Solitons

shock waves

ex: Burger's equation (inviscid fluid dyn) $\quad u_t + u u_x = 0 \quad (*)$

$u(c(t), t) \qquad x = c(t)$ (a characteristic curve)

$$\frac{d}{dt} u(c(t), t) = u_x(c(t), t)\, c'(t) + u_t(c(t), t)$$

if $c(t) = f(x_0)t + x_0$ then $\quad u_x(f(x_0)t + x_0, t) f(x_0) + u_t(f(x_0)t + x_0, t)$

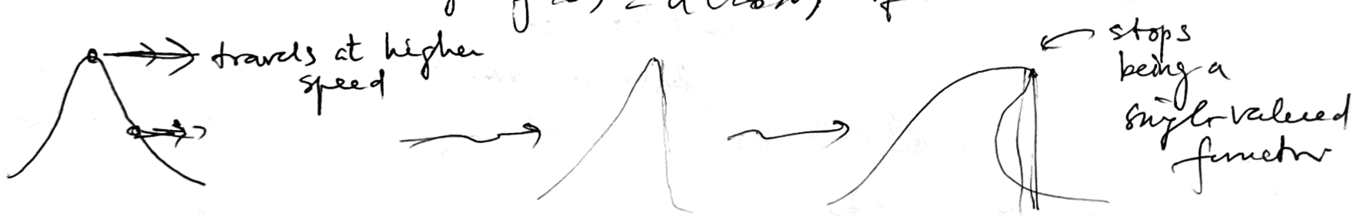
(at $t = 0$ $(x_0, 0)$ with $u(x_0, 0) = f(x_0)$) $\qquad$ at $t = 0$

$$u_x(x_0, 0) f(x_0) + u_t(x_0, 0)$$
$$= \underbrace{u_x(x_0, 0)\, u(x_0, 0) + u_t(x_0, 0) = 0}_{\text{as } u \text{ solution of } (*)}$$

$\frac{d}{dt} u(c(t), t) = 0$

value of u constant along char. curve

in this case a wave traveling at speed equal to initial height $f(x_0) = u(x_0, 0)$ of wave



KdV equation again

$$u_t = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx} \quad (**)$$

Note: similarity to Burger's eqn nonlin

$u_t + u u_x = 0$ where have shock waves

also similarity to lin eq. $u_t = u_{xxx}$

that has sol'ns

$$u_k(x, t) = \cos(kx - k^3 t)$$
$$= \cos(k(x - k^2 t))$$

dispersive equation where speed of traveling wave depends on frequency k

linear combin's w/ different frequencies separate out in time

(3) but surprisingly KdV has family of solutions that avoid both problems

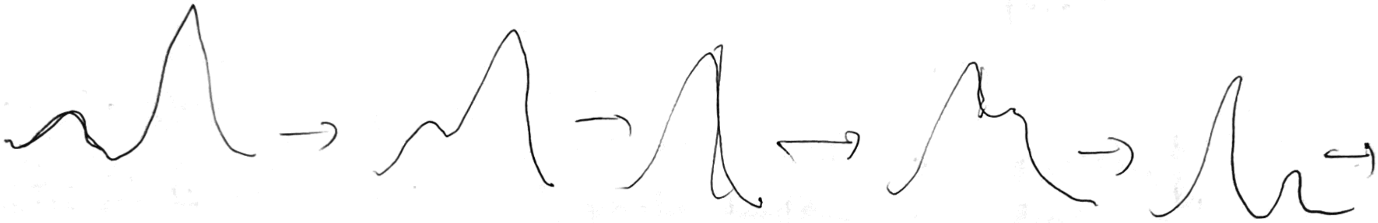
$$u_k(x,t) = \frac{8k^2}{\left(e^{kx+k^3t} + e^{-kx-k^3t}\right)^2}$$

solitary wave traveling at speed k^2 and height $2k^2$

1-soliton ↗

n-soliton solution

example $$u(x,t) = \frac{24\left(e^{2x+2t} + 6e^{6x+18t} + 4e^{4x+16t} + 4e^{8x+20t} + e^{10x+34t}\right)}{\left(1+3e^{2x+2t} + e^{6x+18t} + 3e^{4x+16t}\right)^2}$$



two interacting waves ... scattering theory of solitons
how multiple waves interact
and bounce off each other
(replaces the simpler superposition of the linear case)

