Noncommutative Geometry, Quantum Fields, and Motives: a bird eye view

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Lecture 1: Tuesday September 30, 2008

General information about the class:

The material covered in this class is mostly based on (the first chapter of)

- Alain Connes and Matilde Marcolli, *Noncommutative Geometry, Quantum Fields, and Motives*, Colloquium Publications, Vol.55, American Mathematical Society, 2008.

Other reading material will be distributed in class and listed on the course webpage, along with notes of the lectures.

Course webpage:

`http://www.its.caltech.edu/ matilde/course2008.html`

Other information:

Office hours: by appointment

Research Seminar: Meets weekly
(time to be assigned)

Perturbative renormalization in Quantum Field Theory

Action $S(A) = \int \mathcal{L}(A)\, d^D x$,
Lagrangian $\mathcal{L}\,(A) = \frac{1}{2}(\partial A)^2 - \frac{m^2}{2}\, A^2 - \mathcal{L}_{\text{int}}(A)$

$$\langle \mathcal{O} \rangle = \mathcal{N} \int \mathcal{O}(A)\, e^{i\,\frac{S(A)}{\hbar}}\, [dA]$$

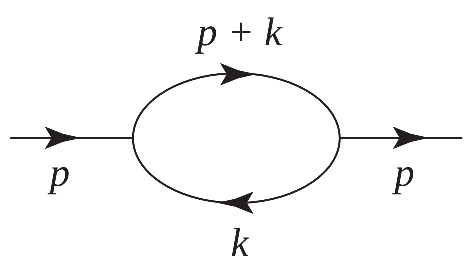
$$S_{eff}(A) = S_0(A) + \sum_{\Gamma \in 1PI} \frac{\Gamma(A)}{\sigma(\Gamma)}$$

$$\Gamma(A) = \frac{1}{N!} \int_{\sum p_j = 0} \widehat{A}(p_1) \ldots \widehat{A}(p_N)\, U(\Gamma(p_1, \ldots, p_N))\, dp_1 \ldots dp_N$$

Regularization and Renormalization

$$\int \frac{1}{k^2 + m^2} \frac{1}{((p+k)^2 + m^2)}\, d^D k$$

ϕ^3-theory $D = 4$ divergent



Dimensional Regularization:

$$\int e^{-\lambda\, q^2}\, d^D q \;=\; \pi^{D/2}\, \lambda^{-D/2}$$

Hopf algebras and quantum field theory (Connes–Kreimer theory)

BPHZ renormalization:

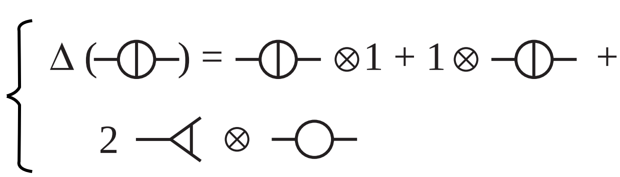
$$\overline{R}(\Gamma) = U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

$$C(\Gamma) = -T(\overline{R}(\Gamma)) = -T\left(U(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma) \right)$$

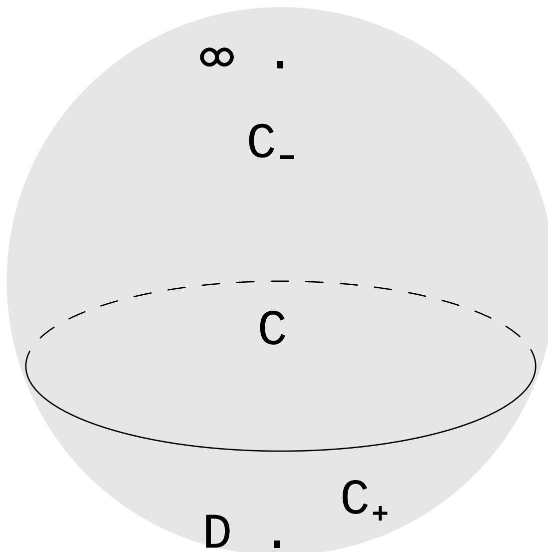
$$R(\Gamma) = \overline{R}(\Gamma) + C(\Gamma) = U(\Gamma) + C(\Gamma) + \sum_{\gamma \subset \Gamma} C(\gamma) U(\Gamma/\gamma)$$

