THE TAME GEOMETRY OF O-MINIMAL STRUCTURES

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Abstract. The motivation for o-minimal structures comes from Grothendieck's *Equisse d'un programme* wherein he proposes a new branch of topology that generalizes the semialgebraic sets and exhibits an overall geometric tameness. Model theorists independently developed the theory of o-minimal structures and it was soon considered a prime candidate for Grothendieck's sought-after tame geometry. In this expository paper we will define o-minimal structures and give proofs of the Monotonicity Theorem as well as the Cell Decomposition Theorem. Along the way we will explore what makes o-minimal structures compelling.

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1. Motivation

In the 1980's, Alexander Grothendieck published a paper in which he outlines his reasons for the creation of a new branch of topology. He wanted a framework of topology and geometry that was tamer than the standard one, which he noted had heavy influence from analysts. The analytic foundations gave rise to false problems, as he saw them, that need not occur because they were not linked to any geometric intuition in the first place. Grothendieck was also studying moduli spaces at the time and wanted a theory that would include stratification. Stratification is a nice geometric property that breaks up sets into sets of lower dimension. Many structures give way to natural stratifications, moduli spaces of algebraic curves being one such example. Simpler ones include polygons, systems of straight lines in a projective plane, general immersed curves with normal crossings, and as we will see, semialgebraic sets.

Other mathematicians had been developing generalizations of the semialgebraic sets in a manner that was in line with Grothendieck's wishes before and after

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the paper was released. That work informed the creation of o-minimal structures, the topic of this paper, which is studied by both topologist-geometers and model theorists and logicians. Many people think of o-minimal structures as being a good candidate for Grothendieck's program of a tame topology, but not all of what he sought out is satisfied by this theory.

That being said, o-minimal structures are heavily studied for their model theoretic properties as well as their applications to fields such as algebraic geometry. In this paper we will look at some of the basic facts about o-minimal structures and give proofs of two foundational theorems.

2. Semialgebraic Sets

O-minimal structures are a generalization of the semialebraic sets, so we will discuss their desirable properties and contrast them with less well-behaved sets. In this section, we will omit proofs, but several of the key statements will be generalized and proven later.

Definition 2.1. A *semialgebraic set*, $X \subset \mathbb{R}^n$, is a finite boolean combination of sets of the form

$$\{x \in \mathbb{R}^n \mid f(x) = 0\} \text{ or } \{x \in \mathbb{R}^n \mid g(x) > 0\},$$

where $f, g \in \mathbb{R}[X_1, \ldots, X_n]$.

Remark 2.2. A finite boolean combination of a family of sets is obtained by a finite application of intersections, unions, and complements to a finite subset of the family. Note that for a semialgebraic set $X \subset \mathbb{R}$ we always obtain a finite collection of points and intervals after applying this operation. This property will be of importance later.

Semialgebraic sets are desirable to work with because we can take finite unions, finite intersections, and complements without degenerating.

Lemma 2.3. *Let $A \in \mathbb{R}^n$ be a semialgebraic set.*

- *The closure, $\overline{A}$, the interior,* $\text{int}(A)$*, and the boundary, ∂A are all semialgebraic.*
- *A has finitely many connected components, each of which is semialgebraic.*

The preceding results and remarks show the stability of the semialgebraic sets. An even more powerful stability result comes to us in the form of the Tarski-Seidenberg Theorem. It states that we can take projections of semialgebraic sets without degenerating. It is an example of a *quantifier elimination* result. That notion comes to us from model theory and we will see in the next section how projections and quantifiers are related.

Theorem 2.4 (Tarski-Seidenberg)**.** *Let $A \in \mathbb{R}^{n+1}$ be a semialgebraic set and let $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ be the projection map onto the first n-coordinates. Then $\pi(A) \in \mathbb{R}^n$ is semialgebraic.*

Now we see that the geometry of the semialgebraic sets is quite tame. The sets are stable under several desirable operations. Furthermore, they can be decomposed into finitely many connected components of the same type.

In the one dimensional case, the semialgebraic sets can be decomposed into finitely many connected components, namely points and intervals. It is useful to note the fact that points and intervals are themselves semialgebraic sets. A generalization of these facts lets us partition a given semialgebraic set of arbitrary dimension into the finite union of two special semialgebraic sets which serve as analogues to points and sets.

Definition 2.5.
- The first set is the familiar *graph* of a function. Given a continuous function, $f : A \subset \mathbb{R}^n \to \mathbb{R}$, the set $\Gamma(f) = \{(x, f(x) \mid x \in A)\}$ is the graph of f.
- The next set is the *cylinder* determined by f, g, A. Given two continuous functions f, g such that $f < g$, both defined from $A \subset \mathbb{R}^m$ to $\mathbb{R}$, the cylinder they define is the set
$$(f, g)_A = \{(x, r) \in \mathbb{R}^{n+1} \mid f(x) < r < g(x)\}.$$

Graphs are the analogue of points and cylinders are the analogue of intervals, because they are the set[s] of points between two graphs.