Weierstrass $\wp$-function

$y^2 = 4x^3 - k_1 x - k_2$ ell. curve $(k_1, k_2) \longleftrightarrow \Lambda$

$\wp(z; \omega_1, \omega_2)$ $\Lambda = \omega_1\mathbb{Z} + \omega_2\mathbb{Z}$

$$\wp(z;\Lambda) = \frac{1}{z^2} + \sum_{\lambda\in\Lambda\setminus\{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2}\right)$$

nonlin ODE

$(w')^2 = 4w^3 - k_1 w - k_2$ parameterization of the elliptic curve

has sol'n $w(z) = \wp(z+r; \Lambda(k_1,k_2))$ constant of integration

KdV travelling waves $\quad u_t = \frac{3}{2} u u_x + \frac{1}{4} u_{xxx}$

$u(x,t) = w(x+ct)$

$$cw' = \frac{3}{2} w w' + \frac{1}{4} w'''$$

integrate

$$cw = \frac{3}{4} w^2 + \frac{1}{4} w'' + \gamma_1$$

mult. by w' & integrate

$$\frac{c}{2} w^2 = \frac{1}{4} w^3 + \frac{1}{8} (w')^2 + \gamma_1 w + \gamma_2$$

$$\Rightarrow \quad (w')^2 = -2w^3 + 4cw^2 - 8\gamma_1 w - 8\gamma_2$$

$$k_1 = \frac{4}{3}(c^2 - 3\gamma_1)$$

$$k_2 = \frac{8c^3}{27} - \frac{4c\gamma_1}{3} - 2\gamma_2$$

$$\Rightarrow u(x,t) = -2\wp(x+ct+\omega, \Lambda_{k_1,k_2}) + 2c/3$$

Solution of KdV for <u>any</u> <u>choice of ω, k_1, k_2, c</u>

= travelling waves of any speed and profile $\wp$ of an arbitrary ell. curve

(periodic)

<u>More general algebro-geometric picture:</u>

choice of any algebraic curve C (not only elliptic curve) $\rightsquigarrow$ $J(C)$ Jacobian $\cong H^1(C,\mathcal{O})/H^1(C,\mathbb{Z})$

(for elliptic curve $J(E) \cong E$)

+ choice of pt. in Jacobian $\Rightarrow$ Solution of KdV if C superelliptic

Solution of KP otherwise

(5) [illegible]

1) continuous at [illegible]

2) lim [illegible]

3) [illegible]-exponential form

[illegible]

[illegible]

3) for large [illegible], graph of [illegible] has a small [illegible]

[illegible]

[illegible] such that

[illegible]

solution

[illegible]

[illegible]

(5) n - solitons 1) $u(x,t)$ sol'n of KdV

2) continuous in x,t

3) $\lim_{x\to\pm\infty} u(x,t) = 0 \quad \forall t$

4) "rational-exponential" form

$$u(x,t) = \frac{\sum_{i=1}^{m} c_i e^{a_i x + b_i t}}{\sum_{j=1}^{n} \tilde{c}_j e^{\alpha_j x + \beta_j t}}$$

$m,n \in \mathbb{N}$

$a_i, b_i, \alpha_j, \beta_j \in \mathbb{R}$

5) for large $|t|$ graph of $u(x,t)$ has n local maxima

τ function

$$\tau(x,t) = \sum_{i=1}^{m} c_i e^{a_i x + b_i t} \quad \text{such that}$$

$$u(x,t) = 2\partial_x^2 \log \tau(x,t) = \frac{2\tau\tau_{xx} - 2\tau_x^2}{\tau^2}$$

is solution

$$u(x,t) = 2\partial_x^2 \log\left(e^{kx+k^3t} + e^{-kx-k^3t}\right)$$

$$= 2\partial_x^2 \frac{k e^{kx+k^3t} - k e^{-kx-k^3t}}{e^{kx+k^3t} + e^{-kx-k^3t}}$$

$$= \frac{8k^2}{\left(e^{kx+k^3t} + e^{-kx-k^3t}\right)^2}$$

[illegible]

[illegible]

[illegible]

$$[illegible]$$

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

[illegible]

τ-functions for n-soliton solutions

choose $\{\varepsilon_i\}$ $\{k_i\}$ $i=1,\dots,n$ real constants

$n\times n$ matrix M

$$M_{ij} = \frac{\partial^{i-1}}{\partial x^{i-1}}\left(e^{k_j x + k_j^3 t} + \varepsilon_j e^{-k_j x - k_j^3 t}\right)$$

Wronskian matrix $\overbrace{\qquad}$ $W(\underline{\phi})$

$$W(\underline{f}) = \begin{pmatrix} f_1 & \cdots & f_n \\ f_1' & \cdots & f_n' \\ \vdots & & \\ f_1^{(n-1)} & \cdots & f_n^{(n-1)} \end{pmatrix} \qquad \phi_j = e^{k_j x + k_j^3 t} + \varepsilon_j e^{-k_j x - k_j^3 t}$$

