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The Riemann–Hilbert problem

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Contents

Introduction

1. For each linear differential equation of order p

$$y^{(p)} + q_1(z)\, y^{(p-1)} + \ldots + q_p(z)\, y = 0 \tag{1}$$

with rational coefficients $q_i(z)$ defined on the complex plane C, there is an associated group called the *monodromy group* of equation (1). We denote by D the set $\{a_1, ..., a_n\}$ of singular points of (1) on the Riemann spheres CP^1. D consists of the poles of $q_1(z), ..., q_p(z)$ that belong to the finite complex plane and perhaps of the point ∞ (in the case when some of the coefficients of the reduced equation obtained from (1) by the substitution $\zeta = 1/z$ have a pole at $\zeta = 0$). We fix a point z_0 in $CP^1 \backslash D$ and a basis $(y_1, ..., y_n)$ in the space of solutions of (1) in a neighbourhood $O(z_0) \subset CP^1\backslash D$. The functions $y_1, ..., y_p$ admit analytic continuations along any path that does not intersect D. Let γ be a loop contained in $CP^1\backslash D$ with the initial and final point z_0. By performing the analytic continuation of $y_1, \ldots, y_p$ along γ, we obtain functions $\tilde{y}_1, \ldots, \tilde{y}_p$, which again form a basis of the space of solutions to (1), and so they are connected with the original functions by a relation of the form

$$(\tilde{y}_1, \ldots, \tilde{y}_p)\cdot G = (y_1, \ldots, y_p).$$

The correspondence $\gamma \mapsto G$ depends only on the homotopy class $[\gamma]$ of the loop γ and it defines the homomorphism

$$\chi: \quad \pi_1 (CP^1 \setminus D, z_0) \to GL(p; C) \tag{2}$$

of the fundamental group of $CP^1 \backslash D$ into the group of non-singular complex-valued matrices of order p.(1) The homomorphism (2) is called the *monodromy representation* or simply the *monodromy* of equation (1), and the group Im χ is called the *monodromy group* of this equation. If z_0 is replaced by z_0' or the basis $(y_1, \ldots, y_n)$ is changed, then the monodromy matrices G turn into $S^{-1}GS$, where S is a non-singular constant matrix. Thus, the monodromy of equation (1) is defined to within this equivalence.

Analogously, we can define the monodromy representation for a system of linear differential equations

$$df = \omega f, \tag{3}$$

where $\omega = ||\omega_{ij}||$ with $1 \leqslant i, j \leqslant p$ is the matrix of a differential 1-form that is holomorphic on $CP^1 \backslash D$. (For more details, see [30].)

Among systems (3) there is the class of *systems of Fuchsian type*. The latter systems are those whose matrix differential forms ω have simple poles at the points in D. We denote by $B^i = \mathrm{res}_{a_i} \omega$ the residues of ω at the points a_i. If ∞ is not among the singular points of a Fuchsian system (3), then the system is of the form

$$df = \Big(\sum_{i=1}^{n} B^i \frac{dz}{z - a_i} \Big) f, \tag{4}$$

where

$$\sum_{i=1}^{n} B^i = 0. \tag{5}$$

All singular points of a Fuchsian system (4) are regular. This means that any solution f increases no faster than a power of the distance $|z - a_i|$ as z tends to the singular point a_i over any sectorial neighbourhood distinct from C with vertex at a_i.

The class of systems (3) with regular singular points contains the class of Fuchsian systems as a proper subset (see [1]). In contrast to the systems for equation (1), in this case the notion of a Fuchsian system is equivalent to that of a system with regular singular points. If $a_i \in D$ is a regular singular point for (1), then one can show that

$$q_j(z) = \frac{r_j(z)}{(z - a_i)^j}, \quad j = 1, \ldots, p, \tag{6}$$

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(1)Strictly speaking, it is a question of a homorphism of the sliding group of the universal covering $\tilde{S}$ of the space $\mathrm{CP}^1 \backslash D$ into $GL(p; C)$. However, as a rule it is more convenient to deal with an object defined in a more concrete form such as the fundamental group, which can, in the well-known way, be identified by the sliding group.

where $r_j(z)$ are holomorphic functions in a neighbourhood of a_i. Equations (1) and (6) are said to be *Fuchsian* at a_i.

2. Riemann was the first to mention the problem of the reconstruction of a Fuchsian equation from its monodromy representation (2) in a note at the end of the 1850's. In 1900 Hilbert included it as Problem XXI on his list of 'Mathematical problems'. It was formulated as follows [2]:

'*Prove that there always exists a linear differential equation of Fuchsian type with given singular points and with a given monodromy group.*'

A tradition has been established in the literature that the problem for Fuchsian systems is called the Riemann – Hilbert problem.(1) By applying a conformal transformation of CP^1, we can always make sure that ∞ is not among the singular points of (3). Thus, the Riemann – Hilbert problem can be formulated as follows:

'*Let the representation* (2) *be given. Prove that there is always a system* (4), (5) *with the given monodromy* (2).'

The Riemann – Hilbert problem has been investigated by many mathematicians. There are several affirmative results on the solubility of the problem. (In the class of systems with regular singular points the analogous problem is always soluble.) A short review of these results is given in §1 of the present article. In §2 we present results due to Levelt [3] on the construction of spaces of solutions for Fuchsian systems and for systems with regular singular points. §3 has a preparatory character. In this section some facts of the theory of reducible systems are stated and basic technical results are described, which will be used in the following sections. In §5 the notion of the Fuchsian weight of the representation (2) is introduced and its properties are studied.

The following theorems represent the basic results of this article (see [4]):

***Theorem* 1.** *For any irreducible representation* (2) *of dimension* $p = 3$, *the Riemann – Hilbert problem is soluble.*

***Theorem* 2.** *For any points* a_1, a_2, a_3 *and any representation* (2) *of dimension* $p = 3$, *the Riemann – Hilbert problem is soluble.*

***Theorem* 3.** *For any* $n > 3$, *any sequence of points* $a_1, \ldots, a_n$, *and any* $p \geqslant 3$, *there is a representation* (2) *for which there are no Fuchsian systems that realize the representation.*

Theorem 3 means that the Riemann – Hilbert problem has a negative solution.

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(1)There is another problem with the same name, but it is not considered here (see, for example, [11]).

Theorem 1 is proved in §4. Theorems 2 and 3 are proved in §6 on the basis of the results of §5. In §6 we also describe all representations of dimension $p = 3$ that cannot be realized by Fuchsian systems (Corollary 6.1) and we give a concrete example of such a representation for $n = 4$ (Example 6.1). In §7 Grothendieck indices [5] for a two-dimensional holomorphic vector bundle on CP^1 with an irreducible connection are interpreted in terms of the Fuchsian weight of the monodromy of the connection. In the Conclusion we show the relations between the solubility of the Riemann–Hilbert problem in various settings, including the problem in the class of systems with regular singular points and equations with dummy singularities. We also give a short review of the literature on the multidimensional Riemann–Hilbert problem.

I wish to express my deep gratitude to V.A. Golubeva and V.P. Leksin, who let me know of the article [6], and also to D.V. Anosov and A.V. Chernavskii, who made a number of valuable remarks on improvements to the proofs.

§1. A short review of results

1.1. For a long time it was believed that the Riemann–Hilbert problem had been completely solved by Plemelj [7] in 1908. However, at the beginning of the 1980's some gaps were discovered in his proof (see [8] and [9]). The method of solution proposed by Plemelj consisted in reducing the Riemann–Hilbert problem to the so-called homogeneous Hilbert boundary-value problem of the theory of singular integral equations. A detailed study of the latter problem is contained in the books by Muskhelishvili [10] and Vekua [11]. The reduction is carried out in the following way.

Let all the points $a_1, \ldots, a_n$ lie on the finite complex plane. We join the points by a simple closed contour L (Fig. 1),

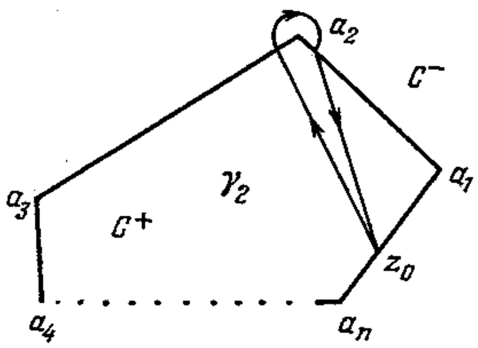


Fig. 1

and we define a piecewise constant non-singular matrix-valued function $g(t)$ on L by

$$g(t) = G_i \cdot \ldots \cdot G_1, \quad t \in [a_i; a_{i+1}),$$

where G_i is the matrix of the monodromy (2) corresponding to a "small" loop around a_i. We denote by C^+ the domain of the finite complex plane bounded by L, and by C^- the complement of $\overline{C}^+$ in CP^1.

We consider the following problem: find all pairs of vector-valued functions $\varphi^+ = (\varphi^+_{(1)}, \ldots, \varphi^+_{(p)})$ and $\varphi^- = (\varphi^-_{(1)}, \ldots, \varphi^-_{(p)})$ such that:

1) $\varphi^+(z)$ is holomorphic in C^+ and $\varphi^-(z)$ is holomorphic in C^- with a finite order at ∞;

2) $\varphi^+(z)$ and $\varphi^-(z)$ are continuous right up to the contour L with the exception of the points $a_1, \ldots, a_n$, and on $(a_i; a_{i+1})$ the functions are connected by the relation

$$\varphi^+(t)\, g(t) = \varphi^-(t);$$

3) $\varphi^\pm(z)(z-a_i)^\varepsilon$ tends to zero for some $0 \leqslant \varepsilon < 1$ as z approaches a_i over C^+ and C^-, respectively.

This problem can be reduced to the problem with a continuous function $g(t)$ (Plemelj and Vekua realize the reduction in different ways), which can be solved by the methods of the theory of singular integral equations. In this connection it turns out that there is always a system of solutions $\varphi_1^\pm, \ldots, \varphi_p^\pm$ such that the following conditions are satisfied:

a) the determinant of the matrix T with rows composed of the vector-valued functions $\varphi_1^\pm(z), \ldots, \varphi_p^\pm(z)$ is non-zero at any point of the complex plane C, except at the points belonging to D;

b) the matrix $z^S.T(z)$, where S is an integral diagonal matrix, is holomorphically invertible at ∞.