Theorem 2.6. *Let $X \subset \mathbb{R}^n$ be a semialgebraic set. Then*
- *X can be written as the union of finitely many connected components, each of which is semialgebraic.*
- *There is a decomposition D of $\mathbb{R}^{n-1}$ into semialgebraic sets such that for each $A_i \in D$ there exist functions*
$$f_{i0} = -\infty < f_{i1} < \ldots < f_{in(i)} < +\infty = f_{in(i)+1}$$
from A to $\mathbb{R} \cup \{-\infty, +\infty\}$ such that each f_{ij} is continuous and both $\Gamma(f_{ij})$ and (f_{ij}, f_{ij+1}) are semialgebraic.

The graphs and cylinders partition X.

This decomposition theorem will be generalized and proven later as the Cell Decomposition Theorem.

3. O-minimal Structures

In order to generalize the behavior of the semialgebraic sets, mathematicians developed a slick set of axioms that both characterized and generalized the semi-algebraic sets. Before we can define o-minimal structures, we must first define structures more generally.

Definition 3.1. Given a nonempty set R, we define a *structure* $\mathcal{S}$ on R to be a sequence $(S_n)_{n\in\mathbb{N}}$ satisfying the following axioms for each $n \geq 0$:

(1) S_n is a boolean algebra of subsets of $\mathbb{R}^n$
(2) if $A \in S_n$, then the sets $A \times R$ and $R \times A$ are *definable* in S_{n+1}, i.e., they are elements of the structure
(3) $\{(x_1, \ldots, x_n) \in R^n \mid x_1 = x_n\} \in S_n$
(4) if $A \in S_{n+1}$ and if $\pi : R^{n+1} \to R^n$ is the projection map onto the first n coordinates, then $\pi(A) \in S_n$.

Definition 3.2. A function $f : A \to B$, where $A \subset R^m$ and $B \subset R^n$ are definable sets, is definable if its graph, $\Gamma(f) \subset R^m \times R^n$ is definable.

Remark 3.3. We say an object is definable in a structure to mean it is a member of the structure.

As a consequence of the projection axiom, we may end up with sets that do not behave like the semialgebraic sets, because pathologies are propogated down through the lower dimensions. Another way to look at this issue is that the conditions of the lower dimensions are too loose and will give rise to untame or pathological sets in higher dimensions.

It is from this characterization of the problem that we find our solution, namely, a fifth axiom that restricts the 1-dimensional definable sets.

Definition 3.4. A structure is said to be *o-minimal* if the graph $\{(x, y) \in R^2 \mid x < y\}$ is definable and if the definable sets in S_1 are exactly the finite unions of points and intervals.

This condition vastly restricts the definable sets in our structure, and for good reason too. We would like our geometric objects to be of finite type and it is this axiom that helps us achieve that goal.

In order for this definition to make sense, we must work with a *dense linear order* without endpoints. That means a linear order such that there is a point between any two points and there is no least or greatest element. However, sometimes we will add $-\infty, +\infty$ as endpoints for convenience.

Intervals are defined in the standard way, i.e.,

$$(a, b) := \{x \in R \mid a < x < b\}.$$

We then place the interval topology on the structure. Open sets are generated by the base of intervals and we place the product topology on R^m.

Most of the useful examples of structures are extensions of the semialgebraic sets. Often proving that the projection axiom is fulfilled constitutes much of the work, as is the case with the semi-algebraic sets and the Tarski-Seidenberg Theorem, but that is not always the case.

We are also primarily concerned with structures on a the real closed field, where $(\mathbb{R}, \cdot, +)$ denotes a theory from the axioms of a real closed field. One can also work with ordered groups or ordered rings. However, in the latter case, the only ordered rings that admit o-minimal structures are real closed fields. See [1].

Here are examples of o-minimal structures.

Example 3.5. The semialgebraic sets, denoted $\mathbb{R}_{\text{alg}}$ are the smallest o-minimal structure on $(\mathbb{R}, \cdot, +)$.

We can determine whether adding a function to the structure results in an o-minimal structure.

Given a family of real valued functions, $\mathcal{F}$, the smallest structure on $(\mathbb{R}, \cdot, +)$ containing $\Gamma(f)$ for all $f \in \mathcal{F}$, is denoted $(\mathbb{R}, \cdot, +, \mathcal{F})$.

Definition 3.6. An *$\mathcal{F}$-set* is a set of the form

$$\{x \in \mathbb{R}^n \mid P(x, f_1(x), f_2(x), \dots, f_n(x) = 0\},$$

for $f_i \in \mathcal{F}$ and $P \in \mathbb{R}[X_1, \dots, X_n, Y_1, \dots, Y_n]$.

Definition 3.7. A structure is said to be *model complete* if taking complements is superfluous in generating the definable sets.

The following theorem due to Khovanskii (see [5] and [7]) is useful in determining whether a given structure with added functions is o-minimal.

Theorem 3.8. *If $\mathcal{F}$ is such that the structure $\mathbb{R}_{\mathcal{F}}$ is model complete and each $\mathcal{F}$-set has finitely many connected components, then $\mathbb{R}_{\mathcal{F}}$ is o-minimal.*

More discussion of this theorem and its uses can be found in [5].

Example 3.9. The structure $R_{\exp}$, the smallest structure on $(\mathbb{R}, \cdot, +, e^x)$, is o-minimal. Wilkie proved this fact in 1991 using the theorem of model completeness.

Example 3.10. The structure $\mathbb{R}_{\text{an}}$, the smallest structure on $(\mathbb{R}, \cdot, +, \mathcal{A})$ where $\mathcal{A}$ is the family of analytic functions restricted to $[-1, 1]^n$, is o-minimal.

