-2-n < j \leq -2, 0 \leq k \leq 2$, with $|\alpha| - j - k = n$

- setting $\zeta_{\mu+1} = \sum_\nu e^\nu_\mu \xi_{\nu+1}$ find inductively for $n \geq 2$

$$\sigma_{-2-n}(x,\xi)|_{S^*(M\times\mathbb{T}^{n-2})} = \sigma_{-2-n}(x,\xi(\zeta))|_{\zeta\in\mathbb{S}^{n+1}} = (w_1 w_2 w_3)^{-\frac{3}{2}n} P_n(\zeta)$$

polynomials $P_n(\zeta)$ coefficients functions of w_i and derivatives

- these explicitly give

$$a_{2n}(D^2) = (w_1 w_2 w_3)^{1-3n} Q_{2n}\left(w_1, w_2, w_3, w_1', w_2', w_3', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}\right)$$

with Q_{2n} polynomials with rational coefficients

$$Q_{2n} = \frac{1}{2\pi^{n+1}} \int_{\mathbb{S}^{2n+1}} \mathrm{Tr}(P_{2n}(\zeta)(\Delta^{-1}))\, d^{2n+1}\zeta$$

Question: is this rationality a sign of an arithmetic structure of Bianchi IX gravitational instantons that persists in the Spectral Action?

Bianchi IX gravitational instantons and Painlevé VI

• Euclidean Bianchi IX metrics with $SU(2)$-symmetry that are
- self-dual (Weyl curvature tensor W self-dual)
- Einstein metrics (Ricci tensor proportional to the metric)

• Self-dual equations for a Riemannian 4-manifold are PDEs; with $SU(2)$-symmetry reduce to ODEs

• This ODE is a Painlevé VI equation with

$$(\alpha, \beta, \gamma, \delta) = (\frac{1}{8}, -\frac{1}{8}, \frac{1}{8}, \frac{3}{8})$$

• N.J. Hitchin. *Twistor spaces, Einstein metrics and isomonodromic deformations*, J.Diff.Geom., Vol. 42, No. 1 (1995), 30–112.

• K.P. Tod. *Self–dual Einstein metrics from the Painlevé VI equation*, Phys. Lett. A 190 (1994), 221–224.

• S. Okumura. *The self–dual Einstein–Weyl metric and classical solutions of Painlevé VI*, Lett. in Math. Phys., 46 (1998), 219–232.

• M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337

Painlevé VI equations

• *Painlevé transcendents*: solutions of nonlinear second-order ODEs in the plane with *Painlevé property* (the only movable singularities are poles) not solvable in terms of elementary functions; classification in types

• *Painlevé VI*: 4-parameter family $(\alpha, \beta, \gamma, \delta)$

$$\frac{d^2X}{dt^2} = \frac{1}{2}\left(\frac{1}{X} + \frac{1}{X-1} + \frac{1}{X-t}\right)\left(\frac{dX}{dt}\right)^2$$

$$-\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{X-t}\right)\frac{dX}{dt} +$$

$$+\frac{X(X-1)(X-t)}{t^2(t-1)^2}\left(\alpha + \beta\frac{t}{X^2} + \gamma\frac{t-1}{(X-1)^2} + \delta\frac{t(t-1)}{(X-t)^2}\right).$$

Painlevé VI and elliptic curves

- Painlevé VI rewritten as (Fuchs)

$$t(1-t)\left[t(1-t)\frac{d^2}{dt^2}+(1-2t)\frac{d}{dt}-\frac{1}{4}\right]\int_\infty^{(X,Y)}\frac{dx}{\sqrt{x(x-1)(x-t)}}=$$

$$=\alpha Y+\beta\frac{tY}{X^2}+\gamma\frac{(t-1)Y}{(X-1)^2}+(\delta-\frac{1}{2})\frac{t(t-1)Y}{(X-t)^2}$$

where $(X,Y):=(X(t),Y(t))$ is a section
(local and/or multivalued) $P:=(X(t),Y(t))$
of the generic elliptic curve $E=E(t):\;\; Y^2=X(X-1)(X-t)$

- left-hand-side $\mu(P)$ satisfies $\mu(P+Q)=\mu(P)+\mu(Q)$ for $P+Q$ addition on the elliptic curve E (in particular $\mu(Q)=0$ for points of finite order)