Each function $\varphi_i^+(z)$ of the system of solutions admits an analytic continuation to any point in $C\backslash D$ along any path that does not intersect the set D of singular points. It follows from condition 2) that φ_i^+ turns into $\varphi_i^+(G_i^{-1})$ under the analytic continuation along a small loop around a_i (in Fig. 1, $i = 2$), and so the same is true for the matrix $T(z)$. Therefore, under the analytic continuation along a loop with the initial and final point z_0, the matrix T turns into $\widetilde{T}$, where

$$\widetilde{T}G = T, \quad G = \chi([\gamma]).$$

The same property holds for the matrix

$$T' = (z - a_1)^S T,$$

which, according to property b) of the system of solutions in question, is now holomorphically invertible at ∞.

Thus, it follows from property a) that the matrix form

$$\omega = dT'\,(T')^{-1} \tag{1.1}$$

is single-valued on CP^1 and holomorphic everywhere, except at the singular points $a_1, \ldots, a_n$ belonging to D. The system (3) with matrix form (1.1) has the given monodromy (2), and $a_1, \ldots, a_n$ are regular singular points for the system.

Next, Plemelj applies a procedure that enables him to proceed from the constructed system to another system with the same monodromy and the same singular points, which is now a Fuchsian system for all except perhaps one of the points. This part of Plemelj's proof does not give rise to any objections. However, as regards the assertion that for the remaining point the system can also be reduced to a Fuchsian one, there is no rigorous proof for the general case in Plemelj's work. Nevertheless, Plemelj's argument is valid if one of the monodromy matrices G_i can be diagonalized (see [9]).

Thus, the solubility of the Riemann–Hilbert problem is proved in Plemelj's work in the case when one of the matrices G_i, namely that corresponding to a "small" loop around a_i, can be diagonalized.

Plemelj was also the first to solve the analogue of the Riemann–Hilbert problem in the class of systems with regular singular points.

1.2. After the publication of Plemelj's result [7], the subject matter of articles connected with the Riemann–Hilbert problem moved, basically, towards the effective construction of a Fuchsian system with given monodromy matrices $G_1, \ldots, G_n$. At the end of the 1920's Lappo-Danilevskii used his own method of analytic functions of matrices to express the solutions of a Fuchsian system and the monodromy matrices $G_1, \ldots, G_n$ in the form of convergent series of the matrix coefficients of the system (see [12]). In this case the effective solution of the Riemann–Hilbert problem was reduced to the reversion of the resulting series and to the study of the question of convergence. An affirmative solution to this question was found in [12] for matrices $G_1, \ldots, G_n$ close to the identity matrix. Thereby, Lappo-Danilevskii proved the solubility of the Riemann–Hilbert problem for representations (2) such that the monodromy matrices corresponding to "small" loops around the points a_i are close to the identity matrix.

In 1956 Krylov [13] proved by constructing an effective solution that the Riemann–Hilbert problem for a representation (2) of dimension $p = 2$ is soluble in the case of three singular points. Erugin [14] considered the analogous problem with four singular points. In particular, he established a connection between this problem and the Painlevé equation.

The problem concerning the effective construction of an equation of Fuchsian type is closely related to the problem of evaluation of accessory coefficients, which is extremely important for applications, but it is not considered in the present article. Information about this problem can be found in [15], [16], and [14].

1.3. In 1957 a new stage in the study of the Riemann–Hilbert problem was opened by the article [17] by Röhrl, who was the first to apply the methods of the theory of fibre bundles to the solution of the problem.[(1)] From the

(1) In fact considerations of this kind go back to Birkhoff, who proved Plemelj's result in [21]. However, at that time an adequate geometric language was not available to describe the problem in an appropriate way.

representation (2) Röhrl constructs a principal bundle on $CP^1\backslash D$ with the structure group $GL(p;\ C)$. Later, we shall need the description of this bundle in coordinates.

Let us consider a finite covering $\{U_\alpha\}$ of $CP^1\backslash D$ by simply-connected open sets U_α with simply-connected intersections. Let us fix paths γ_α from z_0 to some fixed points $z_\alpha \in U_\alpha$. For $z \in U_\alpha \cap U_\beta$, we denote by $t_\alpha(z)$ a path from z_α to z contained in U_α. We define the transition function

$$g_{\alpha\beta}(z) = \chi\,([\gamma_\alpha t_\alpha(z)\, t_\beta^{-1}(z)\, \gamma_\beta^{-1}]) \tag{1.2}$$

on $U_\alpha \cap U_\beta$. It is obvious that the functions $g_{\alpha\beta}(z)$ are constant on $U_\alpha \cap U_\beta$. It is not difficult to check that $g_{\alpha\beta} = g_{\beta\alpha}^{-1}$ and the cocycle condition

$$g_{\alpha\beta}\cdot g_{\beta\gamma} = g_{\alpha\gamma}$$

holds. We denote by F' the holomorphic bundle on $CP^1\backslash D$ constructed above.

The base $CP^1\backslash D$ of F' is contractible to its one-dimensional skeleton, hence the complex bundle F' is topologically equivalent to the trivial bundle (see [18]). Since $CP^1\backslash D$ is a Stein manifold, F' is holomorphically trivial (see [19]). Let $\{T_\alpha(z)\}$ be a holomorphic trivialization of the bundle F'. We shall consider the family $\{\omega_\alpha\}$ of matrix differential 1-forms, where

$$\omega_\alpha = -T_\alpha^{-1}\cdot dT_\alpha. \tag{1.3}$$

Since $T_\alpha^{-1}(z)g_{\alpha\beta}(z) = T_\beta^{-1}(z)$ on $U_\alpha \cap U_\beta$ and $g_{\alpha\beta}(z)$ is constant, it follows that $\omega_\alpha = \omega_\beta$ on $U_\alpha \cap U_\beta$. Thus, (1.3) defines a global matrix differential form ω that is holomorphic everywhere on $CP^1\backslash D$. It follows for the construction that the system (3) with the form ω has the given monodromy (2). Next, Röhrl extends F' onto the entire Riemann sphere CP^1 using the section $\{T_\alpha\}$. The extended bundle always has a meromorphic section that is holomorphically invertible everywhere except for the points in D. The system (3) constructed from this section has the given monodromy and $a_1, \ldots, a_n$ are regular singular points for the system.

In this way, Röhrl proved Plemelj's results as well as the solubility of the Riemann–Hilbert problem on a non-compact Riemann surfrace. Moreover, in [17] Röhrl proved also that the analogue of the Riemann–Hilbert problem for an arbitrary Riemann surface is soluble in the class of systems with regular singular points. (In fact, generally speaking, additional "dummy" singularities, which do not contribute to the monodromy representation, appear for the constructed system.)

The "topological" approach enables one to evaluate the set of representations for which a slightly modified Riemann–Hilbert problem is soluble (Il'yashenko [9]): "There is a countable union of proper analytic submanifolds of $(GL(p;\ C))^{n-1}$ such that any sequence $G_1, \ldots, G_{n-1}$ contained in the complement of the union has the following property: for any sequence $(z-a_1)^{E_1}, \ldots, (z-a_n)^{E_n}$ of Fuchsian principal parts with $\exp 2\pi i E_j = G_j$ for

$j = 1, \ldots, n$, $G_n = (G_1 \cdot \ldots \cdot G_{n-1})^{-1}$, and $\sum_{j=1}^{n} \operatorname{Sp} E_j = 0$, there is a Fuchsian system whose fundamental matrix of solutions has the given Fuchsian principal parts."

1.4. In 1979 the paper [6] by Dekkers appeared. It follows from the results of this paper that the Riemann–Hilbert problem is soluble for any sequence of points $a_1, \ldots, a_n$ and any representation (2) of dimension $p = 2$. The method of solution consists in modifying the system (3) with regular singular points and with a given monodromy representation so that it becomes a Fuchsian system. This is done with the aid of the scale transformations

$$g = \Gamma(z) f, \tag{1.4}$$

under which (3) turns into the system $dg = \omega' g$ with the form

$$\omega' = d\Gamma \cdot \Gamma^{-1} + \Gamma \omega \Gamma^{-1}. \tag{1.5}$$

If the matrix-valued function $\Gamma(z)$ is meromorphic on CP^1 and holomorphically invertible outside D, then (3) with (1.5) is also a system with regular singular points $a_1, \ldots, a_n$ and with the same monodromy representation. Using Deligne's result [20] on the form of the matrix of coefficients of a system with regular singular points, Dekkers proved that there is a matrix $\Gamma(z)$ that reduces ω to the matrix ω' in (1.5), which has singularities of the type of simple poles at the points belonging to D.

§2. Criteria for Fuchsian systems with regular singular points

2.1. Let us consider an arbitrary system (3) whose form ω is holomorphic on $CP^1 \setminus D$. The set of solutions X of such a system is a p-dimensional vector space of vector-valued holomorphic functions on the universal covering

$$(\tilde{S}, y_0) \xrightarrow{\pi} (CP^1 \setminus D, z_0).$$

In what follows we shall write y to denote points in $\tilde{S}$ and we shall write z to denote the corresponding points $\pi(y)$ in CP^1.

Once the points $z_0 \in CP^1 \setminus D$ and $y_0 \in \pi^{-1}(z_0)$ are fixed, the fundamental group $\pi_1(CP^1 \setminus D, z_0)$ can be identified with the sliding group of the covering. Thus, we can define the action g^* of an element $g \in \pi_1(CP^1 \setminus D, z_0)$ on a function $f(y) \in X$ in the following way:

$$(g^* f)(y) = f(g^{-1} y). \tag{2.1}$$

The action defines the monodromy representation (2) for the system (3). For if $(e) = (e_1(y), \ldots, e_p(y))$ is a basis in X, then according to (2.1) we have

$$(g^* T)(y) = T(g^{-1} y) = T(y)\, \chi(g),$$

where $T(y)$ is the fundamental matrix of X constructed from (e). (This means, in particular, that the space of solutions X is invariant under the action of the fundamental group $\pi_1(CP^1 \setminus D, z_0)$ given by (2.1).)

Let O_i be simply-connected pairwise disjoint neighbourhoods of the points a_i in CP^1, and let

$$(\tilde{S}_i, y_i^0) \xrightarrow{\pi_i} (O_i \setminus a_i, z_i^0)$$

be the corresponding universal coverings. Let δ_i be a loop in $O_i \setminus a_i$ passing once around a_i in the clockwise direction with the initial and final point z_i^0. We fix paths γ_i from z_0 to z_i^0 contained in $CP^1 \setminus D$ in such a way that the loop $\prod_{i=1}^{n} (\gamma_i \cdot \delta_i \cdot \gamma_i^{-1})$ will be homotopic with the constant path (the revolution relation). We denote the class of loops $\gamma_i \cdot \delta_i \cdot \gamma_i^{-1}$ by g_i. Hence, embeddings $\tilde{S}_i \subset \tilde{S}$ are defined (under such an embedding y_i^0 turns into $y_0\gamma_i$, the final point of the lifted path γ_i) and

$$(2.2) \qquad \gamma_i^*\colon \pi_1 (O_i \setminus a_i, z_i^0) \to \pi_1 (CP^1 \setminus D, z_0).$$

We shall call the homomorphism

$$(2.3) \qquad \chi_i\colon \pi_1 (O_i \setminus a_i, z_i^0) \to GL (p; C),$$

where $\chi_i = \chi \circ \gamma_i^*$, the *i-th local representation* constructed from the representation (2).