Hopf algebra of Feynman graphs:

$$\Delta(\Gamma) = \sum_{\gamma \subset \Gamma} \gamma \otimes \Gamma/\gamma$$



The BPHZ renormalization and Birkhoff factorization of loops (Connes–Kreimer)



$$\gamma(z) = \gamma_-(z)^{-1}\,\gamma_+(z),$$

Δ = small disk, $\Delta^* = \Delta \setminus \{0\}$, $z \in \Delta^*$

Dimensional regularization $z \in \mathbb{C}^*$

$$\gamma(z) \Leftrightarrow U(\Gamma), \quad \gamma_-(z) \Leftrightarrow C(\Gamma), \quad \gamma_+(z) \Leftrightarrow R(\Gamma)$$

Summary of the Connes–Kreimer theory

• $\mathcal{H}$ dual to affine group scheme G (diffeographisms)

• $G(\mathbb{C})$ pro-unipotent Lie group $\Rightarrow$

$$\gamma(z) = \gamma_-(z)^{-1}\gamma_+(z)$$

Birkhoff factorization of loops exists

• Recursive formula for Birkhoff = BPHZ

• loop = $\phi \in \mathrm{Hom}(\mathcal{H}, \mathbb{C}(\{z\}))$
(germs of meromorphic functions)

• Feynman integral $U(\Gamma) = \phi(\Gamma)$
counterterms $C(\Gamma) = \phi_-(\Gamma)$
renormalized value $R(\Gamma) = \phi_+(\Gamma)|_{z=0}$

Introducing motives

• *pure* motives: cutting out pieces of algebraic varietes $h^i(X)$

$$\mathrm{Hom}((X,p,m),(Y,q,n)) = q\,\mathrm{Corr}^{m-n}_{/\sim}(X,Y)\,p$$

$p^2 = p$, $q^2 = q$, $\mathbb{Q}(m) =$ Tate motives

$$\omega : \mathcal{M}_{\mathbb{K}} \to Vect_{\mathbb{Q}} \quad X \mapsto H_B(X,\mathbb{Q})$$

Motivic Galois groups: (Tate $G = \mathbb{G}_m$)

• A much more complicated story for *mixed* motives

• Periods: $\int_X \omega$

Feynman integrals $\overset{??}{\rightsquigarrow}$ Multiple zeta values

Main question:

• Why periods of motives occur in quantum field theory?

A main open problem:

• Are these periods coming from QFT always periods of *mixed Tate motives*?

Very special motives, but very general varieties X_Γ

Two approaches:

• Bottom-up approach: for each graph Γ show that the part of the cohomology

$$H^{n-1}(\mathbb{P}^{n-1} \smallsetminus X_\Gamma, \Sigma_n \smallsetminus (\Sigma_n \cap X_\Gamma))$$

involved in the Feynman integral computation is a realization of a mixed Tate motive.

• Top-down approach: show that there is an equivalence of categories between a category of mixed Tate motives and one that encodes the Feynman diagrams computations.

Bottom-up: Feynman motives and their periods (Bloch-Esnault-Kreimer)

Feynman trick:

$$\frac{1}{ab} = \int_0^1 \frac{dt}{(ta+(1-t)b)^2}$$

More generally: integral on a simplex

Feynman rules for graph $\Gamma \Rightarrow$

$$\int_\Sigma \frac{dv}{\Psi_\Gamma^2}$$

$\Psi_\Gamma =$ graph polynomial

Graph hypersurface:

$$X_\Gamma = \{t \in \mathbb{P}^N \,:\, \Psi_\Gamma(t) = 0\}$$

Cohomology of $\mathbb{P}^N \smallsetminus X_\Gamma$

A simple example: Banana graphs (Aluffi-M.)

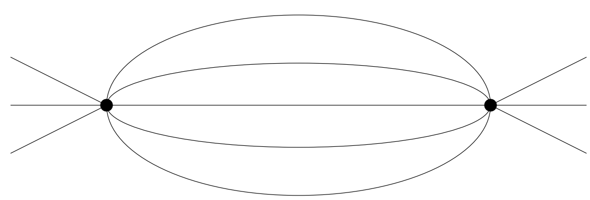
Class in the Grothendieck group

$$[X_{\Gamma_n}] = \frac{\mathbb{L}^n - 1}{\mathbb{L} - 1} - \frac{(\mathbb{L}-1)^n - (-1)^n}{\mathbb{L}} - n\,(\mathbb{L}-1)^{n-2}$$

where $\mathbb{L} = [\mathbb{A}^1] \in K_0(\mathcal{V})$ Lefschetz motive

$$\int_{\sigma_n} \frac{(t_1 \cdots t_n)^{(\frac{D}{2}-1)(n-1)-1}\,\omega_n}{\Psi_\Gamma(t)^{(\frac{D}{2}-1)n}}$$

$$\Psi_\Gamma(t) = t_1 \cdots t_n (\frac{1}{t_1} + \cdots + \frac{1}{t_n})$$



More information on X_Γ from other invariants e.g. Characteristic classes of singular varieties