- analytic description of the elliptic curve $E_\tau = \mathbb{C}/\Lambda$ with $\Lambda = \mathbb{Z} + \tau\mathbb{Z}$, with $\tau \in \mathbb{H}$
- then Painlevé VI rewritten as (Manin)

$$\frac{d^2 z}{d\tau^2} = \frac{1}{(2\pi i)^2} \sum_{j=0}^{3} \alpha_j \wp_z(z + \frac{T_j}{2}, \tau)$$

with $(\alpha_0, \ldots, \alpha_3) := (\alpha, -\beta, \gamma, \frac{1}{2} - \delta)$ and $(T_0, T_1, T_2, T_3) := (0, 1, \tau, 1 + \tau)$, and

$$\wp(z, \tau) := \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

• also have, for $e_i(\tau) = \wp(\frac{T_i}{2}, \tau)$

$$\wp_z(z,\tau)^2 = 4(\wp(z,\tau) - e_1(\tau))(\wp(z,\tau) - e_2(\tau))(\wp(z,\tau) - e_3(\tau))$$

so $e_1 + e_2 + e_3 = 0$

• a multivalued solution $z = z(\tau)$ defines a multi–section of the family, which is a covering of $\mathbb{H}$

• is know ramification and monodromy can study behavior over geodesics in $\mathbb{H}$

- Yu.I. Manin, *Sixth Painlevé equation, universal elliptic curve, and mirror of* $\mathbb{P}^2$, in "Geometry of Differential Equations", Amer. Math. Soc. Transl. (2) Vol. 186 (1998) 131–151

Theta characteristics

• explicit paremeterization of solutions for coefficients W_i of the Bianchi IX gravitational instantons (from solutions of Painlevé VI)

• theta–characteristics with parameters (p, q):

$$\vartheta[p,q](z,i\mu) := \sum_{m\in\mathbb{Z}} \exp\left(-\pi(m+p)^2\mu + 2\pi i(m+p)(z+q)\right)$$

• theta-characteristics and theta functions with vanishing characteristics

$$\vartheta[p,q](z,i\mu) = \exp\left(-\pi p^2\mu + 2\pi ipq\right)\cdot\vartheta[0,0](z+pi\mu+q,i\mu)$$

Gravitational instantons and theta characteristics

- use notation $\vartheta[p,q] := \vartheta[p,q](0,i\mu)$, and

$$\vartheta_2 := \vartheta[1/2,0], \qquad \vartheta_3 := \vartheta[0,0], \qquad \vartheta_4 := \vartheta[0,1/2]$$

- self-dual metrics

$$g = F\,(d\mu^2 + \frac{\sigma_1^2}{w_1^2} + \frac{\sigma_2^2}{w_2^2} + \frac{\sigma_3^2}{w_3^3})$$

with

$$w_1 = -\frac{i}{2}\vartheta_3\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p,q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]}, \quad w_2 = \frac{i}{2}\vartheta_2\vartheta_4 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2},q+\frac{1}{2}]}{e^{\pi i p}\vartheta[p,q]},$$

$$w_3 = -\frac{1}{2}\vartheta_2\vartheta_3 \frac{\frac{\partial}{\partial q}\vartheta[p+\frac{1}{2},q]}{\vartheta[p,q]},$$

- with non-zero cosmological constant Λ:

$$F = \frac{2}{\pi\Lambda}\frac{w_1 w_2 w_3}{(\frac{\partial}{\partial q}\log\vartheta[p,q])^2}$$

- these metrics also satisfy Einstein equation if either
  1. $\Lambda < 0$ with $p \in \mathbb{R}$ and $q \in \frac{1}{2} + i\mathbb{R}$
  2. $\Lambda > 0$ with $q \in \mathbb{R}$ and $p \in \frac{1}{2} + i\mathbb{R}$

- also case with vanishing cosmological constant:

$$w_1 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_2, \quad w_2 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_3,$$

$$w_3 = \frac{1}{\mu + q_0} + 2\frac{d}{d\mu}\log\vartheta_4, \quad F = C(\mu + q_0)^2\, w_1 w_2 w_3$$

with $q_0, C \in \mathbb{R}$, $C > 0$.