The embedding (2.2) enables us to define the action of the element $g_i \in \pi_1(CP^1 \setminus D, z_0)$ described above on any function defined on $\tilde{S}_i$, in particular on the function $\ln(y-a_i)$ defined as follows:

$$\ln (y - a_i) = \int_{\gamma} \frac{d(z - a_i)}{z - a_i},$$

where γ is a path in $O_i \setminus a_i$ connecting z_i^0 and $\pi_i(y)$ such that $y_i^0\gamma = y$. It follows from (2.1) and (2.2) that

$$(2.4) \qquad g_i^* \ln (y - a_i) = \ln (y - a_i) + 2\pi i.$$

From now on, $(y-a_i)^{E_i}$ will be understood to be the function

$$(y - a_i)^{E_i} = \exp (E_i \ln (y - a_i))$$

defined on $\tilde{S}_i$.

The fundamental group $\pi_1(CP^1 \setminus D, z_0)$ can be regarded as a group with n generators $g_1, \ldots, g_n$ satisfying the identity relation $g_1 \cdot \ldots \cdot g_n = e$. We denote by G_i and E_i the matrices

$$(2.5) \qquad G_i = \chi(g_i), \quad E_i = \frac{1}{2\pi i} \ln G_i, \quad 0 \leqslant \operatorname{Re} \rho_i^j < 1,$$

where ρ_i^j are the eigenvalues of E_i.

2.2. We proceed to the description due to Levelt [3] of the construction of the space X of solutions of (3) in a neighbourhood of a singular point a_i.

Any component $f_j(y)$ of a vector-valued function $f(y) \in X|_{\tilde{S}_i}$ can be represented as the so-called logarithmic sum (see [1]):

$$f_j(y) = \sum_{k,l \in \sigma} (y - a_i)^{\rho_k} h_{kl}(z) \ln^{b_l}(y - a_i), \tag{2.6}$$

where the sum is finite, $0 \leqslant \operatorname{Re} \rho_k < 1$, $b_l \in \mathbb{Z}$, $b_l > 0$, and each $h_{kl}(z)$ is a Laurent series with a finite principal part, and where similar terms (with respect to the pairs (ρ_k, b_l)) are collected.

Definition 2.1. The order of the zero (the order of the pole with a "minus" sign) of $h_{kl}(z)$ at a_i is called the *normalization* $\varphi_i(h_{kl}(z))$ *at* a_i or the *i-th normalization of the Laurent series* $h_{kl}(z)$ *with a finite principal part.*

Definition 2.2. The number

$$\varphi_i(f_j) = \min_{k,l \in \sigma} \varphi_i(h_{kl})$$

is called the *i-th normalization* $\varphi_i(f_j)$ *of the finite logarithmic sum* (2.6). The number

$$\varphi_i(f(y)) = \min_{j=1,\dots,p} \varphi_i(f_j)$$

is called the *i-th normalization of the vector-valued function* $f(y) \in X|_{\tilde{S}_i}$.

By definition, we set ∞ as the normalization of $f(y) \equiv 0$.

Example 2.1. For $f_1(z) = z^2 - \frac{1}{z}$, $f_2(z) = y^{1/2} \frac{1}{z^3} \ln^2 y$, and $f_3(y) = (z, z \ln y, z^2)$, the normalizations $\varphi(f)$ at 0 are

$$\varphi(f_1) = -1, \quad \varphi(f_2) = -3, \quad \varphi(f_3) = 1.$$

By Definition 2.2 and the form of (2.6), it is easy to establish that the normalizations have the following properties:

1) $\varphi_i(g_i^* f) = \varphi_i(f)$ for $i = 1, \dots, n$;

2) $\varphi_i(cf) = \varphi_i(f)$ for all $c \in C \backslash 0$, $\varphi_i(f+g) \geqslant \min(\varphi_i(f), \varphi_i(g))$, and if $\varphi_i(f) \neq \varphi_i(g)$, then $\varphi_i(f+g) = \min(\varphi_i(f), \varphi_i(g))$;

3) $f(y)(y-a_i)^{-\lambda} \to 0$ for all $\lambda < \varphi_i(f)$ as z tends to a_i over any sectorial neighbourhood O distinct from the complex plane C with vertex at a_i. (Here we assume that $y \in \pi_i^{-1}(z)$ remains inside one sheet of $\pi_i^{-1}(O)$.) Moreover, $\varphi_i(f)$ is the greatest of all integers k satisfying the above condition for all $\lambda < k$.

The space $X|_{\tilde{S}_i}$ can be decomposed into a direct sum of subspaces ${}^mX$ of dimension p_m invariant under the action of g_i^* and corresponding to the eigenvalues $\varkappa_m$ of g_i^*. The normalization φ_i takes k_m distinct finite values ${}^m\varphi_i^1 > \dots > {}^m\varphi_i^{k_m}$ on ${}^mX$ ($k_m \leqslant p_n$) and defines a filtration of ${}^mX$ by subspaces

$$0 \subset {}^mX^1 \subset \dots \subset {}^mX^{k_m} = {}^mX,$$

where ${}^mX^l = \{f \in {}^mX | \varphi_i(f) \geqslant {}^m\varphi_i^l\}$. Moreover, according to property 1), the monodromy operator g_i^* preserves the filtration. We choose a basis ${}^me_1, \ldots, {}^me_{l_1}$ in ${}^mX^1$ in which the operator $g_i^*|_{{}^mX^1}$ has the upper triangular form, and we complement the basis so that it becomes a basis in ${}^mX^2$ in which $g_i^*|_{{}^mX^2}$ has the upper triangular form again, and so on. In this way, we obtain a basis $({}^me)$ of ${}^mX$. From $({}^me)$ we form a basis (e) of X. This basis, which is called the *associated basis* in [3], has the following properties:

a) φ_i takes all of its values on the elements of (e), with all multiple values accounted for;

b) $\varphi_i({}^me_j) \geqslant \varphi_i({}^me_{j+1})$ for all m, j;

c) the matrix G_i of g_i^* has the upper triangular form in the basis (e).

We denote by A_i the diagonal matrix of normalizations $A_i = \operatorname{diag}(\varphi_i(e_1), \ldots, \ldots, \varphi_i(e_p))$ for the associated basis (e) of the space $X|_{\tilde{S}_i}$. From properties b) and c) we immediately obtain the following result.

***Proposition* 2.1.** *The matrix-valued function*

$$(z-a_i)^{A_i} E_i (z-a_i)^{-A_i}, \tag{2.7}$$

where E_i is given by (2.5), *is holomorphic in a neighbourhood O_i of a_i. For any $\varepsilon > 0$ the matrix*

$$(z-a_i)^{A_i} (y-a_i)^{E_i} (z-a_i)^{-A_i} (y-a_i)^{\varepsilon}$$

tends to zero as z tends to a_i over any sectorial neighbourhood O distinct from the entire plane C with vertex at a_i. (Here we assume that $y \in \pi_i^{-1}(z)$ remains in one sheet of $\pi_i^{-1}(O)$.)

We denote by $T_i(y)$ the fundamental matrix of $X|_{\tilde{S}_i}$ constructed from the associated basis (e).

***Proposition* 2.2.** *$T_i(y)$ can be expressed in the form*

$$T_i(y) = U_i(z)(z-a_i)^{A_i}(y-a_i)^{E_i}, \tag{2.8}$$

where E_i is given by (2.5) *and $U_i(z)$ is single-valued and holomorphic in a neighbourhood of a_i.*

Proof. $U_i(z)$ is single-valued, since by (2.1) and (2.4)

$$(g_i^* T_i)(y) = G_i = g_i^*(y-a_i)^{E_i}.$$

To show that $U_i(z)$ is holomorphic, it is sufficient to prove that $U_i(z)(y-a_i)^{\varepsilon} \to 0$ for any $\varepsilon > 0$ as z tends to a_i over any sectorial neighbourhood distinct from C. But

$$U_i(z)(y-a_i)^{\varepsilon} = T_i(y)(y-a_i)^{-E_i}(z-a_i)^{-A_i}(y-a_i)^{\varepsilon} = S_1(y)\,S_2(y),$$

where

$$S_1(y) = T_i(y)(y-a_i)^{-A_i+(\varepsilon/2)I},$$
$$S_2(y) = (z-a_i)^{A_i}(y-a_i)^{-E_i}(z-a_i)^{-A_i}(y-a_i)^{\varepsilon/2}.$$

Since the j-th column of $S_1(y)$ is equal to $e_j(y)(y-a_i)^{-\varphi_i(e_j)+\varepsilon/2}$, it follows from property 3) of the normalizations that $S_1(y) \to 0$ as $z = \pi_1(y)$ tends to a_i over any sectorial neighbourhood. From Proposition 2.1 we obtain the same property for $S_2(y)$.

Remark 2.1. It follows from the properties of the associated basis and from Definition 2.2 of the normalizations that the first column of U_i in (2.8) does not vanish at a_i.

Definition 2.3. The eigenvalues $\beta_i^j = \varphi_i^j + \rho_i^j$ of $A_i + E_i$ are called the *i-th exponents of X at a_i*.

***Proposition* 2.3.** *The system* (3) *with a regular singular point a_i is Fuchsian at a_i if and only if*

$$\det U_i(a_i) \neq 0 \tag{2.9}$$

in the factorization (2.8) *for the space X of solutions of the system.*

Proof. Sufficiency. According to (2.8), the matrix form ω in (3) can be expressed as

$$\begin{aligned}(2.10)\quad \omega = dT_i \cdot T_i^{-1} = \\ = \left(\frac{dU_i}{dz} U_i^{-1} + \frac{U_i}{z-a_i}(A_i + (z-a_i)^{A_i} E_i (z-a_i)^{-A_i}) U_i^{-1}\right) dz\end{aligned}$$

in a neighbourhood of a_i. If $\det U_i(a_i) \neq 0$, then ω has a pole of the first order at a_i, that is, (3) is a Fuchsian system at a_i.