More complicated examples: wheels with n-spokes (Bloch–Esnault–Kreimer)

• General result: Classes $[X_\Gamma]$ generate $K_0(\mathcal{V})$ Grothendieck ring of varieties (Belkale-Brosnan)

Observations on the bottom-up method

• $\mathbb{P}^{n-1} \smallsetminus X_\Gamma$ can be very complicated motivically (by Belkale-Brosnan)

• but since $\Psi_\Gamma(t) = \det M_\Gamma(t)$

$$\mathbb{P}^{n-1} \smallsetminus X_\Gamma \hookrightarrow \mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$$

$\mathcal{D}_\ell =$ determinant variety (w/ conditions on Γ)

• The motive of $\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell$ is mixed Tate

• Feynman integral as period computation. Divergent case: Igusa local L-functions (Belkale–Brosnan)

• Main difficulty: explicit control of $\hat{\Sigma} \cap \mathcal{D}_\ell$ to show

$$H^*(\mathbb{P}^{\ell^2-1} \smallsetminus \mathcal{D}_\ell, \hat{\Sigma} \smallsetminus (\hat{\Sigma} \cap \mathcal{D}_\ell))$$

mixed Tate (Aluffi-M. work in progress)

Top-down: Counterterms and beta function (Connes-M.)

Generator of renormalization group

$$\beta = \frac{d}{dt} F_t|_{t=0}$$

Counterterms reconstructed from the beta function ('t Hooft–Gross relations):

$$\gamma_-(z) = \mathrm{T} e^{-\frac{1}{z} \int_0^\infty \theta_{-t}(\beta)\, dt}$$

Time ordered exponential

$$\mathrm{T} e^{\int_a^b \alpha(t)\, dt} = 1 + \sum_1^\infty \int_{a \le s_1 \le \cdots \le s_n \le b} \alpha(s_1) \cdots \alpha(s_n) \prod ds_j$$

Renormalization and iterated integrals

Data of renormalization ⇒ loops with

$$\gamma_\mu(z) = \mathrm{T}e^{-\frac{1}{z}\int_\infty^{-z\log\mu}\theta_{-t}(\beta)\,dt}\,\theta_{z\log\mu}(\gamma_{\mathrm{reg}}(z))$$

$$\gamma_{\mu^+}(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{-z\log\mu}\theta_{-t}(\beta)\,dt}\,\theta_{z\log\mu}(\gamma_{\mathrm{reg}}(z))$$

$$\gamma_-(z) = \mathrm{T}e^{-\frac{1}{z}\int_0^{\infty}\theta_{-t}(\beta)\,dt}$$

Time ordered exponential $\mathrm{T}e^{\int_a^b \alpha(t)\,dt} = g(b)$ solution of diff equation

$$dg(t) = g(t)\alpha(t)dt \quad \text{with } g(a) = 1$$

Divergences of QFT ⇒ Differential systems

Differential Galois theory

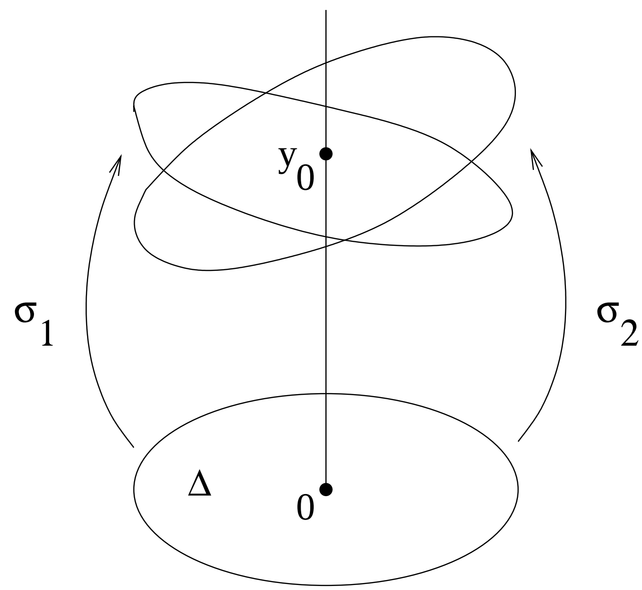
Hilbert 21st problem: Reconstruct differential equations from monodromy representation