- M.V. Babich, D.A. Korotkin, *Self–dual $SU(2)$–Invariant Einstein Metrics and Modular Dependence of Theta–Functions.* Lett. Math. Phys. 46 (1998), 323–337
- Yuri Manin, Matilde Marcolli, *Symbolic Dynamics, Modular Curves, and Bianchi IX Cosmologies*, arXiv:1504.04005 [gr-qc]

Bianchi IX: time-dependent conformal perturbations

- original triaxial Bianchi IX:

$$ds^2 = w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1}\sigma_1^2 + \frac{w_3 w_1}{w_2}\sigma_2^2 + \frac{w_1 w_2}{w_3}\sigma_3^2$$

$w_i = w_i(\mu)$ cosmic time μ

- time-dependent conformal perturbation:

$$d\tilde{s}^2 = F ds^2 = F\left(w_1 w_2 w_3 \, d\mu^2 + \frac{w_2 w_3}{w_1}\sigma_1^2 + \frac{w_3 w_1}{w_2}\sigma_2^2 + \frac{w_1 w_2}{w_3}\sigma_3^2\right)$$

with $F = F(\mu)$

- effect on Dirac operator:

$$\tilde{D} = \frac{1}{\sqrt{F}} D + \frac{3F'}{4F^{\frac{3}{2}} w_1 w_2 w_3}\gamma^0$$

D Dirac operator of unperturbed Bianchi IX

- spectral action expansion for $\tilde{D}$ from heat kernel

$$\mathrm{Tr}\left(\exp(-t\tilde{D}^2)\right) \sim t^{-2}\sum_{n=0}^{\infty} \tilde{a}_{2n} t^n, \qquad t \to 0^+$$

- rationality result for coefficients of the spectral action

$$\tilde{a}_{2n} = \frac{\tilde{Q}_{2n}\left(w_1, w_2, w_3, F, w_1', w_2', w_3', F', \ldots, w_1^{(2n)}, w_2^{(2n)}, w_3^{(2n)}, F^{(2n)}\right)}{F^{2n}(w_1 w_2 w_3)^{3n-1}}$$

$\tilde{Q}_{2n}$ polynomial with rational coefficients

• zeroth coefficient: volume form (cosmological term)

$$\tilde{a}_0 = 4F^2 w_1 w_2 w_3$$

• second coefficient $\tilde{a}_2$: Einstein-Hilbert action

$$-\frac{F}{3}\left(w_1^2+w_2^2+w_3^2\right)+\frac{F}{6}\left(\frac{w_1^2 w_2^2 - w_3'^2}{w_3^2}+\frac{w_1^2 w_3^2 - w_2'^2}{w_2^2}+\frac{w_2^2 w_3^2 - w_1'^2}{w_1^2}\right)$$

$$-\frac{F}{3}\left(\frac{w_1' w_2'}{w_1 w_2}+\frac{w_1' w_3'}{w_1 w_3}+\frac{w_2' w_3'}{w_2 w_3}\right)+\frac{F}{3}\left(\frac{w_1''}{w_1}+\frac{w_2''}{w_2}+\frac{w_3''}{w_3}\right)-\frac{F'^2}{2F}+F''$$

• much longer and more complicated explicit formula for $\tilde{a}_4$ (Weyl conformal gravity and Gauss-Bonnet gravity)

Gravitational Instantons

- now assuming conformally perturbed Bianchi IX is self-dual Einstin metric and use parameterization by theta functions
- two-parameter family with non-vanishing cosmological constant:

$$w_1[p,q](i\mu) = -\frac{i}{2}\vartheta_3(i\mu)\vartheta_4(i\mu)\frac{\partial_q \vartheta[p, q+\frac{1}{2}](i\mu)}{e^{\pi i p}\vartheta[p,q](i\mu)}$$

$$w_2[p,q](i\mu) = \frac{i}{2}\vartheta_2(i\mu)\vartheta_4(i\mu)\frac{\partial_q \vartheta[p+\frac{1}{2}, q+\frac{1}{2}](i\mu)}{e^{\pi i p}\vartheta[p,q](i\mu)}$$

$$w_3[p,q](i\mu) = -\frac{1}{2}\vartheta_2(i\mu)\vartheta_3(i\mu)\frac{\partial_q \vartheta[p+\frac{1}{2}, q](i\mu)}{\vartheta[p,q](i\mu)}$$

$$F[p,q](i\mu) = \frac{2}{\pi\Lambda}\frac{1}{(\partial_q \ln \vartheta[p,q](i\mu))^2} = \frac{2}{\pi\Lambda}\left(\frac{\vartheta[p,q](i\mu)}{\partial_q\vartheta[p,q](i\mu)}\right)^2$$