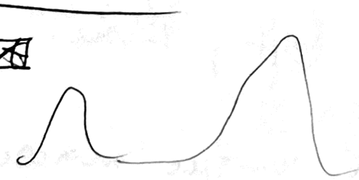
$\det W(\underline{f}) = 0$ iff some lin combin of f_i
$\sum_i a_i f_i = 0$

if $\tau(x,t) = \det M \neq 0$

then $u(x,t) = 2\partial_x^2 \log \tau(x,t)$

sol'n of KdV n-soliton sol'

2-soliton soln $\boxtimes$

for $|t|$ large $\rightsquigarrow$  two peaks separated

each like a 1-soliton

surface over (x,t) plane

scattering of one wave by other

incoming/outgoing waves

(7) Consider again Weyl algebra of differential operators
now not nec. polyn. coeff's
instead of $W_n(\mathbb{R}[\underline{x}])$ take $W_n(C^\infty(\Omega))$ $\Omega \in \mathbb{R}^n$

D diff op in one-var x <u>linear</u>

$f(x)$ in $\mathrm{Ker}(D)$ iff

$D = Q \circ (\partial - \frac{f'}{f})$

because $(\partial - \frac{f'}{f}) f = f' - \frac{f'}{f} \cdot f = 0$

so if $D = Q \circ (\partial - \frac{f'}{f})$ then $f \in \mathrm{Ker}(D)$

conversely; if in dim arbitrary

$V \subset \mathrm{Ker}(D)$ k-dim subspace of $\mathrm{Ker}(D)$

B basis for V

$\Rightarrow D = Q \circ R$ ← quotient produced from $B = \{\phi_1, \dots, \phi_n\}$

diff op 1-variable $R(f) = \dfrac{W(\phi_1, \dots, \phi_n, f)}{W(\phi_1, \dots, \phi_n)}$

R unique monic nth order OD_B w/ Kernel $= V(B)$ span of B

$B' \cup B$ basis of Ker D

$Q = $ (leading coeff of D) $\cdot$ unique monic op w/ Ker $=$ Image of B' under R then if $f \in \ker(D) \Rightarrow f$ also in $\ker(D - Q \circ K)$

$$\underbrace{(D - Q \circ K)}(f) = D(f) - Q \circ K(f)$$

$$= Q(\underbrace{K(f)}_{\text{either } 0 \text{ or}}) = 0$$

Strictly less order than D
(as Q has leading ord term of D)
so ker should be smaller $\leadsto D - Q \circ K \equiv 0$

Isospectrality: matrices $n \times n$

C with $Cv = \lambda v$

A, M with $AC = MA$

$\Rightarrow w = Av$ satisfies $Mw = \lambda w$

} $M = ACA^{-1}$ if A invertible obvious but true more generally

So C & M have same spectrum

$M(t) \underset{A(t)}{=} A(t)\, C$ continuous isosp. deform

if invertible: $M(t) = A(t)\, C\, A(t)^{-1}$

$B = \dot{A}(t)\, A^{-1}(t)$ $\qquad \boxed{\dot{M} = [B, M]}$

from $\dot{M} = \frac{d}{dt}(ACA^{-1})$

For differential operators: D_0 ordinary linear diff op

V fin dim invariant subspace $D_0 V \subset V$

(a) K unique monic diff op
w/ kernel $= V$
$\exists$ $\mathcal{D}$ ordinary diff op

$$K \circ \mathcal{D}_0 = \mathcal{D} \circ K$$

$\forall\, \varphi_0$ eigenfunction $\mathcal{D}_0 \varphi_0 = \lambda_0 \varphi_0$

$\varphi_0 \notin V \Rightarrow \varphi = K(\varphi_0)$ eigenf of $\mathcal{D}$ w/ λ_0 eigenvalue

for $\phi \in V$ $\quad K \circ \underbrace{\mathcal{D}_0(\phi)}_{\in\, V = \ker(K)} = 0$

so $K \circ \mathcal{D}_0 = \underset{\mathcal{D}}{\varphi} \circ K$ factors

$$\mathcal{D}(\varphi) = \mathcal{D}(K(\varphi_0))$$
$$= K\, \mathcal{D}_0 \varphi_0 = \lambda_0 K(\varphi_0)$$

<u>example</u>: $\mathcal{D}_0 = \partial^2 + 2\partial + 1$

$V = \operatorname{span}(x^2(1+x)e^x, (1+3x)e^x)$ inv. subsp.

$$\rightsquigarrow K = \partial^2 - \frac{6x^3+15x^2+8x+1}{3x^3+3x^2+x}\partial + \frac{3x^3+12x^2+16x+4}{3x^3+3x^2+x}$$

monic ord diff op deg 2 with V as Kernel

$$= \partial^2 + q(x)\partial + c_0(x)$$

look for $\mathcal{D} = \partial^2 + \alpha(x)\partial + \beta(x)$ s.t.