$\Rightarrow$ Riemann–Hilbert correspondence: Classify differential systems with singularities by representations

Differential Galois group

Flat equisingular connections

Principal $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^*$-bundle $\mathbb{G}_m \to B \xrightarrow{\pi} \Delta$



Restrictions to different sections same type of singularity

Equisingular vector bundles $\Leftrightarrow Rep_{\mathbb{U}^*}$

$\mathbb{U}^*$ and the renormalization group

Free graded Lie algebra $\mathcal{L}_\bullet = \mathcal{F}(1, 2, 3, \cdots)_\bullet$ generators e_{-n} deg $n > 0$

Hopf algebra $\mathcal{H}_u = \mathcal{U}(\mathcal{L}_\bullet)^\vee$ dual to $\mathcal{U}$

$$\mathcal{U}^* = \mathcal{U} \rtimes \mathbb{G}_m$$

In a given physical theory: generator $e_{-n} \mapsto \beta_n$

$$\beta = \sum_n \beta_n$$

n-loop component of the beta function

$$e = \sum_n e_{-n} \mapsto \beta$$

renormalization group as subgroup of $\mathbb{U}^*$

Main results on the top-down approach
(Connes–M.)

• Counterterms as iterated integrals
('t Hooft–Gross relations)
• Solutions of irregular singular differential equations (flat equisingular connections)
• Flat equisingular vector bundles
form a neutral Tannakian category $\mathcal{E}$
• Free graded Lie algebra $\mathcal{L} = \mathcal{F}(e_{-n}; n \in \mathbb{N})$

$$\mathcal{E} \simeq Rep_{\mathbb{U}^*}, \quad \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

$\mathbb{U} = \mathrm{Hom}(\mathcal{H}_{\mathbb{U}}, -)$, with $\mathcal{H}_{\mathbb{U}} = U(\mathcal{L})^\vee$
• Motivic Galois group (Deligne–Goncharov)

$$\mathbb{U}^* \simeq \mathrm{Gal}(\mathcal{M}_S)$$

$\mathcal{M}_S$ mixed Tate motives on $S = \mathrm{Spec}(\mathbb{Z}[i][1/2])$

Renormalization and motives: summary

- Periods of mixed Tate motives from Feynman integrals (Broadhurst–Kreimer)

- Graph hypersurfaces can be arbitrary motives (Belkale–Brosnan)

- Motives from Feynman integrals (Bloch–Esnault–Kreimer)

- Mixed Tate motives with

$$G = \mathbb{U}^* = \mathbb{U} \rtimes \mathbb{G}_m$$

(Deligne–Goncharov)

II.Part of the course: Noncommutative spaces

Equivalence relation $\mathcal{R}$ on X:
quotient $Y = X/\mathcal{R}$
Even for "good" X usually "bad" Y

Classical: functions on the quotient
$\mathcal{A}(Y) := \{f \in \mathcal{A}(X) \,|\, f \text{ is } \mathcal{R}-\text{invariant}\}$
$\Rightarrow$ often too few functions
$\mathcal{A}(Y) = \mathbb{C}$ only constants

NCG: $\mathcal{A}(Y)$ noncommutative algebra

$$\mathcal{A}(Y) := \mathcal{A}(\Gamma_{\mathcal{R}})$$

functions on the graph $\Gamma_{\mathcal{R}} \subset X \times X$ of the equivalence relation

Convolution product

$$(f_1 * f_2)(x,y) = \sum_{x \sim u \sim y} f_1(x,u) f_2(u,y)$$

involution $f^*(x,y) = \overline{f(y,x)}$.