Necessity. Suppose that the form ω in (3) can be written as

$$\omega = \frac{B^i}{z-a_i} dz + \psi$$

in a neighbourhood of a_i, where ψ is a holomorphic form. From (2.10) we find that

$$\begin{aligned}(2.11)\qquad B^i U_i(a_i) = U_i(a_i) L_i, \\ L_i = A_i + \lim_{z \to a_i} (z-a_i)^{A_i} E_i (z-a_i)^{-A_i}.\end{aligned}$$

Suppose that $\det U_i(a_i) = 0$. We denote by Y the kernel of $U_i(a_i)$:

$$Y = \{v \in C^p / U_i(a_i) v = 0\}.$$

It follows from (2.11) that L_i transforms Y into itself. Let v be an eigenvector of L_i belonging to Y. Then

$$L_i v = \beta_i^j v \tag{2.12}$$

for some j, where β_i^j is one of the exponents of X at a_i. It follows from (2.12) and the form of L_i that the vector-valued function $w = T_i \cdot v$, represented in (e) by the column vector v, belongs to one of the subspaces ${}^mX^l$ and is a linear combination of the base vectors (e) with zero coefficients corresponding to the part of the basis (e) that is contained in ${}^mX^{j-1}$. Therefore, $\varphi_i(w) = \varphi_i^j = {}^m\varphi_i^l$ and

$$(2.13) \qquad (z-a_i)^{-A_i}v = v(z-a_i)^{-\varphi_i^j}.$$

We denote by $\widetilde{A}_i$ the matrix $\widetilde{A}_i = \operatorname{diag}(\beta_i^1, \ldots, \beta_i^p)$ and by $\widetilde{E}_i$ the nilpotent matrix $E_i - R$, where $R = \operatorname{diag}(\rho_i^1, \ldots, \rho_i^p)$. From (2.13) we obtain

$$w(y)(y-a_i)^{-\beta_i^j} = T_i(y)v(y-a_i)^{-\beta_i^j} = U_i(z)(y-a_i)^{\widetilde{A}_i}(y-a_i)^{\widetilde{E}_i}(y-a_i)^{-\widetilde{A}_i}v =$$

$$= U_i(z)\Big[\sum_{l=0}^{p-1}\frac{1}{l!}\ln^l(y-a_i)(L_i-\widetilde{A}_i+o(z-a_i))^l\Big]v =$$

$$= U_i(z)v + \sum_{l=1}^{p-1}\frac{1}{l!}\ln^l(y-a_i)[(L_i-\widetilde{A}_i)^l+o(z-a_i)]v =$$

$$= U_i(z)v + o(z-a_i)\Big[\sum_{l=1}^{p-1}\frac{1}{l!}\ln^l(y-a_i)\Big]v.$$

Since $0 \leqslant \operatorname{Re}\rho_i^j < 1$ and $U_i(a_i)v = 0$, it follows that $w(y)(y-a_i)^{-\lambda} \to 0$ for all $\lambda < \varphi_i^j + 1$ as z tends to a_i over any sectorial neighbourhood of a_i. Thus, property 3) of normalizations implies that $\varphi_i(w) \geqslant \varphi_i^j + 1$, which contradicts the equality $\varphi_i(w) = \varphi_i^j$. The contradiction means that $\det U_i(a_i) \neq 0$.

***Corollary* 2.1.** *The eigenvalues of each matrix B^i of a Fuchsian system* (4) *coincide with the i-th exponents of the space X of solutions of the system.*

The proof follows from (2.10), since

$$(2.14) \qquad B^i = \lim_{z\to a_i}(z-a_i)\frac{dT_i}{dz}T_i^{-1} = U_i(a_i)L_iU_i^{-1}(a_i).$$

***Proposition* 2.4.** *The sum Σ of all exponents of the space X of solutions of the system* (3) *with regular singular points $a_1, \ldots, a_n$ is a non-positive integer*:

$$(2.15) \qquad \Sigma = \sum_{i=1}^{n}\sum_{j=1}^{p}\beta_i^j \leqslant 0.$$

The system (3) *with regular singular points is a system of Fuchsian type on CP^1 if and only if*

$$(2.16) \qquad \Sigma = 0.$$

Proof. Let us consider the form $\operatorname{Sp}\omega$ on CP^1. If $T(y)$ is the fundamental matrix of the space of solutions X, then

$$\operatorname{Sp}\omega = d\ln\det T,$$

hence we find from (2.8) that

$$\operatorname*{res}_{a_i} \operatorname{Sp}\omega = b_i + \sum_{j=1}^{p} \beta_i^j,$$

where b_i is the order of the zero of det $U_i(z)$ at a_i. By the theorem on the sum of residues,

$$\sum_{i=1}^{n}\sum_{j=1}^{p} \beta_i^j + \sum_{i=1}^{n} b_i = 0.$$

Proposition 2.4 follows now from Proposition 2.3 and the fact that $b_i \in \mathbb{Z}$ and $b_i \geqslant 0$ for $i = 1, \ldots, n$.

§3. Preliminary information. The method of solution

3.1. The method used to prove Theorems 1 and 2 stated in the Introduction consists in modifying the system (3) with regular singular points $a_1, \ldots, a_n$ and with given monodromy representation (2), the existence of which was proved in [7], by meromorphic transformations (1.4) on CP^1 that are holomorphically invertible outside the set of points $a_1, \ldots, a_n$ in such a way that it becomes a Fuchsian system. In so doing we use the criteria for Fuchsian systems stated in Propositions 2.3 and 2.4 and a technical procedure called the "$A(a_i, l, j, k)$ procedure" and described in the following lemma.

***Lemma* 3.1.** *Let the l-th row $u_l(z)$ of a matrix $U(z)$ that is holomorphic in a neighbourhood O_i of a_i be of the form*

$$u_l(z) = (z - a_i)^k\, v_l(z), \tag{3.1}$$

where the j-th component $v_{lj}(z)$ of the row vector $v_l(z)$ is holomorphic in O_i and $v_{lj}(a_i) \neq 0$. There is a meromorphic matrix-valued function $\Gamma(z)$ on CP^1 holomorphically invertible at any point except a_i such that the j-th row $x_j(z)$ of the matrix $U' = \Gamma U$ has the form

$$x_j(z) = (z - a_i)^k\, x_j'(z), \tag{3.2}$$

where $x_j'(z)$ is holomorphic in O_i, $x_{lj}'(a_i) \neq 0$, and *$x_{mj}'(a_j) = 0$ for $m \neq l$, x_{mk}' being the m-th component of the row vector x_j'.*

If the row vector $v_l(z)$ in (3.1) *is holomorphic in O_i, then all of the elements of U' are also holomorphic in O_i.*

Proof. Let us consider an element $u_{mj}(z)$ of the matrix $U(z)$ with $m \neq l$. If $u_{mj}(z) \not\equiv 0$, then the element can be written as $(z-a_i)^s t(z)$, where $t(z)$ is a holomorphic function in O_i and $t(a_i) \neq 0$. Let $0 \leqslant s \leqslant k$. There is a polynomial $Q_m\left(\frac{1}{z-a_i}\right)$ of degree $k-s$ such that

$$Q_m\left(\frac{1}{z-a_i}\right) u_{lj}(z) + u_{mj}(z) = (z - a_i)^{k+1} f(z), \tag{3.3}$$

where $f(z)$ is a holomorphic function in O_i.

The coefficients of the polynomial $Q_m\left(\frac{1}{z-a_i}\right)$ are defined as follows. As the leading coefficient c_1 corresponding to $\left(\frac{1}{z-a_i}\right)^{k-s}$ we take

$$c_1 = -\frac{t(a_i)}{v_{lj}(a_i)}.$$

It is obvious that

$$\frac{c_1}{(z-a_i)^{k-s}} u_{lj}(z) + u_{mj}(z) = (z-a_i)^r t'(z),$$

where $s < r$ and $t'(z)$ is a holomorphic function in O_i such that $t'(a_i) \neq 0$. If $r \leqslant k$, then we set $c_2 = -\frac{t'(a_i)}{v_{lj}(a_i)}$ for the coefficient c_2 corresponding to the next term $\frac{1}{(z-a_i)^{k-r}}$, and so on. But if the order s of the zero of $u_{mj}(z)$ at a_i exceeds k, then we set $Q_m \equiv 0$. We consider the matrix

$$\text{(3.4)} \qquad \Gamma(z) = \begin{pmatrix} 1 & 0 & \dots & 0 & Q_1 & 0 & \dots & 0 \\ 0 & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & \cdot & \cdot & & & \cdot \\ \cdot & & & & Q_{l-1} & & & \\ & & & & 1 & & & \\ & & & & Q_{l+1} & & & \\ & & & & \cdot & & & \\ & & & & \cdot & & & \\ & & & & \cdot & & & 0 \\ 0 & & \dots & 0 & Q_p & 0 & \dots & 1 \end{pmatrix}.$$

From the construction of $\Gamma(z)$ and (3.3) we obtain assertion (3.2) of the lemma.

If the row $v_l(z)$ in (3.1) is holomorphic in O_i, it follows that $U' = \Gamma U$ is holomorphic, since $U(z)$ is holomorphic and $\deg Q_m \leqslant k$ for $m = 1, \dots, \hat{l}, \dots, p$.

The passage from U to $U' = \Gamma U$ and from X to $X' = \Gamma X$, where $\Gamma(z)$ is given by (3.4), will from now onwards be called the "*$A(a_i, l, j, k)$ procedure*".

Let us consider the space X of solutions of a system (3) that is Fuchsian at $a_1, \dots, \hat{i}, \dots, a_n$ and has a regular singularity at a_i. We consider factorizations (2.8) for X at $a_1, \dots, a_n$.

***Lemma* 3.2.** *Let the l-th row u_l of the matrix U_i in the factorization* (2.8) *for X at a_i be of the form*

$$\text{(3.5)} \qquad u_l(z) = (z-a_i)^k v_l(z),$$

where $k > 0$ and $v_l(z)$ is a row vector holomorphic in O_i with $v_{lt}(a_i) \neq 0$. If the normalizations φ_i^t for X satisfy the inequality $\varphi_i^{t-1} \geqslant \varphi_i^t + k$, then the space $X' = \Gamma(z)X$ obtained from X by applying the $A(a_i, l, t, k)$ procedure is the *space of solutions of a system* (3) *that is Fuchsian at the points* $a_1, \ldots, \hat{i}, \ldots, a_n$, *and the normalizations* $'\varphi_m^j$ *and* φ_m^j *for* X' *and* X *are connected by the following relations*:

$$(3.6)\qquad \begin{aligned} &'\varphi_m^j = \varphi_m^j,\ j = 1, \ldots, p;\quad m \neq i;\\ &'\varphi_i^t = \varphi_i^t + k,\quad '\varphi_i^j \geqslant \varphi_i^j,\quad j = 1, \ldots, \hat{t}, \ldots, p. \end{aligned}$$

Proof. The system (3) with the space of solutions X' is Fuchsian at $a_1, \ldots, \hat{i}, \ldots, a_n$ and the first equality in (3.6) holds by virtue of Propositions 2.2 and 2.3 and the fact that the factorizations (2.8) for X' have the form

$$T'_m(y) = \Gamma(z) U_m(z) (z - a_m)^{A_m} (y - a_m)^{E_m}$$

with $\det \Gamma(a_m) \neq 0$ for $m \neq i$, where $\Gamma(z)$ is given by (3.4).