Example 3.11. The structure $\mathbb{R}_{\text{an,exp}}$, the smallest structure on $(\mathbb{R}, \cdot, +, \mathcal{A}, e^x)$, is also o-minimal. This was shown by van der Dries and Miller in a manner similar to Wilkie. (see [9]).

In general, one cannot combine o-minimal structures in this manner to create new ones.

Those familiar with model theory may wonder whether we can use ultraproducts to generate new o-minimal structures, but this is in fact not the case.

One can construct an ultraproduct of o-minimal structures that is not itself o-minimal. By Loz's Theorem, which states that first order properties are preserved through ultraproducts, we see that o-minimality is not a first order property. It is a weaker notion known as elementary. For further reading see [8].

Now we will discuss the use of logical notation in characterizing definable sets in a structure. There is a helpful correspondence between logical notation and boolean algebra operations and projection that make checking definability simpler.

Pick arbitrary sets X, Y, Z and consider a proposition $\phi(x, y, z)$ which takes x, y, z as variables that range over X, Y, Z respectively. Then we can define the set $\Phi := \{(x, y, z \mid \phi(x, y, z) \text{ is true})\}$ as a subset of $X \times Y \times Z$.

- Now we can see that the complement, Φ^c, is defined by $\not\phi$ which evaluates to true exactly when ϕ evaluates to false.
- If we consider two sets, Φ, Ψ, the set $\Phi \cup \Psi$ is defined by the proposition $\phi \vee \psi$ which evaluates to true exactly when either ϕ evaluates to true, ψ evaluates to true, or both evaluate to true.
- Similarly, the set $\Phi \cap \Psi$ is defined by the proposition $\phi \wedge \psi$ which evaluates to true exactly when both ϕ and ψ are true.
- We can also conisder the set defined by the proposition $\exists z \phi(x, y, z)$. It corresponds to the set

  $$\{(x, y) \in X \times Y \mid \text{there exists } z \in Z \text{ such that } \phi(x, y, z) \text{ is true}\}.$$

  This means that the existential quantifier acts like a projection.

The other common logical operations can all be defined using the operations listed above. Because these operations correspond exactly to the ones allowed in a structure, we can use logical notation to describe the definable sets. This means that any definable set can be written as a proposition in first-order logic and any proposition in first order logic describes a definable set. This correspondence makes it simpler to determine whether a given set is definable.

To see how this would be useful, we will show the following lemma.

Lemma 3.12. *Given a definable set $X \subset R^n$, the closure, $\overline{X}$, is also definable.*

Proof. The statement $(x_1, \ldots, x_n) \in \overline{X}$ is equivalent to the statement

$$\forall y_1 \ldots \forall y_n \forall z_1 \ldots \forall z_n[(y_1 < x_1 < z_1 \wedge \ldots \wedge y_n < x_n < z_n) \implies$$
$$\exists a_1 \ldots \exists a_n \, (y_1 < a_1 < z_1 \wedge \ldots \wedge y_n < a_n < z_n \wedge (a_1, \ldots, a_n) \in X)].$$

Because X is a definable set and the rest of the statement consists of projections and boolean algebra operations on X, we have that the closure is also definable. □

Definable sets also have suprema and infima.

Lemma 3.13. *Given a definable set $X \subset R^n$ bounded above,* $\sup(X)$ *exists. Similarly for* $\inf$.

Proof. If X is bounded above, we know the definable set $\{x \in R^n \mid \forall y \in X, y \leq x\}$ is nonempty. We can then take the minimum to be the supremum, i.e., $\min\{x \in R^n \mid \forall y \in X, y \leq x\} = \sup(X)$. □

4. Monotonicity

In this section we will be discussing definable functions in an o-minimal structure and characterizing their behavior. Luckily for us, the definable functions behave nicely.

Now we consider one of the most important theorem concerning definable functions.

Theorem 4.1. *Let $f : (a, b) \to R$ be a definable function. Then there is a finite partition of (a, b) given by $a = a_0 < a_1 < a_2 < \ldots < a_n = b$ such that the restriction, $f : (a_i, a_{i+1}) \to R$, is either constant or continuous and strictly monotone on each subinterval. (Note: $a_0 = -\infty, a_{n+1} = +\infty$ are possible).*

To complete the proof, we need three lemmas.

Assume that $f : I \to R$ is a definable function and I is an interval.

Lemma 4.2. *There is a subinterval of I on which f is constant or injective.*

Proof. If a single point $y \in R$ had an infinite preimage under f, then the preimage would contain a subinterval on which f takes the value y for all input and is thus constant. We now assume that $f^{-1}(y)$ is finite for all $y \in R$. This means that $f(I)$ is infinite so it contains an interval J. Let $g : J \to I$ be given by

$$g(y) := \min\{x \in I \mid f(x) = y\}.$$

This function is injective by construction and $g(J)$ is also infinite. It contains a subinterval of I on which f is injective. □

Lemma 4.3. *If f is injective, then f is strictly monotone on a subinterval of I.*

Proof. Let $I = (a, b)$ and pick $x \in I$. Then we can partition (a, x) into two sets

$$\{y \in (a, x) \mid f(y) < f(x)\} \text{ and } \{y \in (a, x) \mid f(y) > f(x)\}.$$