Spectral triples

Riemannian geometry: X with metric tensor g

$$(C^\infty(X), L^2(X,S), \not{D})$$

Dirac operator and spinors

Noncommutative Riemannian geometries

$$(\mathcal{A}, \mathcal{H}, \mathcal{D})$$

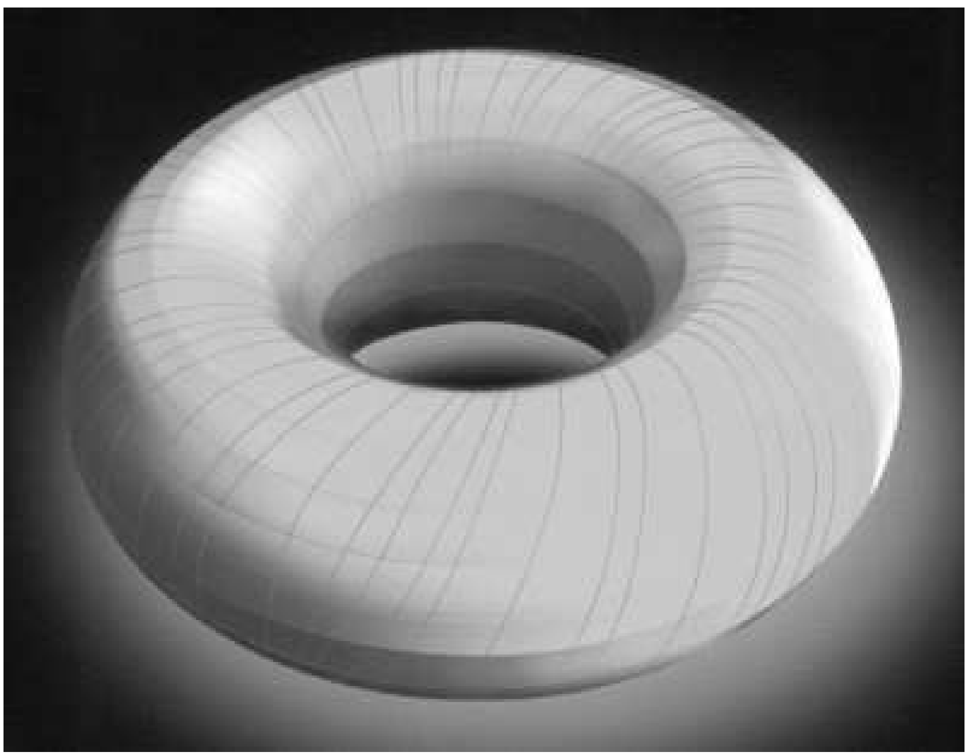
NC algebra $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$

Unbounded operator $\mathcal{D}$ with $\mathcal{D}^* = \mathcal{D}$

$$[\mathcal{D}, a] \quad \text{bounded} \quad \forall a \in \mathcal{A}_\infty \subset \mathcal{A}$$

Examples of noncommutative spaces

Noncommutative tori: $S^1/\mathbb{Z}$ irrational rotation



Elliptic curves $E_q = \mathbb{C}^*/q^{\mathbb{Z}}$ with $|q| < 1$

Degeneration for $q \to e^{2\pi i\theta} \in S^1$ and $\theta \in \mathbb{R} \smallsetminus \mathbb{Q}$