We set $C = \mathrm{diag}(0, \ldots, 0, k_t, 0, \ldots, 0)$. It follows from Lemma 3.1 that $\Gamma(z)U_i(z)$ can be written in the form

$$\Gamma(z)\, U_i(z) = U'_i(z)\,(z - a_i)^C,$$

where $U'_i(z)$ is holomorphic in O_i and $u'_{lt}(a_i) \neq 0$. Thus, T'_i can be represented in the form

$$T'_i(y) = U'_i(z)(z - a_i)^{A_i + C}(y - a_i)^{E_i}.$$

From the condition $\varphi_i^{t-1} \geqslant \varphi_i^t + k$ and Definition 2.2 of the normalizations we obtain

$$'\varphi_i^t = \varphi_i^t + k,\ '\varphi_i^j \geqslant \varphi_i^j,\ j \neq t.$$

Remark 3.1. If in the assumption of Lemma 3.2 we drop the condition that the components $v_{l\,t+1}(z), \ldots, v_{lp}(z)$ of the row vector $v_l(z)$ in (3.5) are holomorphic, then (3.6) will take the form

$$(3.6')\quad '\varphi_i^j \geqslant \varphi_i^j,\ j < t;\ '\varphi_i^t = \varphi_i^t + k,\ '\varphi_i^{t+s} \geqslant \varphi_i^{t+s} - k_{t+s},\ s = 1, \ldots, p - t,$$

with

$$k_{t+1} = \max(0,\ k - l_{t+1}),\ k_{t+2} = \max(0,\ k - l_{t+1},\ k - l_{t+2}), \ldots,$$
$$k_p = \max(0,\ k - l_{t+1}, \ldots, k - l_p),$$

where l_j is the order of the zero of $v_{lj}(z)$ at a_j if $v_{lj}(z) \not\equiv 0$, and $l_j = k$ if $v_{lj}(z) \equiv 0$.

The inequalities (3.6′) follow from the fact that in this case the matrix $\Gamma(z)U_i(z)$ can be written in the form

$$\Gamma(z)\, U_i(z) = U'_i(z)\,(z - a_i)^{C'},$$

where $C' = \mathrm{diag}(0, \ldots, 0, k, -k_{t+1}, \ldots, -k_p)$, $U'_i(z)$ is holomorphic in O_i, and $k \geqslant -k_{t+1} \geqslant \ldots \geqslant -k_p$.

In addition, we shall describe another procedure that will be applied in the proofs of Theorems 1–3 stated in the Introduction. This procedure consists in passing from the space X to $X' = \Gamma(z)X$, where

$$\Gamma(z) = \left(\frac{z-a_j}{z-a_i}\right)^C, \tag{3.7}$$

$$C = \operatorname{diag}(c_{11}, \ldots, c_{pp}),\ c_{ll} \in \mathbb{Z},\ l = 1, \ldots, p.$$

We call it the "$B(a_i, a_j, c_{11}, \ldots, c_{pp})$ *procedure*".

3.2. In what follows we shall need some properties of the system (3) connected with the reducibility of its monodromy representation (2).

***Lemma* 3.3.** *If for some component $f_j(y)$ of any function $f(y)$ in the space X of solutions of a system* (3) *with monodromy* (2) *the identity*

$$f_j(y) \equiv 0$$

holds, then the representation (2) *is reducible.*

Proof. We consider a basis $(e_1(y), \ldots, e_p(y))$ of X such that

$$e_1(y) = f(y),\ e_{j1}(y) \equiv \ldots \equiv e_{jl}(y) \equiv 0$$

and the functions $e_{jl+1}(y), \ldots, e_{jp}(y)$ are linearly independent. It is obvious that l must satisfy the inequality $1 \leqslant l < p$. Let $m \leqslant l$ and $g \in \pi_1(CP^1 \setminus D, z_0)$. Let us consider $g^* e_m = \sum_{i=1}^{p} \lambda_i e_i$. Since, by construction, $e_{jm}(y) \equiv 0$ for $1 \leqslant m \leqslant l$, it follows from (2.1) that

$$(g^* e_{jm})(y) = e_{jm}(g^{-1}y) \equiv 0 = \sum_{i=l+1}^{p} \lambda_i e_{ji}(y).$$

Since $e_{jl+1}(y), \ldots, e_{jp}(y)$ are linearly independent, we have $\lambda_{l+1} = \ldots = \lambda_p = 0$. This means that the subspace $X_l \subset X$ generated by $e_1(y), \ldots, e_l(y)$ is a common invariant subspace for the monodromy operators, and so the monodromy representation (2) for the system (3) is reducible.

***Lemma* 3.4.** *If the matrix differential form ω for the system* (3) *satisfies the condition*

$$\omega_{ij} \equiv 0,\ i = l+1, \ldots, p;\ j = 1, \ldots, l;\ l < p, \tag{3.8}$$

then the monodromy representation (2) *for the system* (3) *is reducible.*

Proof. Let us consider the system $df = \omega' f$, where $\omega' = \|\omega_{ij}\|$ with $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant l$. If f' is a solution of the system, then the vector-valued function $f = (f', 0, \ldots, 0)$ is a solution of the original system (3). Lemma 3.4 follows now from Lemma 3.3.

Once again, let us consider a system (3) that is Fuchsian at $a_1, \ldots, \hat{i}, \ldots, a_n$ and has a regular singularity at a_i. We shall consider factorizations (2.8) for the space X of solutions of the system.

Lemma 3.5. *Let the matrix $U_i(z)$ in the factorization* (2.8) *for X at a_i be of the form*

$$U_i(z) = (z - a_i)^C V(z) \tag{3.9}$$

with $C = \operatorname{diag}(0, \ldots, 0, c_{l+1\,l+1}, \ldots, c_{pp})$, where $c_{jj} \in \mathbb{Z}$ and $c_{jj} \geqslant 0$ for $j = l+1, \ldots, p$, and $V(z) = \|v_{km}\|$ is a matrix that is holomorphically invertible in a neighbourhood O_i of a_i. If the elements u^j_{km} of the matrices U_j ($j = 1, \ldots, n$) in the factorizations (2.8) *at $a_1, \ldots, a_n$ and the elements v_{km} of V satisfy the equalities*

$$u^j_{km}(a_j) = 0, \; v_{km}(a_i) = 0, \; l+1 \leqslant k \leqslant p, \; 1 \leqslant m \leqslant l, \quad j = 1, \ldots, n, \tag{3.10}$$

then the monodromy representation (2) *for the system* (3) *is reducible.*

Proof. It follows from the form of $U_i(z)$ that the elements u^i_{km} of $U_i^{-1}(z)$ for $1 \leqslant k \leqslant p$ and $1 \leqslant m \leqslant l$ are holomorphic in a neighbourhood O_i of a_i and satisfy (3.10). Of course, (3.10) holds for the elements of the matrices $U_1^{-1}(z), \ldots, \hat{i}, \ldots, U_n^{-1}(z)$, which, by virtue of Proposition 2.3, are holomorphic in some neighbourhoods of $a_1, \ldots, \hat{i}, \ldots, a_n$, respectively. We find from (2.10) that for $l+1 \leqslant k \leqslant p$ and $1 \leqslant m \leqslant l$ the elements ω_{km} of the matrix differential form ω are holomorphic at each of the points $a_1, \ldots, a_n$. Since ω is holomorphic on $CP^1 \backslash D$, it follows that the forms ω_{km} in question are holomorphic everywhere on CP^1, hence $\omega_{km} \equiv 0$ for the indices k and m defined above. Now Lemma 3.5 follows from Lemma 3.4.

Lemma 3.6. *Let the monodromy representation* (2) *for the system* (3) *with regular singular points $a_1, \ldots, a_n$ be reducible and let X_l be a common invariant subspace for the monodromy operators in the space X of solutions of the system. Then the sum s_l of the exponents of X_l over all the points $a_1, \ldots, a_n$ is an integer and satisfies the inequality*

$$s_l = \sum_{i=1}^{n} \sum_{j=1}^{l} \beta_i^j \leqslant 0. \tag{3.11}$$

Proof. Let us choose a basis $(e_1, \ldots, e_l)$ in X_l and let us consider the fundamental matrix $T(y)$ constructed from this basis. We denote by $T'(y)$ the matrix consisting of the elements of the principal minor of $T(y)$ at $y_0 \in \widetilde{S}$. Then $\det T'(y_0) \neq 0$. The space X' generated by the columns of $T'(y)$ is the space of solutions of the system $df = \omega' f$ of l linear differential equations with $\omega' = dT' \cdot (T')^{-1}$. The set of singular points of the system consists of $a_1, \ldots, a_n$ and of additional "dummy" singularities $a'_{n+1}, \ldots, a'_r$. The latter set of singularities contains points such that $\det T'(y) = 0$ for $y \in \pi^{-1}(a'_t)$. We remark that if $\det T'(y) = 0$ for some $y \in \pi^{-1}(a'_t)$, then by (2.1) $\det T'(y) = 0$ for all $y \in \pi^{-1}(a'_t)$.

It follows from this remark that the number of additional singular points is finite, because otherwise the set $\{a'_{n+i}\}$ has a point of accumulation $\tilde{y} \in \tilde{S}$, or the set $\{\pi(a'_{n+i})\}$ has one of the points a_j ($j = 1, ..., n$) as a point of accumulation. The uniqueness theorem for analytic functions applied to $\det T'(y)$ in the former case and to $\det U_j(z)$ in the latter, $U_j(z)$ being the matrix in (2.8), yields $\det T'(y) \equiv 0$, which contradicts the condition $\det T'(y_0) \neq 0$.