By o-minimality, one of these sets contains an interval like $(x - \epsilon, x)$. Similarly for (x, b). This partitioning gives a set of four propositions of which each $x \in I$ must satisfy exactly one. They are

$$\phi_{++}(x) := \exists\epsilon > 0 \forall y \in I[y \in (x-\epsilon, x) \implies f(y) > f(x) \wedge$$
$$y \in (x, x+\epsilon) \implies f(y) > f(x)]$$
$$\phi_{+-}(x) := \exists\epsilon > 0 \forall y \in I[y \in (x-\epsilon, x) \implies f(y) > f(x) \wedge$$
$$y \in (x, x+\epsilon) \implies f(y) < f(x)]$$
$$\phi_{--}(x) := \exists\epsilon > 0 \forall y \in I[y \in (x-\epsilon, x) \implies f(y) < f(x) \wedge$$
$$y \in (x, x+\epsilon) \implies f(y) < f(x)]$$
$$\phi_{-+}(x) := \exists\epsilon > 0 \forall y \in I[y \in (x-\epsilon, x) \implies f(y) < f(x) \wedge$$
$$y \in (x, x+\epsilon) \implies f(y) > f(x)].$$

These propositions say either that f is monotone near x or $f(x)$ is a local extremum. We want to show that there are finite local extrema. It is sufficient to consider ϕ_{++}. Assume that the set is infinite. By o-minimality, it contains a subinterval I. Let

$$B := \{x \in I \mid \forall y \in I(y > x \implies f(y) > f(x))\}.$$

If this set were infinite, it would contain an interval on which f is strictly increasing, so we can assume that B is finite. Now we pass to an interval, also called I, to the right of B. In this interval

$$\forall x \in I \exists y \in I \mid y > x \wedge f(y) < f(x).$$

Pick $x \in I$ and define the set $C_x := \{y \in I \mid y > x \wedge f(y) < f(x)\}$. This set is not finite because if it were, its maximum could not be in the interval. So it is infinite and contains an interval. Let $z \inf(C_x)$. We know $z > x$ because $f > f(x)$ on a neighborhood to the right of x, i.e., $(x, x+\epsilon)$, so there are no elements of C_x there. We can then see that $f > f(x)$ on $(z-\epsilon, z)$ and $f < f(x)$ on $(z, z+\epsilon)$. The proposition $\psi_{+-}(z)$ is true, given by

$$\exists\epsilon > 0 \forall u, v \in I\,(z-\epsilon, u, z, v, z+\epsilon \implies f(u) > f(v))\,.$$

So we have that $\forall x \in I, \exists z\,(x < z \wedge \psi_{+-}(z))$. The set of points satisfying ψ_{+-} is also infinite, because if not, its maximum would contradict the preceding statement. We can then replace I by a subinterval contained in the set of points satisfying ψ_{+-} so that it holds on an interval. Now define $h : -I \to R$ with $h(x) := f(-x)$. It is true that ϕ_{++} applies to h by construction. So by our argument, there is a subinterval of $-I$ where ψ_{+-} is true. Thus there is a subinterval of I where ψ_{-+} holds for f. This is a contradiction and ψ_{++} must be finite. We know that both ϕ_{++} and ϕ_{--} are finite, so it is sufficient to look at ϕ_{-+} on an interval $I = (a, b)$. For each $x \in I$, let $s(x) := \sup\{s \in (x, b) \mid f > f(x) \text{ on } (x, s]\}$. Then $s(x) = b$ because $s(x) < b$ leads to a contradiction of $\phi_{-+}(s(x))$. Thus f is strictly increasing on I. □

Lemma 4.4. *If f is strictly monotone, then f is continuous on a subinterval of I.*

Proof. Assume that f is strictly increasing. The image $f(I)$ is infinite, so there is a subinterval $J \subset f(I)$. Take two arbitrary points in J, $u < v$ and let s, t be their preimages. So we have $f(s) = u, f(t) = v, s < t$. The function f is an order preserving bijection, so by definition of the order topology, f is continuous on (s, t). □

Proof. To prove the theorem, we start with

$X := \{x \in (a,b) \mid \ f$ is either constant or continouos and strictly monotone near $x.\}$.

If $(a,b) \setminus X$ were infinite, it would contain an interval I. We could then apply the preceding lemmas to shrink I so that f is either constant, or strictly monotone and continuous which would result in a contradiction. So $(a,b) \setminus X$ is finite. So we can partition (a,b) into the finitely many intervals of which X consists. Once we have done that, we can check two cases: f is constant near x for all $x \in (a,b)$ or f is strictly monotone near x for all $x \in (a,b)$.

In the first case, pick $y \in (a,b)$ and let

$$s := \sup\{x \mid y < x < b, f \text{ is constant on } [y,x)\}.$$

If $s < b$, we have a contradiction, so $s = b$ and f is constant on $[y,b)$. Similarly, we show that f is constant on $(a,y]$ so it is constant on (a,b).