The spectral action

Action functional for spectral triples $(\mathcal{A}, \mathcal{H}, \mathcal{D})$

$$\text{Tr}\,(f(\mathcal{D}/\Lambda))$$

Λ mass scale, $f > 0$ even function

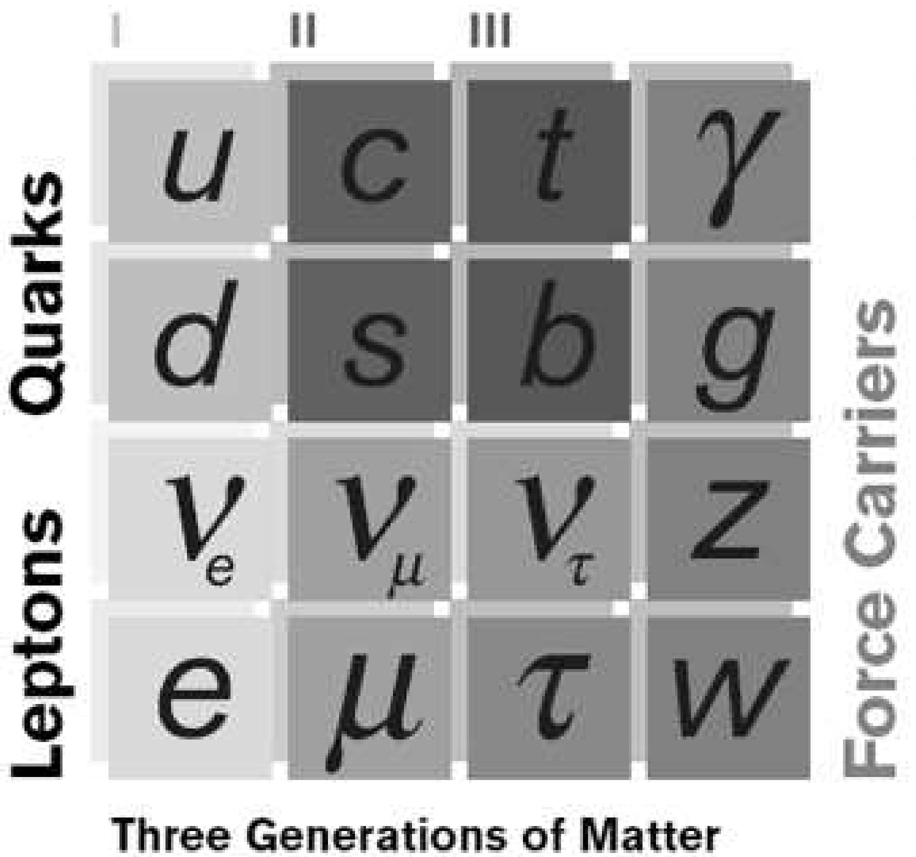
Asymptotic expansion

$$\text{Tr}\,(f(D/\Lambda)) \sim \sum_k f_k\,\Lambda^k \int\!\!\!\!\!- |D|^{-k} + f(0)\,\zeta_D(0) + o(1)$$

with $f_k = \int_0^\infty f(v)\,v^{k-1}\,dv$

Contributions from $k \in$ Dimension Spectrum

The Standard Model of elementary particle physics



- coupling with gravity $S_{EH} + S_{SM}$
- neutrino mixing

The problem: Standard Model Lagrangian

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc}f^{ade}g^b_\mu g^c_\nu g^d_\mu g^e_\nu -$$
$$\partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu -$$
$$igc_w(\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) +$$
$$Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) - igs_w(\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) -$$
$$A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)) -$$
$$\frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu -$$
$$Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) +$$
$$g^2 s_w c_w(A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu) - \frac{1}{2}\partial_\mu H \partial_\mu H -$$
$$2M^2\alpha_h H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 -$$
$$\beta_h\left(\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)\right) + \frac{2M^4}{g^2}\alpha_h -$$
$$g\alpha_h M\left(H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-\right) -$$
$$\frac{1}{8}g^2\alpha_h\left(H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2H^2\right) -$$
$$gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H -$$
$$\frac{1}{2}ig\left(W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)\right) +$$
$$\frac{1}{2}g\left(W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) + W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)\right) +$$
$$\frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) + M\,(\frac{1}{c_w}Z^0_\mu\partial_\mu\phi^0 + W^+_\mu\partial_\mu\phi^- + W^-_\mu\partial_\mu\phi^+) -$$
$$ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) -$$
$$ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) -$$
$$\frac{1}{4}g^2 W^+_\mu W^-_\mu\left(H^2 + (\phi^0)^2 + 2\phi^+\phi^-\right) -$$
$$\frac{1}{8}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu\left(H^2 + (\phi^0)^2 + 2(2s_w^2 - 1)^2\phi^+\phi^-\right) - \frac{1}{2}g^2\frac{s_w^2}{c_w}Z^0_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) - \frac{1}{2}ig^2\frac{s_w^2}{c_w}Z^0_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) + \frac{1}{2}g^2 s_w A_\mu\phi^0(W^+_\mu\phi^- +$$
$$W^-_\mu\phi^+) + \frac{1}{2}ig^2 s_w A_\mu H(W^+_\mu\phi^- - W^-_\mu\phi^+) - g^2\frac{s_w}{c_w}(2c_w^2 - 1)Z^0_\mu A_\mu\phi^+\phi^- -$$
$$g^2 s_w^2 A_\mu A_\mu\phi^+\phi^- + \frac{1}{2}ig_s\,\lambda^a_{ij}(\bar{q}^\sigma_i\gamma^\mu q^\sigma_j)g^a_\mu - \bar{e}^\lambda(\gamma\partial + m^\lambda_e)e^\lambda - \bar{\nu}^\lambda(\gamma\partial +$$
$$m^\lambda_\nu)\nu^\lambda - \bar{u}^\lambda_j(\gamma\partial + m^\lambda_u)u^\lambda_j - \bar{d}^\lambda_j(\gamma\partial + m^\lambda_d)d^\lambda_j +$$
$$igs_w A_\mu\left(-(\bar{e}^\lambda\gamma^\mu e^\lambda) + \frac{2}{3}(\bar{u}^\lambda_j\gamma^\mu u^\lambda_j) - \frac{1}{3}(\bar{d}^\lambda_j\gamma^\mu d^\lambda_j)\right) + \frac{ig}{4c_w}Z^0_\mu\{(\bar{\nu}^\lambda\gamma^\mu(1 +$$