The exponents $'\beta^j_{n+i}$ of the space X' at the points a'_{n+i} coincide with the normalizations $'\varphi^j_{n+i}$, which in turn are non-negative, since $T'(y)$ is analytic at the points $\pi^{-1}(a'_{n+1})$:

$$'\beta^j_{n+i} = {}'\varphi^j_{n+i} \geqslant 0. \tag{3.12}$$

The normalizations $'\varphi^j_i$ for X' are connected with the normalizations φ^j_i for X_l at the points $a_1, ..., a_n$ by the inequalities

$$'\varphi^j_i = \varphi_i(e'_j) \geqslant \varphi_i(e_j) = \varphi^j_i, \tag{3.13}$$

which follow from Definition 2.2 of the normalizations and the fact that the column vector $e'_j(y)$ of the matrix $T'(y)$ can be obtained from the vector-valued function $e_j(y)$ by crossing out some of its components.

From Proposition 2.4 and inequalities (3.12) and (3.13) we obtain

$$s_l = \sum_{i=1}^{n} \sum_{j=1}^{l} \beta^j_i \leqslant \sum_{i=1}^{n} \sum_{j=1}^{l} {}'\beta^j_i + \sum_{i=n+1}^{r} \sum_{j=1}^{l} {}'\varphi^j_i = s'_l \leqslant 0,$$

where s'_l is the sum of exponents of X'.

Lemma 3.7. *Let a representation* (2) *of dimension $p = 3$ be reducible and let all the monodromy operators have a common invariant subspace of dimension $l = 1$ or $l = 2$. Then there is a system of equations* (3) *with regular singular points $a_1, ..., a_n$, Fuchsian at $a_2, ..., a_n$, whose monodromy representation coincides with* (2), *and there is a fundamental matrix $T(y)$ for the space X of solutions of the system whose elements e_{km} satisfy the identities*

$$e_{km}(y) \equiv 0, \; l+1 \leqslant k \leqslant 3, \; 1 \leqslant m \leqslant l. \tag{3.14}$$

Proof. We consider a system (3) with regular singularities at $a_1, ..., a_n$ that is Fuchsian at $a_2, ..., a_n$ and whose monodromy coincides with (2). (The existence of such a system is proved in [7].) Let us choose an associated basis (e) in the space $X|_{\tilde{S}_1}$ for the system in such a way that the first l elements of (e) constitute a basis of a subspace $X_l \subset X$ invariant with respect to the monodromy operators. We can assume that for the elements u_{km} of the matrix $U_1(z)$ in the factorization (2.8) of the fundamental matrix $T(y)$ constructed from (e), the equalities

$$u_{21}(a_1) = u_{31}(a_1) = u_{32}(a_1) = 0 \tag{3.15}$$

hold. We can make sure that conditions (3.15) are saatisfied by applying linear transformations of the rows of $U_1(z)$, which corresponds to the passage from the system (3) with the space of solutions X to a system with the space of solutions $X' = SX$, where S is a constant non-singular matrix.

1. To start with, we consider the case $l = 1$. Suppose that

$$u_{j1}(a_1) \not\equiv 0, \; j = 2 \text{ or } j = 3. \tag{3.16}$$

We denote by r the order of the zero of $u_{j1}(z)$ at a_1. It follows from (3.15) that $r > 0$.

Applying the $A(a_1, j, 1, r)$ procedure to the space X, we obtain $X' = \Gamma_1(z)X$. According to Lemma 3.1, the sum s' of the exponents of $X_1' = \Gamma_1(z)X_1$ is related to the sum s of the exponents of X_1 by the inequality

$$s' = s + r > s. \tag{3.17}$$

We shall bring the space X' back to the form (3.15). If (3.16) holds for an element $u_{j1}'(z)$ of the matrix $U_1'(z)$, we apply the $A(a_1, j, 1, r')$ procedure again, and so on. It follows from Lemma 3.6 and (3.17) that after a finite number of steps we shall obtain the space $\widetilde{X}$ of solutions of a system (3) with regular singularities at $a_1, ..., a_n$ that is Fuchsian at $a_2, ..., a_n$ with given monodromy representation (2) such that the elements of the matrix $\widetilde{U}_1(z)$ in (2.8) satisfy the identities $u_{21}(z) \equiv u_{31}(z) \equiv 0$, and so $e_{21}(z) \equiv e_{31}(z) \equiv 0$.

2. Now we consider the case $l = 2$. The following subcases are possible:

a) the two-dimensional monodromy representation χ_2 for the space X_2 is reducible;

b) χ_2 is irreducible.

In case a), according to the case $l = 1$ discussed above, we can assume that $T(y)$ already has the form

$$T(y) = \begin{pmatrix} e_{11} & * \; * \\ 0 & \\ 0 & \widetilde{T}_2(y) \end{pmatrix}, \tag{3.18}$$

where $\widetilde{T}_2(y)$ is the fundamental matrix of the space of solutions of a system (3) whose monodromy representation coincides with the corresponding quotient representation of the representation (2). For the quotient representation there is a one-dimensional invariant subspace generated by $e_2 = (e_{22}, e_{32})$. Hence, it follows from the discussion of the case $l = 1$ that there is a matrix $\Gamma'(z)$ such that the element t_{21} of the matrix $\widetilde{T}_2' = \Gamma'\widetilde{T}_2$ satisfies the equality $t_{21}(y) \equiv 0$. Let us consider the matrix

$$\Gamma(z) = \begin{pmatrix} 1 & 0 \; 0 \\ 0 & \\ 0 & \Gamma'(z) \end{pmatrix}.$$

For $T'(y) = \Gamma(z)T(y)$ we find that

$$e_{21}'(y) \equiv e_{31}'(y) \equiv e_{32}'(y) \equiv 0. \tag{3.19}$$

In case b) we assume that $u_{31}(z) \not\equiv 0$ and we denote the orders of the zeros of $u_{31}(z)$ and $u_{32}(z)$ at a_1 by r_1 and r_2. It follows from (3.15) that $r_1 > 0$ and $r_2 > 0$. (If $u_{32}(z) \equiv 0$, we set $r_2 = r_1$.) Applying the $A(a_1, 3, 1, r_1)$ procedure to X, we obtain the space $X' = \Gamma_1(z)X$. According to Remark 3.1, the sum s' of the exponents of the space $X_2' = \Gamma_1'(z)X_2$ is related to the sum s of the exponents of X_2 by the inequality

$$s' \geqslant s + r_1 - \max(0, r_1 - r_2) > s. \tag{3.20}$$

We shall bring the space X' back to the form (3.15). If the element $u_{31}'(z)$ of the matrix $U_1'(z) = \Gamma_1 U_1(z)$ does not vanish identically, then we apply the $A(a_1, 3, 1, r_1')$ procedure again, and so on. It follows from Lemma 3.6 and (3.20) that after a finite number of steps we shall obtain the space $\widetilde{X}$ of solutions of a system (3) with regular singular points $a_1, \ldots, a_n$ that is a Fuchsian system at $a_2, \ldots, a_n$ with the given monodromy representation (2) such that the element $\widetilde{u}_{31}(z)$ of the matrix $\widetilde{U}_1(z)$ in (2.8) satisfies the identity $\widetilde{u}_{31}(z) \equiv 0$. Therefore $\widetilde{e}_{31}(y) \equiv 0$, and since χ_2 is irreducible, $\widetilde{e}_{32}(y) \equiv 0$. (Otherwise, according to Lemma 3.3, $e_1(y)$ would be a common eigenvector for the monodromy operators and χ_2 would be reducible.)

***Corollary* 3.1.** *If all the monodromy matrices of a representation* (2) *of dimension p can be simultaneously reduced to the upper triangular form, then there is a system* (3) *with regular singular points $a_1, \ldots, a_n$ that is Fuchsian at $a_2, \ldots, a_n$ and such that the monodromy representation of the system coincides with* (2) *and the fundamental matrix $T(y)$ of the space of solutions of the system has the upper triangular form.*

The proof can easily be carried out by induction, the first step of which was realized during our discussion of case 2a) of Lemma 3.7.

§4. The solubility of the Riemann–Hilbert problem for an irreducible representation of dimension three

We consider the space X of solutions of system (3) with a regular singular point a_i and factorization (2.8) for X at a_i.

***Lemma* 4.1.** *There is a matrix $\Gamma(z)$ meromorphic on CP^1 and holomorphically invertible at any point except a_i, and there is a matrix $V_i(z)$ holomorphically invertible in a neighbourhood O_i of a_i, such that for the matrix $U_i(z)$ in* (2.8) *the factorization*

$$\Gamma(z)U_i(z) = (z - a_i)^C V_i(z) \tag{4.1}$$

holds, with $C = \operatorname{diag}(c_{11}, \ldots, c_{pp})$, where $c_{ll} \in \mathbb{Z}$ for $l = 1, \ldots, p$ and $0 \leqslant c_{11} \leqslant \ldots \leqslant c_{pp}$.

Proof. The assertion about the existence of a factorization (4.1) with some integral diagonal matrix C for $U_i(z)$, which is a holomorphically invertible matrix on $O_i \backslash a_i$, is equivalent to the Birkhoff–Grothendieck theorem [5] on the decomposition of a holomorphic bundle on the Riemann sphere into a sum of one-dimensional bundles, since $U_i(z)$ can be regarded as the transition function for the holomorphic bundle $E(CP^1 \backslash a_i, O_i, U_i(z))$.

We claim that in (4.1) $c_{ll} \geqslant 0$ for all l. Let us rewrite (4.1) in the form

$$U_i(z) V_i^{-1}(z) = \Gamma^{-1}(z)(z - a_i)^C. \tag{4.2}$$

Suppose that $c_{ll} < 0$ for some l. The matrix on the left-hand side of (4.2) is holomorphic at a_i, hence the l-th column $t(z)$ of the matrix on the right-hand side of (4.2) is also holomorphic at a_i. But

$$t(z) = (z - a_i)^{c_{ll}} \gamma(z),$$

where $\gamma(z)$ is the l-th column of the matrix $\Gamma^{-1}(z)$, and so $\gamma(z) \to 0$ as $z \to a_i$. Thus, by virtue of Liouville's theorem, it follows that $\gamma(z) \equiv 0$, which contradicts the fact that $\Gamma(z)$ is holomorphically invertible everywhere except at a_i. The contradiction means that $c_{ll} \geqslant 0$. By applying a linear transformation of the rows of $\Gamma(z)U_i(z)$ (that is, by multiplying $\Gamma(z)U_i(z)$ on the left by a constant non-singular matrix S), we can make sure that the condition $0 \leqslant c_{11} \leqslant \ldots \leqslant c_{pp}$ holds.