In the second case, we will assume without loss of generality that f is strictly increasing. Pick $y \in (a,b)$ and let

$$s := \sup\{x \mid y < x < b, f \text{ is strictly increasing on } [y,x)\}.$$

If $s < b$, we get another contradiction, so $s = b$ and f is strictly increasing on $[y,b)$. As above, we conclude that f is strictly increasing on (a,b). □

Lemma 4.5. *Let $f : R \to R$ be a definable function and let $x \in R_\infty$. Then* $\lim_{t \nearrow x} f(t)$ *and* $\lim_{t \searrow x} f(t)$ *exist in R_∞.*

Proof. We will show the case for $x = +\infty$. The others are similar. For each $m \in R$, the set $S_m = \{t \in R \mid f(t) \geq m\}$ is a finite union of points and intervals so there exists some $a \in R$ such that $(a,+\infty) \subset S_m$ or $(a,+\infty)$ is disjoint from S_m, i.e., eventually $f \geq m$ or eventually $f < m$. Let G be the set of $m \in R$ such that f is eventually greater than or equal to m. If G is empty, then $f < m$ for all m and $\lim_{t \nearrow \infty} f(t) = -\infty$. If $G = R$ then f is eventually greater than all m, so $\lim_{t \nearrow \infty} f(t) = +\infty$. If G is between these two, then the set $R \setminus G$ is a set of upper bounds for f, so the supremum, s, must equal the limit. □

The Finiteness Lemma pairs nicely with the Monotonicity Theorem to characterize the tame behavior of sets and functions in the structure.

Lemma 4.6 (Finiteness Lemma)**.** *Let $A \subset R^2$ be a definable set such that for each $x \in R$, the fiber above x, $A_x = \{y \in R \mid (x,y) \in A\}$ is finite. Then there exists a natural number N such that $|A_x| \leq N$ for all $x \in R$.*

Definition 4.7. A point $(a,b) \subset R^2$ is *normal* if there is a box $I \times J$ around (a,b) so that

- either $(I \times J) \cap A = \emptyset$
- or $(a,b) \in A$ and $(I \times J) \cap A = \Gamma(f)$ for some continuous function $f : I \to R$.

A point of the form $(a,-\infty) \in R \times R_\infty$ is *normal* if there is a box $I \times J$ disjoint from A such that $a \in I$ and $J = (-\infty,b)$ for some b. And a point of the form $(a,+\infty) \in R \times R_\infty$ is *normal* if there is a box $I \times J$ disjoint from A with $a \in I$ and $J = (b,+\infty)$ for some b.

Remark 4.8. From our definitions, we have that the sets

$$\{(a,b) \in R^2 \mid (a,b) \text{ is normal}\},$$
$$\{a \in R \mid (a,-\infty) \text{ is normal}\}$$
$$\{a \in R \mid (a,+\infty) \text{ is normal}\}$$

are all definable.

Essentially, the normal points are what would compose a set that is uniformly finite, because such a set locally looks like a stack of graphs of continuous functions. If we can show that all but finitely many points of our set are normal, then we can uniformly bound the size of the fibers.

Proof. To begin the proof, we define functions on the fibers of the set A. Given a point in R, the function will return a point in the fiber above it, provided it exists. Let $f_1, f_2, \dots, f_n, \dots$ be functions with domain $\{x \in R \mid |A_x| \geq i\}$, for the i-th function, and with $f_i(x)$ equal to the i-th element of the fiber above x. By construction, these are definable functions.

Now we will split our set of points into good and bad points. Pick some $a \in R$ and let $n \geq 0$ be the maximal number such that the first n functions, as defined above, have non empty domain and are continuous in an interval around a.

A point a is *good* if $a \notin \overline{\mathrm{dom}(f_{n+1})}$ and a point a is bad if $a \in \overline{\mathrm{dom}(f_{n+1})}$.

In other words, the good points are those that map to the largest element of the fiber or are close to an element that does.

Then we look at the set G of good points and the set B of bad points. These sets are parameterized by an $n \in \mathbb{N}$ which depends on our choice of a so it is not obvious that the sets are definable.

We do have, however, that if a is a good point, then the domain of f_{n+1} is disjoint from an entire interval around a on which the functions $f_1, \dots, f_n$ are defined and continuous. Then by construction, we have that $|A_x|$ is constant on an interval around a and that (a,b) is normal for all $b \in R_\infty$.

Thus we can say that if $a \in B$, then there is at least one $b \in R_\infty$ such that (a,b) is not normal. Once the sets B, G are characterized in terms of normality, an existing definable characteristic, we can say they are definable themselves.

Pick $a \in B$ and consider the following elements: $\lambda(a,-), \lambda(a,0), \lambda(a,+)$. They are given by

$$\begin{aligned}
\lambda(a,-) &:= \lim_{x \nearrow a} f_{n+1}(x) \text{ if } f_{n+1} \text{ is defined on some interval } (t,a) \\
&:= +\infty \text{ otherwise} \\
\lambda(a,0) &:= f_{n+1}(a) \text{ if } a \in \mathrm{dom}(f_{n+1}) \\
&:= +\infty \text{ otherwise} \\
\lambda(a,+) &:= \lim_{x \searrow a} f_{n+1} \text{ if } f_{n+1} \text{ is defined on an interval } (a,t) \\
&:= +\infty \text{ otherwise.}
\end{aligned}$$

If we let $\beta(a) := \min\{\lambda(a,-), \lambda(a,0), \lambda(a,+)\}$ then it is the least $b \in R_\infty$ such that (a,b) is not normal. Therefore, B and G are definable.