$$
\begin{aligned}
&\gamma^5)\nu^\lambda)+(\bar{e}^\lambda\gamma^\mu(4s_w^2-1-\gamma^5)e^\lambda)+(\bar{d}_j^\lambda\gamma^\mu(\tfrac{4}{3}s_w^2-1-\gamma^5)d_j^\lambda)+(\bar{u}_j^\lambda\gamma^\mu(1-\tfrac{8}{3}s_w^2+\\
&\gamma^5)u_j^\lambda)\}+\tfrac{ig}{2\sqrt{2}}W_\mu^+\left((\bar{\nu}^\lambda\gamma^\mu(1+\gamma^5)U^{lep}{}_{\lambda\kappa}e^\kappa)+(\bar{u}_j^\lambda\gamma^\mu(1+\gamma^5)C_{\lambda\kappa}d_j^\kappa)\right)+\\
&\tfrac{ig}{2\sqrt{2}}W_\mu^-\left((\bar{e}^\kappa U^{lep\dagger}{}_{\kappa\lambda}\gamma^\mu(1+\gamma^5)\nu^\lambda)+(\bar{d}_j^\kappa C^\dagger_{\kappa\lambda}\gamma^\mu(1+\gamma^5)u_j^\lambda)\right)+\\
&\tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m_e^\kappa(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1-\gamma^5)e^\kappa)+m_\nu^\lambda(\bar{\nu}^\lambda U^{lep}{}_{\lambda\kappa}(1+\gamma^5)e^\kappa\right)+\\
&\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m_e^\lambda(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1+\gamma^5)\nu^\kappa)-m_\nu^\kappa(\bar{e}^\lambda U^{lep\dagger}{}_{\lambda\kappa}(1-\gamma^5)\nu^\kappa\right)-\\
&\tfrac{g}{2}\tfrac{m_\nu^\lambda}{M}H(\bar{\nu}^\lambda\nu^\lambda)-\tfrac{g}{2}\tfrac{m_e^\lambda}{M}H(\bar{e}^\lambda e^\lambda)+\tfrac{ig}{2}\tfrac{m_\nu^\lambda}{M}\phi^0(\bar{\nu}^\lambda\gamma^5\nu^\lambda)-\tfrac{ig}{2}\tfrac{m_e^\lambda}{M}\phi^0(\bar{e}^\lambda\gamma^5e^\lambda)-\\
&\tfrac{1}{4}\,\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa-\tfrac{1}{4}\,\overline{\bar{\nu}_\lambda\,M^R_{\lambda\kappa}\,(1-\gamma_5)\hat{\nu}_\kappa}+\\
&\tfrac{ig}{2M\sqrt{2}}\phi^+\left(-m_d^\kappa(\bar{u}_j^\lambda C_{\lambda\kappa}(1-\gamma^5)d_j^\kappa)+m_u^\lambda(\bar{u}_j^\lambda C_{\lambda\kappa}(1+\gamma^5)d_j^\kappa\right)+\\
&\tfrac{ig}{2M\sqrt{2}}\phi^-\left(m_d^\lambda(\bar{d}_j^\lambda C^\dagger_{\lambda\kappa}(1+\gamma^5)u_j^\kappa)-m_u^\kappa(\bar{d}_j^\lambda C^\dagger_{\lambda\kappa}(1-\gamma^5)u_j^\kappa\right)-\\
&\tfrac{g}{2}\tfrac{m_u^\lambda}{M}H(\bar{u}_j^\lambda u_j^\lambda)-\tfrac{g}{2}\tfrac{m_d^\lambda}{M}H(\bar{d}_j^\lambda d_j^\lambda)+\tfrac{ig}{2}\tfrac{m_u^\lambda}{M}\phi^0(\bar{u}_j^\lambda\gamma^5u_j^\lambda)-\tfrac{ig}{2}\tfrac{m_d^\lambda}{M}\phi^0(\bar{d}_j^\lambda\gamma^5d_j^\lambda)+\\
&\bar{G}^a\partial^2G^a+g_sf^{abc}\partial_\mu\bar{G}^aG^bg_\mu^c+\bar{X}^+(\partial^2-M^2)X^++\bar{X}^-(\partial^2-M^2)X^-+\\
&\bar{X}^0(\partial^2-\tfrac{M^2}{c_w^2})X^0+\bar{Y}\partial^2Y+igc_wW_\mu^+(\partial_\mu\bar{X}^0X^--\partial_\mu\bar{X}^+X^0)+\\
&igs_wW_\mu^+(\partial_\mu\bar{Y}X^--\partial_\mu\bar{X}^+Y)+igc_wW_\mu^-(\partial_\mu\bar{X}^-X^0-\partial_\mu\bar{X}^0X^+)+\\
&igs_wW_\mu^-(\partial_\mu\bar{X}^-Y-\partial_\mu\bar{Y}X^+)+igc_wZ_\mu^0(\partial_\mu\bar{X}^+X^+-\partial_\mu\bar{X}^-X^-)+\\
&igs_wA_\mu(\partial_\mu\bar{X}^+X^+-\partial_\mu\bar{X}^-X^-)-\\
&\tfrac{1}{2}gM\left(\bar{X}^+X^+H+\bar{X}^-X^-H+\tfrac{1}{c_w^2}\bar{X}^0X^0H\right)+\\
&\tfrac{1-2c_w^2}{2c_w}igM\left(\bar{X}^+X^0\phi^+-\bar{X}^-X^0\phi^-\right)+\tfrac{1}{2c_w}igM\left(\bar{X}^0X^-\phi^+-\bar{X}^0X^+\phi^-\right)+\\
&igMs_w\left(\bar{X}^0X^-\phi^+-\bar{X}^0X^+\phi^-\right)+\tfrac{1}{2}igM\left(\bar{X}^+X^+\phi^0-\bar{X}^-X^-\phi^0\right)\,.
\end{aligned}
$$