Remark 4.1. The factorization (4.1) was used by Plemelj in [7] and by Birkhoff in [21]. The assertion of Lemma 4.1 can also be obtained from a modified lemma of Sauvageot (see [9]).

Let us now proceed directly to the proof of Theorem 1 stated in the Introduction. We shall assume that ∞ is not in D and $a_1 = 0$. (We can always make sure that these conditions are met by applying a conformal transformation of the Riemann sphere.)

Proof of Theorem 1. Let us consider the space X of solutions of a system of three equations (3) with regular singular points $a_1, \ldots, a_n$ that is Fuchsian at $a_2, \ldots, a_n$ and whose monodromy representation (2) is irreducible. (It is proved in [7] that there is such a system.) By the criterion (2.16) for Fuchsian systems and condition (2.15), to prove the theorem it is sufficient to show that there is a meromorphic matrix-valued function $\Gamma(z)$ on CP^1 holomorphically invertible outside the set of points $a_1, \ldots, a_n$ such that the following conditions hold for $X' = \Gamma(z)X$:

(4.3a) X' is the space of solutions of a system (3) that is Fuchsian at all except perhaps one of the points $a_1, \ldots, a_n$;

$$\sum_{i=1}^{n} \sum_{j=1}^{3} {}'\beta_i^j > \sum_{i=1}^{n} \sum_{j=1}^{3} \beta_i^j, \tag{4.3b}$$

where β_i^j and ${}'\beta_i^j$ are the exponents of X and X', respectively.

Let us consider the factorization (2.8) for X in a neighbourhood of $a_1 = 0$. We pass from the space X to $X' = \Gamma(z)X$, where $\Gamma(z)$ is the matrix defined in Lemma 4.1. We set $U_1'(z) = \Gamma(z)U_1(z) = z^C V_1(z)$.

From now on the proof breaks up into three cases:
1) $c_{11} > 0$;
2) $c_{11} = 0$ and $c_{22} > 0$;
3) $c_{11} = c_{22} = 0$ and $c_{33} > 0$.

1. Let us consider case 1. In this case it follows from Definition 2.2 of the normalizations and from the form of the factorization (2.8) for X' that

$$'\varphi_1^j \geqslant \varphi_1^j + c_{11}, \; j = 1, 2, 3,$$

where φ_1^j and $'\varphi_1^j$ are normalizations for X and X', respectively. From these inequalities we obtain conditions (4.3a) and (4.3b).

2. Let us consider case 2. We denote X' anew by X and we denote U_1' by U_1. It follows from the irreducibility of the representation (2) and Lemma 3.5 that either $v_{l1}(0) \neq 0$ for the elements of the matrix $V_1(z)$ in (4.1) with $l = 2$ or $l = 3$, or there are numbers $2 \leqslant i \leqslant n$ and $2 \leqslant l \leqslant 3$ such that the element $u_{l1}(z)$ of the matrix $U_i(z)$ in the factorization (2.8) for X at a_i does not vanish at $a_i : u_{l1}(z) \neq 0$. In the former case the l-th row of $U_1(z)$ has the form (3.5) with $t = 1$. We apply the $A(a_1, l, 1, c_{ll})$ procedure to X, and from Lemma 3.2 we find that conditions (4.3a) and (4.3b) hold for the space $X' = \Gamma(z)X$. If, however, $u_{l1}(a_i) \neq 0$, then we apply the $B(a_1, a_i, 0, c_{22}, c_{33})$ procedure to X to obtain the space of solutions $X' = \Gamma X$ for a system (3) that is Fuchsian at $a_1, \ldots, \hat{i}, \ldots, a_n$. The row u_l' of the holomorphic matrix $U_i' = \Gamma U_i$ will be of the form (3.5) with $t = 1$ and $k = c_{ll} > 0$. Let us apply the $A(a_i, l, 1, c_{ll})$ procedure to X'. For $X'' = \Gamma' X'$ we again obtain conditions (4.3a) and (4.3b).

3. For $1 \leqslant l, m \leqslant 3$, we denote by u_{lm}^i the elements of the matrix $U_i(z)$ in the factorization (2.8) for X at a_i, and we denote by v_{lm} the elements of the matrix $V_1(z)$ in (4.1). The third case is broken up into two subcases:

a) one of the numbers $v_{31}(0), u_{31}^2(a_2), \ldots, u_{31}^n(a_n)$ is non-zero;

b) the functions $v_{31}(z), u_{31}^2(z), \ldots, u_{31}^n(z)$ have zeros of orders $k_1, k_2, \ldots, k_n$ at $a_1, \ldots, a_n$ with $k_i > 0$ for $i = 1, \ldots, n$.

Case 3a) can be reduced to case 2) already discussed. Let us consider case 3b). Since the representation (2) is irreducible, it follows from Lemma 3.3 that $k_1, \ldots, k_n$ are finite numbers. We set

$$s = k_1 + c_{33} + k_2 + \ldots + k_n \tag{4.4}$$

for the sum of the orders of zeros of the functions $u_{31}^i(a)$ at the points a_i, where $i = 1, \ldots, n$. Also, because of the irreducibility of the representation (2), it follows from Lemma 3.5 that either $v_{32}(0) \neq 0$ or there is an i such that $u_{32}^i(a_i) \neq 0$. In the latter case, by applying the $B(a_1, a_i, 0, 0, c_{33})$ procedure to X, we obtain the space $X' = \Gamma(z)X$ of solutions of a system (3) that is Fuchsian at $a_1, \ldots, \hat{i}, \ldots, a_n$. The sum s in (4.4) does not change under this operation. We denote the space again by X. The matrix $U_i(z)$ in the factorization (2.8) for this space has the form (4.1) with $C = \mathrm{diag}(0, 0, c_{33})$.

The following part of the proof is based on the procedure described below, which we call the *L procedure*.

Description of the L procedure. It follows from Remark 2.1 that either $u^i_{11}(a_i) \neq 0$ or $u^i_{21}(a_i) \neq 0$. Let us pass from X to $X' = SX$, where S is a constant non-singular matrix of the form

$$S = \begin{pmatrix} S' & & 0 \\ & & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.5}$$

such that the first column x_1 of $U_i'(a_i) = SU_i(a_i)$ has the form

$$x_1 = (1, 0, 0). \tag{4.6}$$

By applying the $A(a_i, 3, 2, c_{33})$ procedure to X', we obtain the space ${}^1X = \Gamma_1 X'$ such that the matrix ${}^1U_i(z) = \Gamma_1(z)U_i'(z)$ has the form

$${}^1U_i(a_i) = \begin{pmatrix} 1 & 0 & * \\ 0 & 0 & * \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.7}$$

It follows from Lemma 3.5 that there is a number j such that the value ${}^1u^j_{21}(a_j)$ of the element ${}^1u^j_{21}(z)$ of ${}^1U_j(z)$ at a_j is non-zero: ${}^1u^j_{21}(a_j) \neq 0$. By applying the $A(a_j, 2, 1, 0)$ procedure to ${}^1X$, we obtain ${}^2X = \Gamma_2(z){}^1X$. It follows from the form of the transformation applied that the matrices ${}^2U_m(z) = \Gamma_2(z){}^1U_m(z)$ satisfy the following conditions:

1) for $m = 1, \ldots, n$, the third row of ${}^2U_m(z)$ coincides with the third row of $U_m(z)$;

2) ${}^2U_i(a_i)$ has the form (4.7);

3) the first column of ${}^2U_j(a_j)$ has the form $(0, {}^1u^j_{21}(a_j), 0)$.

We apply the $B(a_i, a_j, 0, 1, 0)$ procedure to ${}^2X$, and denote the resulting space by $\widetilde{X}$. The description of the L procedure is completed.

Since ${}^2U_i(a_i)$ has the form (4.7), it follows that

$$\tilde{\varphi}^1_i = \varphi^1_i, \quad \tilde{\varphi}^2_i \geqslant \varphi^2_i, \quad \tilde{\varphi}^3_i \geqslant \varphi^3_i - 1, \tag{4.8}$$

where φ^j_i and $\tilde{\varphi}^j_i$ are the normalizations for X and $\widetilde{X}$, respectively. Since the first column of ${}^2U_j(a_j)$ has the form given in 3), it follows that

$$\tilde{\varphi}^1_j = \varphi^1_j + 1, \quad \tilde{\varphi}^2_j = \varphi^2_j, \quad \tilde{\varphi}^3_j = \varphi^3_j, \tag{4.9}$$

and the factorization (2.8) for $\widetilde{X}$ at a_j has the form

$$\widetilde{T}_j(y) = \widetilde{U}'_j(z)(z - a_j)^{A_j + N}(y - a_j)^{E_j}, \tag{4.10}$$

where $N = \mathrm{diag}(1, 0, 0)$ and $\widetilde{U}'_j(z) = \widetilde{U}_j(z)(z-a_j)^{-N}$. We find from 1) and (4.10) that the sums s and $\tilde{s}$ defined by (4.4) for the spaces X and $\widetilde{X}$ are connected by the relation

$$\tilde{s} = s - 1 < s. \tag{4.11}$$

It follows from (4.8) and (4.9) that $\widetilde{X}$ satisfies condition (4.3a) and

$$\sum_{k=1}^{n}\sum_{m=1}^{3}\tilde{\beta}_k^m \geqslant \sum_{k=1}^{n}\sum_{m=1}^{3}\beta_k^m. \tag{4.12}$$

Let us go back to the proof of Theorem 1. We apply the L procedure to X. If $\widetilde{X}$ satisfies condition 3b), then we apply the L procedure again. It follows from (4.11) that after no more than $\sum_{l=1, l\neq i}^{n}(k_l - 1) + 1$ steps we shall obtain a space satisfying conditon 3a). By virtue of (4.12), in this case the proof of the theorem can also be reduced to case 2 discussed above. Theorem 1 is proved.

Another version of the proof of Theorem 1 is given in [22].

§5. Fuchsian systems of two equations on the Riemann sphere. The Fuchsian weight of a representation

5.1. We shall consider a Fuchsian system (4) of two equations on CP^1 and factorizations (2.8) for the space X of solutions of the system at $a_1, \ldots, a_n$. We can assume that the i-th normalizations for X satisfy the conditions

$$\varphi_i^1 \geqslant \varphi_i^2. \tag{5.1}$$

For if the monodromy matrix G_i defined by (2.5) is non-diagonable, then, according to condition b) for the associated basis at a_i, (5.1) holds. However, if G_i is diagonable and $\varphi_i^1 < \varphi_i^2$ (which is possible in this case since the elements e_1 and e_2 of the associated basis (e) correspond to different blocks of G_i), then we again obtain (5.1) by replacing the basis (e_1, e_2) of $X|_{\tilde{S}_i}$ at a_i by (e_2, e_1).

