A Trip through Knot Theory

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October 2, 2011

What follows is the technical abstract:

A *knot* K is an embedding $K : S^1 \to \mathbb{R}^3$. By abuse of notation, Image$(K) = K$. To avoid pathological cases, we only study knots that have a tubular neighborhood. Such knots are called tame; unless specified otherwise, "knot" means "tame knot." An example of a non-tame knot will be given. We define the unknot as the natural embedding $S^1 = \left\{(x_1, x_2, 0) : x_1^2 + x_2^2 = 1\right\} \subset \mathbb{R}^3$.

Let X be a topological space. An *ambient isotopy* of two subspaces $Y, Y' \subset X$ is a family of homeomorphisms $f_t : X \to X$ such that $f_0 = \mathbb{1}_X$ and $f_1(Y) = Y'$. Two knots K_0 and K_1 are said to be of the same *type* if there is an ambient isotopy from one to the other. Note that the relation of ambient isotopy of knots is an equivalence relation, denoted $\simeq$. The knot problem asks: "If K_0 and K_1 are knots, is $K_0 \simeq K_1$?"

Of course, a category must be chosen to work out of. However, the tameness condition implies the knot problem is essentially the same, regardless of category. For technical reasons (e.g., compactness is useful), it is sometimes better to define knots as maps into S^3 instead of $\mathbb{R}^3$. The knot problem does not change in this setting either. The question can be generalized by considering tame links, where a *link* L is an embedding $L : \coprod_{i=1}^n S_i^1 \to S^3$. The theory of links is clearly much richer (e.g. *Brunnian* links).

There are some reformulations of the knot problem. For example, we define a knot diagram as a planar projection of a knot that has nice crossings–*i.e.*, projections such that the only singularities have preimage of cardinality 2; we then introduce the Reidemeister moves, and show that any two knots K_0, K_1 are of the same type if and only if every admissible diagram of K_0 can be changed to any diagram of K_1 using a sequence of Reidemeister moves. Another way to reformulate the knot problem is with braids. Consequently,

we define the braid group on n-strings B_n following Artin, as well as the operation of braid closure, which sends a braid $b \in B_n$ canonically to a link $\bar{b}$. A result of J.W. Alexander says that every link has a braid representative, *i.e.*, is the closure of some braid. While braid groups are relatively well understood (e.g. Markov moves and the fact that $\bar{b} = \bar{b}' \iff b$ and b' are conjugate in B_n) , the problem of finding a braid representative for a link is highly nontrivial. As an aside, we introduce the braid index $b(K)$ of a knot K (as well as the broader notion of a knot invariant) as the fewest strings necessary to create a braid representative for a knot. This proves existence of nontrivial knots, since b(unknot)=1 and b(trefoil knot)=2.

Next we discuss the knot group $G(K) := \pi_1(S^3 - K)$. We present how to compute it, using a knot diagram and the so-called Wirtinger presentation. Unfortunately, the knot group is not a complete knot invariant (we present examples showing this) since different knots can have isomorphic groups.

Next we present the operation of connect sum $K_0 \# K_1$ of two knots. Naturally accompanying the operation of connect sum is the notion of primeness: a knot K is *prime* if $K = K_1 \# K_2$ then $K = K_1$ or $K = K_2$ (so K_2 =unknot or K_1 =unknot, respectively). It can be shown that any knot is uniquely a connect sum of prime knots. Results of Gordon and Luecke imply that the fundamental group is a complete invariant (up to chirality, which we define) when restricted to prime knots, giving a partial solution to the knot problem.

Finally, we introduce the notions of hyperbolic knot, torus knot and satellite knot, in order that we may quote Thurston's result that every knot is at least one of the three.

Time permitting, we will delve into polynomial invariants of knots via skein relations and Khovanov homology.