NCG models of particle physics

Minimal mathematical input $\Rightarrow$ SM Lagrangian *derived* by calculation

Classification of finite geometries

• Product of spacetime by "finite NC space"

• Real structure on a spectral triple

• Finite space: metric dimension zero but "homological" dimension six

• All possible Dirac operators on the finite space $\Rightarrow$ physical properties (color unbroken, values of hypercharges, etc)

The Yukawa parameters of the Standard Model

• Cabibbo–Kobayashi–Maskawa matrix (quark masses, mixing angles, phase)

• Pontecorvo–Maki–Nagakawa–Sakata matrix (lepton masses, including neutrinos, mixing angles and phase)

• Majorana mass terms for neutrinos

Moduli space of Dirac operators:

$$\mathcal{C}_1 \times \mathcal{C}_3$$

lepton and quark sectors

$$\mathcal{C}_3 = (K \times K)\backslash(G \times G)/K$$

$G = \mathrm{GL}_3(\mathbb{C})$ and $K = U(3)$

$\pi : \mathcal{C}_1 \to \mathcal{C}_3$ surjection fiber symm matrices mod $M_R \mapsto \lambda^2 M_R$; $\dim_{\mathbb{R}}(\mathcal{C}_3 \times \mathcal{C}_1) = 31$

Bosonic and fermionic parts of the action

Bosonic part from asymptotic formula for the spectral action:

- Cosmological terms
- Riemann curvature terms
- Higgs minimal coupling and quartic potential
- Higgs mass terms
- Yang–Mills terms for gauge bosons

Fermionic part: from real structure, Pfaffian (Grassman fields)

- Fermion-Higgs coupling
- Gauge-fermion coupling
- Fermion doubling
- see-saw mechanism for neutrino masses

Physical predictions

- As in grand-unified theories:

$$\frac{g_3^2\, f_0}{2\pi^2} = \frac{1}{4}, \qquad g_3^2 = g_2^2 = \frac{5}{3}\, g_1^2$$

- Mass relation at unification

$$\sum_\sigma (m_\nu^\sigma)^2 + (m_e^\sigma)^2 + 3\,(m_u^\sigma)^2 + 3\,(m_d^\sigma)^2 = 8\,M^2$$

$M = W - mass$

- From mass relation and RGE $\Rightarrow$ top quark mass estimate

- Higgs mass (168 GeV)

Dimensional regularization as a noncommutative geometry

$$\int e^{-\lambda q^2} d^D q = \pi^{D/2} \lambda^{-D/2}$$

NCG space X_z of Dimension Spectrum $z \in \mathbb{C}$

Dimensional Regularization: cup product of spectral triples

$$X \cup X_z$$

$X = (\mathcal{A}, \mathcal{H}, \mathcal{D})$ space time and finite space

$\Rightarrow$ Anomalies computations