- one-parameter family with vanishing cosmological constant:

$$\begin{aligned} w_1[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_2(i\mu), \\ w_2[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_3(i\mu), \\ w_3[q_0](i\mu) &= \frac{1}{\mu+q_0} + 2\frac{d}{d\mu}\log\vartheta_4(i\mu), \\ F[q_0](i\mu) &= C(\mu+q_0)^2, \end{aligned}$$

C arbitrary positive constant

Modularity

• generators of the modular group $\mathrm{PSL}_2(\mathbb{Z})$

$$T_1(\tau) = \tau + 1, \qquad S(\tau) = \frac{-1}{\tau}, \qquad \tau \in \mathbb{H}$$

• using behavior of theta functions and derivatives under modular transformations (two-parameter family):

$$\begin{aligned}
w_1[p,q](i\mu+1) &= w_1[p,q+p+\tfrac{1}{2}](i\mu), & w_1^{(n)}[p,q](i\mu+1) &= w_1^{(n)}[p,q+p+\tfrac{1}{2}](i\mu),\\
w_2[p,q](i\mu+1) &= w_3[p,q+p+\tfrac{1}{2}](i\mu), & w_2^{(n)}[p,q](i\mu+1) &= w_3^{(n)}[p,q+p+\tfrac{1}{2}](i\mu),\\
w_3[p,q](i\mu+1) &= w_2[p,q+p+\tfrac{1}{2}](i\mu), & w_3^{(n)}[p,q](i\mu+1) &= w_2^{(n)}[p,q+p+\tfrac{1}{2}](i\mu).
\end{aligned}$$

- for μ with $\Re(\mu) > 0$:

$$
\begin{aligned}
w_3[p,q](\frac{i}{\mu}) &= -\mu^2 w_1[-q,p](i\mu), \\
w_3'[p,q](\frac{i}{\mu}) &= \mu^4 w_1'[-q,p](i\mu) + 2\mu^3 w_1[-q,p](i\mu), \\
w_3''[p,q](\frac{i}{\mu}) &= -\mu^6 w_1''[-q,p](i\mu) - 6\mu^5 w_1'[-q,p](i\mu) - 6\mu^4 w_1[-q,p](i\mu), \\
w_3^{(3)}[p,q](\frac{i}{\mu}) &= \mu^8 w_1^{(3)}[-q,p](i\mu) + 12\mu^7 w_1''[-q,p](i\mu) + 36\mu^6 w_1'[-q,p](i\mu) \\
&\quad +24\mu^5 w_1[-q,p](i\mu), \\
w_3^{(4)}[p,q](\frac{i}{\mu}) &= -\mu^{10} w_1^{(4)}[-q,p](i\mu) - 20\mu^9 w_1^{(3)}[-q,p](i\mu) - 120\mu^8 w_1''[-q,p](i\mu) \\
&\quad -240\mu^7 w_1'[-q,p](i\mu) - 120\mu^6 w_1[-q,p](i\mu).
\end{aligned}
$$

- similar results for w_2 and w_3 under modular generator S

• conformal factor:

$$\begin{aligned} F[p,q](i\mu+1) &= F[p,q+p+\frac{1}{2}](i\mu), \\ F^{(n)}[p,q](i\mu+1) &= F^{(n)}[p,q+p+\frac{1}{2}](i\mu). \end{aligned}$$

$$\begin{aligned} F[p,q](\frac{i}{\mu}) &= -\mu^{-2}F[-q,p](i\mu), \\ F'[p,q](\frac{i}{\mu}) &= F'[-q,p](i\mu) - 2\mu^{-1}F[-q,p](i\mu), \\ F''[p,q](\frac{i}{\mu}) &= -\mu^{2}F''[-q,p](i\mu) + 2\mu F'[-q,p](i\mu) - 2F[-q,p](i\mu), \\ F^{(3)}[p,q](\frac{i}{\mu}) &= \mu^{4}F^{(3)}[-q,p](i\mu), \\ F^{(4)}[p,q](\frac{i}{\mu}) &= -\mu^{6}F^{(4)}[-q,p](i\mu) - 4\mu^{5}F^{(3)}[-q,p](i\mu). \end{aligned}$$