First suppose that B is finite, i.e. $B = \{a_1, \dots, a_k\}$ with $-\infty < a_1 < \dots < a_k < \infty$. Pick an interval (a_i, a_{i+1}) and fix some a inside it. Let $n = |A_a|$. Consider the two sets

$$\{x \in (a_i, a_{i+1}) \mid |A_x| = n\} \text{ and } \{x \in (a_i, a_{i+1}) \mid |A_x| \neq n\}.$$

Both of these sets are open, because the size of the fiber is constant in an interval around any given point. But the latter must be empty because it cannot be both open and finite. Thus we can find a uniform bound because we have only finitely may points and intervals each with constant fiber size.

Now we want to show that B cannot be infinite. First we let $\beta(a)$, for $a \in B$, be the minimum $b \in R_\infty$ such that (a, b) is not normal.

We now define two new sets as follows

$$B_- := \{a \in B \mid \exists y\, (y < \beta(a) \wedge (a, y) \in A)\}$$
$$B_+ := \{a \in B \mid \exists y\, (y > \beta(a) \wedge (a, y) \in A)\}$$

with corresponding functions $\beta_- : B_- \to R$ and $\beta_+ : B_+ \to R$ defined as follows

$$\beta_-(a) := \max\{y \mid y < \beta(a) \wedge (a, y) \in A\}$$
$$\beta_+(a) := \min\{y \mid y > \beta(a) \wedge (a, y) \in A\}.$$

Since B is infinite by assumption, one of the sets $B_- \cap B_+, B_- \setminus B_+, B_+ \setminus B_-, B \setminus (B_- \cup B_+)$ is infinite. We will show that the case of $B_- \cap B_+$ being infinite leads to a contradiction. The other cases are similar.

By the Monotonicity Theorem, there is an interval $I \subset B_- \cap B_+$ on which β_-, β, β_+ are all continuous. By construction, $\beta_- < \beta < \beta_+$ on I.

We split I into two subsets

$$\{x \in I \mid (x, \beta(x)) \in A\} \text{ and } \{x \in I \mid (x, \beta(x)) \notin A\}.$$

By o-minimality, one of these sets contains an interval. Replace I by this interval. Then we have that either $\Gamma(\beta\big|_I) \subset A$ or $\Gamma\left(\beta\big|_I\right) \cap A = \emptyset$. By continuity of β_-, β, β_+, $\Gamma\left(\beta\big|_I\right)$ will always consist of normal points in either case. This is a contradiction by definition of β, so B must be finite and the proof is complete.

□

As a consequence of both the Monotonicity Theorem and the Finiteness Lemma we can decompose definable sets in R^2 with finite fibers into the union of the graphs of continuous functions. More precisely, There exist points $a_1 < \ldots < a_k$ in R such that A intersected with $(a_i, a_i + 1) \times R$ is the set $\Gamma(f_{i1}) \cup \ldots \cup \Gamma(f_{in(i)})$ for definable continuous functions $f_{ij} : (a_i, a_{i+1}) \to R$ where $f_{i1}(x) < \ldots < f_{in(i)}(x)$ for each $x \in (a_i, a_{i+1})$.

We will see the generalization of the above statement in the next section.

5. Cell Decomposition

Definition 5.1. A (0)-cell is a point $\{r\} \subset R$; a (1)-cell is an interval $(a, b) \subset R$.

Assume that $(i_1, \ldots, i_m)$-cells are already defined. Then an $(i_1, \ldots, i_m, 0)$-cell is the graph of a function, $\Gamma(f)$, where $f : X \to R$ and X is an $(i_1, \ldots, i_m)$-cell; an $(i_1, \ldots, i_m)$-cell is a cylinder set (f, g) where $f, g : X \to R$, $f < g$, and X is an $(i_1, \ldots, i_m)$-cell.

Much like the semialgebraic case, we have that graphs are analogous to points and that cylinders are analogous to intervals. Thus the $(1, 1, \ldots, 1)$-cells are exactly the open cells.

We can also define a homeomorphism from any cell to an open cell by projection. Let $(i_1, \ldots, i_m)$ be a cell. Then define $p : R^m \to R^k$ as follows: let $\lambda(1) < \ldots < \lambda(k)$ be the indices of the cell for which $i_\lambda = 1$ and $k = i_1 + \ldots + i_k$. Then $p(x_1, \ldots, x_m) = \left(x_{\lambda(1)}, \ldots, x_{\lambda(k)}\right)$. We also denote the projection $p(A)$ as p_A.

The cells are tamely defined and as such they possess the amicable property of connectedness.

Proposition 5.2. *Cells are definably connected, i.e., they cannot be written as the union of two disjoint, non-empty, definable sets.*

Definition 5.3. A *decomposition* of R is a finite partition of R of the form

$$\{(-\infty, a_1), (a_1, a_2), \ldots, (a_k, +\infty), \{a_1\}, \ldots, \{a_k\}\}.$$

A decomposition of R^{m+1} is a finite partition of R^{m+1} into cells $A_1, \ldots, A_k$ such that the set of projections, $\pi(A_i)$, gives a partition of R^m.

We can build higher dimensional decompositions from lower dimensional ones by adding cylinder sets and graphs of functions above each cell.