$$K \circ \mathcal{D}_0 = \mathcal{D} \circ K$$

comparing terms of same order

$\alpha(x) q(x) = -2q(x)$
$1 + c_0(x) - 2q(x) =$
$c_0(x) + \beta(x) - 2q(x) + 2q'(x)$

$$\begin{cases} \beta(x) = 1 - 2q'(x) \\ \alpha(x) = -2 \end{cases}$$

Examples: $D_0 = \partial^2$ $\quad D_0\psi_0 = 4\psi_0$

$\psi_0 = e^{2x}$

$V = \mathrm{span}\{x+t^2,\ e^{x+t}+e^{-x}\}$ $\Rightarrow$ $K = \partial^2 + \dfrac{(e^{t+2x}+1)(t^2+x)}{-e^{t+2x}(t^2+x-1)+t^2+x+1}\,\partial - \dfrac{e^{t+2x}+1}{-e^{t+2x}(t^2+x-1)+t^2+x+1}$

$$D = \partial^2 - \frac{2(4t^4e^{t+2x}+8t^2xe^{t+2x}+4x^2e^{t+2x}+2e^{t+2x}+e^{2t+4x}+1)}{(t^2e^{t+2x}-t^2-e^{t+2x}+xe^{t+2x}-x-1)^2}$$

$$\psi(x,t) = -\frac{e^{2x}\left(e^{t+2x}(2t^2+2x-3)-3(2t^2+2x+1)\right)}{-e^{t+2x}(t^2+x-1)+t^2+x+1}$$

$D\psi = 4\psi$

Lax form of KdV and other soliton equations

operator $H = \partial^2 + u(x,t)$ (Hamilt. in Schrödinger eq (as instationary pert. w/ potential $u(x,t)$ & time-dependent))

- if $u(x,t)$ is a ~~pure~~ KdV n-soliton

H ~~depend~~ isospectral when t-variable changes

Gardner-Greene-Kruskal-Miura

- this observation combined w/ techniques for reconstruction of ~~spec~~ potential from spectral data ⇝ inverse scattering technique for solitons solutions

a related approach: Lax pair & Lax equation

(11)

$$\mathcal{D} = \sum_i c_i(x,t)\, \partial_x^i$$

differentiation only in x

coeffs dep on t-param.

ordinary diff op

$$\dot{\mathcal{D}} = \sum \frac{\partial c_i(x,t)}{\partial t} \partial_x^i$$

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if $\mathcal{D} = \partial^2 + u(x,t)$

$$M = \partial^3 + \frac{3}{2} u(x,t)\partial + \frac{3}{4} u_x(x,t)$$

then u solution of KdV iff $\mathcal{D}, M$ satisfy

$$\dot{\mathcal{D}} = [M, \mathcal{D}]$$

Lax equation

$$M \circ \mathcal{D} = \partial^5 + \frac{5}{2} u\partial^3 + \frac{15}{4} u_x \partial^2 + \frac{3}{2}(u^2 + 2u_{xx})\partial + \frac{9}{4} uu_x + u_{xxx}$$

Just $u_t(x,t)$ as ∂^2 constant coeff in $\mathcal{D}$

$$\mathcal{D} \circ M = \partial^5 + \frac{5}{2} u\partial^3 + \frac{15}{4} u_x \partial^2 + \frac{3}{2}(u^2 + 2u_{xx})\partial + \frac{3}{4}(uu_x + u_{xxx})$$

$$\Rightarrow \quad M \circ \mathcal{D} - \mathcal{D} \circ M = \frac{3}{2} uu_x + \frac{1}{4} u_{xxx}$$

KdV

if a PDE can be written in this form it has a Lax form w/ $\mathcal{D}, M$ its Lax pair

$\longrightarrow$ KdV has a Lax form

Use this to compute solutions of KdV!

example: $V = \mathrm{span}(x, x^3+6t)$

$D_0 = \partial^2$ $\quad D_0 x = 0$ $\quad$ so $D_0 V \subset V$

$D_0(x^3+6t) = 6x$

$K = \partial^2 - \frac{3x^3}{x^3-3t}\partial + \frac{3x}{x^3-3t}$ $\quad$ monic second order ordinary K with x, x^3+6t in Kernel

Then want $D = \partial^2 + u(x,t)$

s.t. $K \circ D_0 = D \circ K$ $\quad$ find $u(x,t)$

computing 2nd order terms to match

l.h.s. $= \frac{-3x}{x^3-3t}$ $\quad$ r.h.s.: $\frac{9x(x^3-3t)}{(x^3-3t)^2} + u(x,t)$

$$\Rightarrow u(x,t) = -\frac{6(6xt+x^4)}{(x^3-3t)^2}$$