• similar results for the case of the one-parameter family with vanishing cosmological constant

• modularity of spectral action coefficients:

$$\tilde{a}_0[p,q](i\mu+1) = \tilde{a}_0[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_2[p,q](i\mu+1) = a_2[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_4[p,q](i\mu+1) = \tilde{a}_4[p,q+p+\frac{1}{2}](i\mu)$$

$$\tilde{a}_0[p,q](\frac{i}{\mu}) = -\mu^2 \tilde{a}_0[-q,p](i\mu)$$

$$\tilde{a}_2[p,q](\frac{i}{\mu}) = -\mu^2 \tilde{a}_2[-q,p](i\mu)$$

$$\tilde{a}_4[p,q](\frac{i}{\mu}) = -\mu^2 \tilde{a}_4[-q,p](i\mu)$$

Modularity of remaining coefficients $\tilde{a}_{2n}$

- Dirac operators $\tilde{D}^2[p,q]$, $\tilde{D}^2[p,q+p+\frac{1}{2}]$ and $\tilde{D}^2[-q,p]$ are isospectral

- heat kernel $K_t[p,q]$ of $\exp\left(-t\tilde{D}^2[p,q]\right)$ in terms of eigenvalues and eigenspinors $\Rightarrow$ modularity

$$\begin{aligned} K_t[p,q](i\mu_1+1, i\mu_2+1) &= K_t[p,q+p+\frac{1}{2}](i\mu_1, i\mu_2), \\ K_t[p,q](-\frac{1}{i\mu_1}, -\frac{1}{i\mu_2}) &= (i\mu_2)^2 K_t[-q,p](i\mu_1, i\mu_2). \end{aligned}$$

- then modularity of coefficients $\tilde{a}_{2n}$:

$$\begin{aligned} \tilde{a}_{2n}[p,q](i\mu+1) &= \tilde{a}_{2n}[p,q+p+\frac{1}{2}](i\mu), \\ \tilde{a}_{2n}[p,q](\frac{i}{\mu}) &= (i\mu)^2 \tilde{a}_{2n}[-q,p](i\mu). \end{aligned}$$

Vector valued modular forms

- coefficients satisfy:

$$\tilde{a}_{2n}[p+1,q] = \tilde{a}_{2n}[p,q+1] = \tilde{a}_{2n}[p,q],$$

- $\mathrm{PSL}_2(\mathbb{Z})$ action on $(p,q) \in \mathbb{R}/\mathbb{Z}^2$:

$$\begin{aligned} \tilde{S}(p,q) &= (-q,p) \\ \tilde{T}_1(p,q) &= (p, q+p+\frac{1}{2}) \end{aligned}$$

finite orbits $\mathcal{O}_{(p,q)}$ on rationals

- $\tilde{a}_{2n}[p',q'](i\mu)$, with $(p',q') \in \mathcal{O}_{(p,q)}$, vector-valued modular form of weight 2 for the modular group $\mathrm{PSL}_2(\mathbb{Z})$

- summing over orbits:

$$\tilde{a}_{2n}(i\mu;\mathcal{O}_{(p,q)}) = \sum_{(p',q')\in\mathcal{O}_{(p,q)}} \tilde{a}_{2n}[p',q'](i\mu)$$

is an ordinary modular form of weight 2 for $\mathrm{PSL}_2(\mathbb{Z})$

- Question: which modular form is it?
- analyze zeros and poles structure to find out
  - Example: for all n, modular form $\tilde{a}_{2n}(i\mu;\mathcal{O}_{(0,\frac{1}{3})})$ in one-dimensional space spanned by

$$\frac{G_{14}(i\mu)}{\Delta(i\mu)},$$

    with Δ modular discriminant (cusp form weight 12) and G_{14} is Eisenstein series weight 14
  - Example: for all n, modular form $\tilde{a}_{2n}(i\mu;\mathcal{O}_{(\frac{1}{6},\frac{5}{6})})$ in one-dimensional space spanned by

$$\frac{\Delta(i\mu)G_6(i\mu)}{G_4(i\mu)^4}$$