Suppose we start with a decomposition $D = \{A_1, \ldots, A_k\}$ of R^m. Then for each $i \in [k]$ we can pick functions $f_{i1} < \ldots < f_{in(i)}$ with domain A_i and construct a decomposition of R^{m+1}.

This new decomposition will be given by the union of the partitions of $A_i \times R$, $D_i = \{(-\infty, f_{i1}), (f_{i1}, f_{i2}), \ldots, \left(f_{in(i)}, +\infty\right), \Gamma(f_{i1}), \ldots, \Gamma(f_{in(i)}\}$. This union can be projected down to R^m, and by construction, we recover the original decomomposition. Thus our construction is valid.

Now we will prove the main result of the paper, the Cell Decomposition Theorem. The theorem comes in two parts and we will prove it by induction.

Theorem 5.4 (Cell Decomposition)**.**

- *Let $A_1, \ldots, A_k$ be definable sets in R^m. Then there exists a decomposition of R^m that partitions each set A_i, i.e., each A_i is the union of cells in the decomposition.*
- *Given a definable function $f : A \to R$ with $A \subset R^m$, there exists a decomposition of R^m which partitions A such that f is continuous when restricted to any cell in A.*

The 1-dimensional cases have actually been discussed before. In fact, the first statement is simply given by o-minimality and the second is a consequence of the monotonicity theorem.

Now we assume that the theorem is true up to and including m and prove all the statements simultaneously.

To prove this result we will need an auxilliary lemma. We wish to generalize the Finiteness Lemma. This new lemma is called the Uniform Finiteness Property and it states a natural generalization of the earlier.

Lemma 5.5. *Let $Y \subset R^{m+1}$ be a definable set such that for each $x \in R^m$, the fiber above x, given by $Y_x = \{r \in R \mid (x, r) \in Y\}$ is finite. Then there exists a natural number N such that $|Y_x| \leq N$ for all $x \in R^m$.*

The proof is inductive, of course, and follows a relatively similar idea to that of the first Finiteness Lemma. A proof can be found at [1].

Now we prove the first statement for $m + 1$.

Proof. Let $A_1, \ldots, A_k$ be definable sets in R^{m+1}. The boundary of a set is a definable set, and specifically the boundary of a set in R is finite.

We will modify this idea to create the set

$$\partial_m(A) := \{(x,r) \in R^{m+1} \mid r \in \partial(A_x)\},$$

where $A \subset R^{m+1}$ is definable. This set is definable and finite over R^m so we can use the Uniform Finiteness Property for R^m.

Let $Y := \partial_m(A_1) \cup \ldots \cup \partial_m(A_k)$. The set $Y \subset R^{m+1}$ is definable and finite over R^m, so there is a uniform finite bound, $N \in \mathbb{N}$ such that $|Y_x| \leq N$ for all $x \in R^m$. For each $i \in \{1, \ldots, N\}$ we let $B_i := \{x \in R^m \mid |Y_x| = i\}$ and define functions $f_{i1}, \ldots, f_{ii} : B_i \to R$ such that $Y_x = \{f_{i1}(x), \ldots, f_{ii}(x)\}$ with $f_{i1}(x) < \ldots < f_{ii}(x)$.

We now create two new sets based on B_i. First fix $f_{i0} = -\infty, f_{ii+1} = +\infty$. For each $\lambda \in \{1, \ldots, k\}$, $i \in \{0, N\}$, and $1 \leq j \leq i$ we let

$$C_{\lambda ij} := \{x \in B_i \mid f_{ij}(x) \in (A_\lambda)_x\}.$$

Similarly, for each $\lambda \in \{1, \ldots, k\}$, $i \in \{0, \ldots, N\}$, and $0 \leq j \leq i$ we let

$$D_{\lambda ij} := \{x \in B_i \mid (f_{ij}(x), f_{ij+1}(x)) \subset (A_\lambda)_x\}.$$

We can now use the technique of building up new decompositions to finish the proof.

By our inductive hypotheses, we can take a decomposition D of R^m which partitions all of the $B_i, C_{\lambda ij}, D_{\lambda ij}$ such that if $E \in D$ is contained in B_i, then $f_{i1}\big|_E, \ldots, f_{ii}\big|_E$ are continuous. For each cell $E \in D$, we consider the parition D_E of $E \times R$ given by

$$\{\left(f_{i0}\big|_E, f_{i1}\big|_E\right), \ldots, \left(f_{ii}\big|_E, f_{ii+1}\big|_E\right), \Gamma(f_{i1}\big|_E), \ldots, \Gamma\left(f_{ii}\big|_E\right)\},$$

where i is chosen such that $E \subset B_i$. The union of each D_E gives D^* which is a decomposition of R^{m+1} that partitions each of $A_1, \ldots, A_k$. □

Before the proof of the second statment for $m+1$, we will state a lemma.

Lemma 5.6. *Let X be a topological space and $(R_1, <), (R_2, <)$ be dense linear orderings without endpoints and $f : X \times R_1 \to R_2$ a function such that for each $(x,r) \in X \times R_1$*

- *$f(x, \cdot) : R_1 \to R_2$ is continuous and monotone on R_1,*
- *$f(\cdot, r) : X \to R_2$ is continuous at x.*

Then f is continuous.

A proof can be found at [1].

Now for the proof of the second statement of the Cell Decomposition Theorem:

Proof. Fix a definable function $f : A \to R$ with $A \subset R^{m+1}$. We want to show that f is cellwise continuous. We will use the decomposition we have from the first statement for $m+1$ to show this.

We can break up A into finitely many cells, so we can assume that A is already a cell.

Suppose that the cell A is not open. Take the definable homeomorphic projection, $p_A : A \to p(A)$, and consider the open cell $p(A) \in R^n$ where $n \leq m$. By the inductive hypothesis, we can partition $p(A)$ into cells $B_1, \ldots, B_k$ such that $\left(f \circ p_A^{-1}\right) \big| B_i$ is continuous for each i. Then A must be partitioned into $p_A^{-1}(B_1), \ldots, p_A^{-1}(B_k)$ and f is continuous on each cell.

Now suppose that A is an open cell. We call a function f well-behaved at a point $(p, r) \in A$ if $p \in C$ for some $C \subset R^m$ where C is a box and $a < r < b$ for some $a, b \in R$ such that the following holds

- $C \times (a, b) \subset A$
- for all $x \in C$ the function $f(x, \cdot)$ is continuous and monotone on (a, b)
- the function $f(\cdot, r)$ is continuous at p.

Let A^* be the (definable) set of points for which f is well-behaved. We want to show that A^* is dense in A. To do so, we will show that given any box $B \subset R^m$ and $-\infty < a < c < +\infty$ such that $B \times (a, c) \subset A$, the box has nonempty intersection with A^*. By the Monotonicity Theorem, we can find for each $x \in B$ a largest $\lambda(x) \in (a, c]$ such that $f : (x, \cdot)$ is continuous and monotone on $(a, \lambda(x))$. The function $\lambda : B \to R$ is definable, so we can use the inductive hypothesis to find a box $C \subset B$ such that λ is continuous and monotone on C. We can take C to be small so that we can assume $b \leq \lambda(x)$ for all $x \in C$ and some fixed $b \in (a, c)$. Pick an arbitrary element $r \in (a, b)$. The function $f(\cdot, r) : C \to R$ is continuous on a still smaller box by the inductive hypothesis. We then replace C by this smaller box and obtain the fact that f is well-behaved at each point (p, r) with $p \in C$. Thus A^* is dense in A.

By the proof of the previous section, we can take a decomposition D of R^{m+1} that partitions both A and A^*.

We must show that for arbitrary cell $B \in D$ contained in A, $f\big|_B$ is continuous. Let B be an arbitrary cell contained in A. This implies $B \subset A^*$ because it partitions A^* and has nonempty intersection by density. By our construction of A^*, $f(\cdot, r)$ is continuous at p all $(p, r) \in B$. Finally, B is the union of boxes on which f is well-behaved. By the lemma, we have that f is continuous on B and the proof is complete. □

6. Conclusions

The powerful Cell Decomposition Theorem generalizes the two basic notions of o-minimal structures, namely o-minimality and monotonicity, so it implies many of the important tameness properties that we seek.

We know that every definable set has finitely many connected components and they partition the set.

The *dimension* of a nonempty definable set X is given by

$$\max\{i_1 + \ldots + i_m \mid X \text{ contains an } (i_1, \ldots, i_m)\text{-cell}\}.$$

This definition gives an intuitively pleasing notion of dimension. We can say that this definition is intuitively pleasing because its properties behave as we would expect geometric objects to behave.

For example:

- If $X \subset Y \subset R^m$ and X, Y are definable, then $\dim(X) \leq \dim(Y) \leq m$.
- If $X \subset R^m, Y \subset R^n$ are definable and in definable bijection with each other, then $\dim(X) = \dim(Y)$.
- If $X, Y \subset R^m$ are definable, then $\dim(X \cup Y) = \max\{\dim(X), \dim(Y)\}$.
- If X, Y are definable then $\dim(X \times Y) = \dim(X) + \dim(Y)$.
- Most importantly, if X is a nonempty definable set, then $\dim(\partial X) < \dim(X)$ and $\dim(\overline{X}) = \dim(X)$.

See [1] for proofs.

This last property is key because it lets us develop a stratification of definable sets (along with the Cell Decomposition Theorem).

Definition 6.1. A *stratification* S of a closed definable set $X \subset R^m$ is a partition of X into finitely many cells, called strata, such that for each strata $A \in S$, we have that ∂A is a union of lower dimensional strata.

One can use induction and the Cell Decomposition Theorem to show that such a stratification exists for every closed definable set (see [1]).

The main application of o-minimal structures is in the field of algebraic geometry, in particular Hodge theory.

In some cases, the o-minimal version of a theorem can imply the standard statement. So if the o-minimal theorem is easier to prove, this is helpful.

One example is Chow's Theorem.

The o-minimal statement is

Theorem 6.2 (O-minimal Chow)**.** *Let $Y \subset \mathbb{C}^n$ be a closed analytic subvariety whose underlying set is definable in an o-minimal structure S. Then Y is algebraic.*

Then the standard Chow's Theorem follows as a corollary.

Corollary 6.3. *Let $X \subset \mathbb{P}^n$ be a closed analytic variety. Then X is algebraic.*

See [10] for a proof and further discussion.

A famous conjecture from Hodge theory was recently proved in [11] so the interested reader can read there for more information about how o-minimal structures are used in an algebraic geometry context.

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