Definition 5.1. The number

$$\gamma_\omega = \sum_{i=1}^{n}(\varphi_i^1 - \varphi_i^2)$$

will be called the *Fuchsian weight of a Fuchsian system* (4) of two equations.

Let us consider the set Ω of all Fuchsian systems of two equations on CP^1 with given monodromy representation (2). (According to [6], this set is non-empty.)

Definition 5.2. We call the number

$$\gamma_\chi = \min_{\Omega}\gamma_\omega$$

the *Fuchsian weight of the representation* (2).

***Proposition* 5.1.** *For any representation* (2) *of dimension* $p = 2$, *the inequalities*

$$\gamma_\omega \geqslant 0, \quad \gamma_\chi \geqslant 0 \tag{5.2}$$

hold, and the parity of γ_ω *and* γ_χ *coincides with that of* $\sum_{i=1}^{n} \operatorname{Sp} E_i$. *If* (2) *is a commutative representation that cannot be decomposed into a direct sum of one-dimensional representations, then* $\gamma_\chi = 0$.

Proof. The first part of the proposition follows from (5.1) and the fact that

$$\sum_{i=1}^{n} (\beta_i^1 + \beta_i^2) = \sum_{i=1}^{n} \operatorname{Sp} E_i + \sum_{i=1}^{n} \varphi_i^1 + \sum_{i=1}^{n} \varphi_i^2.$$

For we find from (2.16) that

$$\sum_{i=1}^{n} \varphi_i^1 + \sum_{i=1}^{n} \varphi_i^2 = -\sum_{i=1}^{n} \operatorname{Sp} E_i,$$

and so

$$\gamma_\omega = \sum_{i=1}^{n} (\varphi_i^1 - \varphi_i^2) = -\sum_{i=1}^{n} \operatorname{Sp} E_i - 2\sum_{i=1}^{n} \varphi_i^2.$$

If (2) is a commutative representation that cannot be decomposed into a direct sum of one-dimensional representations, then the matrices G_i can be simultaneously reduced to the form

$$\begin{pmatrix} \lambda_i & * \\ 0 & \lambda_i \end{pmatrix}.$$

Therefore,

$$T(y) = (z - a_1)^{-\frac{1}{2}\sum_{i=1}^{n} \operatorname{Sp} E_i} \prod_{i=1}^{n} (z - a_i)^{E_i} \tag{5.3}$$

is the fundamental matrix for a Fuchsian system (4) with the given monodromy and with weight 0.

Lemma 5.1. *For any Fuchsian system* (4) *of Fuchsian weight* γ_ω *with* X *as its space of solutions, there is a point* $a_l \in D$ *and a matrix-valued function* $\Gamma(z)$ *meromorphic on* CP^1 *and holomorphically invertible outside* D *such that* $X' = \Gamma(z)X$ *satisfies the following conditions*:

a) X' *is the space of solutions of a Fuchsian system* (4);

b) $'\varphi_i^1 = {'\varphi_i^2} = 0$ *for* $i = 1, \ldots, \hat{l}, \ldots, n$, *and* $'\varphi_l^1 - {'\varphi_l^2} \leqslant \gamma_\omega$, where $'\varphi_i^j$ *are the normalizations for* X'.

Proof. For each point a_i we consider the factorization (2.8) for X such that (5.1) is satisfied. We denote by J the set of indices such that

$$\varphi_i^1 > \varphi_i^2, \quad i \in J.$$

If this set is empty or $J = \{l\}$ for some l, then we apply to X the transformation $\prod_{i \neq l} B(a_i, a_l, \varphi_i^1, \varphi_i^1)$. (If $J \neq \varnothing$, we can choose any index l.) In this case the assertion of the lemma follows easily from the form of the factorizations (2.8).

Suppose that J contains $i, m, \ldots$. To prove the lemma, we only need to show that by substituting $\widetilde{X} = \Gamma X$ one can always reduce the number of elements of J without increasing the Fuchsian weight of the system.

We pass from X to $X' = U_i^{-1}(a_i)X$, which we denote again by X. Now we have

$$U_i(a_i) = I \tag{5.4}$$

in the factorization (2.8) for X. By applying the $B(a_i, a_m, 1, 0)$ procedure to X, we obtain $X' = \Gamma_1(z)X$. It follows from the form of $\Gamma_1(z)U_i(z)$ that the normalizations $'\varphi_i^j$ for X' are connected with the normalizations φ_i^j for X by the following relations:

$$\begin{cases} '\varphi_i^1 = \varphi_i^1 - 1, \, '\varphi_i^2 = \varphi_i^2, \\ \Delta_i' = {}'\varphi_i^1 - {}'\varphi_i^2 = \Delta_i - 1, \end{cases} \tag{5.5}$$

where $\Delta_i = \varphi_i^1 - \varphi_i^2$.

We consider the matrix $U_m(z)$ in the factorization (2.8) for X. The remaining part of the proof breaks up into two cases:

a) $u_{11}^m(a_m) \neq 0$;

b) $u_{11}^m(a_m) = 0$.

First we consider case a). In this case the first row of $\Gamma_1(z)U_m(z)$ has the form (3.5) with $t = 1$ and $k = 1$. By applying the $A(a_m, 1, 1, 1)$ procedure to X', we get $X'' = \Gamma_2 X'$. According to Lemma 3.2, the normalizations $''\varphi_m^j$ of X'' at a_m satisfy the relations

$$''\varphi_m^1 = \varphi_m^1 + 1, \quad ''\varphi_m^2 = \varphi_m^2, \; \Delta_m'' = \Delta_m + 1. \tag{5.6}$$

Let us consider case b). In this case it follows from Proposition 2.3 that $u_{12}^m(a_m) \neq 0$, and so the first row of $\Gamma_1(z)U_m(z)$ is of the form (3.5) with $t = 2$ and $k = 1$. Applying the $A(a_m, 1, 2, 1)$ procedure to X', we obtain $X'' = \Gamma_2(z)X'$. According to Lemma 3.2, the normalizations $''\varphi_m^j$ of X'' satisfy the relations

$$''\varphi_m^1 = \varphi_m^1, \quad ''\varphi_m^2 = \varphi_m^2 + 1, \quad \Delta_m'' = \Delta_m - 1. \tag{5.7}$$

It follows from (5.5)–(5.7) that both in case a) and in case b) we have

$$\sum_{l=1}^{n} {}''\varphi_l^1 - {}''\varphi_l^2 \leqslant \gamma_\omega, \quad \Delta_i'' = \Delta_i - 1 \tag{5.8}$$

for X''.

If $\Delta_i'' > 0$ and $\Delta_m'' > 0$, we repeat the above procedure once again. It follows from (5.8) that after no more than $\Delta_i = \varphi_i^1 - \varphi_i^2$ steps we shall obtain a space $\widetilde{X} = \Gamma(z)X$ such that (5.8) holds and either $\widetilde{\Delta}_i = 0$ or $\widetilde{\Delta}_m = 0$. This means that the number of elements of J has decreased. We find from (5.5)–(5.8) and from Proposition 2.4 that $\widetilde{X}$ is the space of solutions of a Fuchsian system (4) whose Fuchsian weight does not exceed γ_ω.

***Corollary* 5.1.** *Any Fuchsian system* (4) *of two equations and of Fuchsian weight* $\gamma_{\omega'}$ *can be reduced by means of the transformation* (1.4) *to a Fuchsian system* (4) *whose Fuchsian weight* γ_ω *does not exceed* $\gamma_{\omega'}$, *such that the factorizations* (2.8) *for the space* X *of solutions of the latter system are of the following form*:

$$T_i(y) = U_i(z)(y - a_i)^{E_i}, \quad i \neq l, \tag{5.9}$$

$$T_l(y) = U_l(z)(z - a_l)^{A_l}(y - a_l)^{E_l}, \tag{5.10}$$

where all the matrices $U_j(z)$ *are holomorphically invertible at the points* a_j, *the matrices* E_i *have the upper triangular form,*

$$E_l = \begin{pmatrix} \rho_l^1 & \varepsilon \\ 0 & \rho_l^2 \end{pmatrix}, \tag{5.11}$$

$$A_l = \begin{pmatrix} b + \gamma_1 & 0 \\ 0 & b - \gamma_2 \end{pmatrix}, \tag{5.12}$$

and

$$U_l(z) = \begin{pmatrix} 1 & c(z - a_l)^m \\ s(z - a_l)^k & 1 \end{pmatrix}(1 + o(1)), \tag{5.13}$$

where

$$\gamma_1 = \left[\frac{\gamma_\omega + 1}{2}\right], \ \gamma_1 + \gamma_2 = \gamma_\omega, \ m > 0, \ k > 0, \ c \neq 0$$

if $u_{12}^l(z) \not\equiv 0$, *and* $s \neq 0$ *if* $u_{21}^l \not\equiv 0$. $[x]$ *is the integral part of* x.

***Lemma* 5.2.** *Let a Fuchsian system* (4) *be reduced to* (5.9)–(5.13) *and let* $\gamma_\omega > 0$. *If all monodromy matrices* G_i *are non-diagonable, then there is an* $i \neq l$ *such that the value at* a_i *of the element* $u_{11}^i(z)$ *of the matrix* $U_i(z)$ *in* (5.9) *is non-zero*:

$$u_{11}^i(a_i) \neq 0. \tag{5.14}$$

Proof. By assumption, it follows that all matrices E_i in (5.9) and (5.10) have the form of Jordan blocks. Suppose that $u_{11}^i(a_i) = 0$ for $i = 1, \ldots, \hat{l}, \ldots, n$. Then we find from (2.14) and (5.9) that

$$B^i = U_i(a_i)E_iU_i^{-1}(a_i) = \begin{pmatrix} \rho_i & 0 \\ * & \rho_i \end{pmatrix}.$$

Since $\gamma_\omega > 0$, it follows from (2.14) and (5.13) that $B^l = \mathrm{diag}(\beta_l^1, \beta_l^2)$. Consequently, we find from (5) that

$$\sum_{i=1,\, i\neq l}^{n} \rho_i + \beta_l^1 = \sum_{i=1,\, i\neq l}^{n} \rho_i + \beta_l^2 = 0,$$

and so $\beta_l^1 = \beta_l^2$. Since $\rho_l^1 = \rho_l^2$, we have $\varphi_l^1 = \varphi_l^2$, which contradicts the assumption that $\gamma_\omega > 0$.

Lemma 5.3. *Let a Fuchsian system* (4) *with monodromy* (2) *be reduced to the form* (5.9)–(5.13) *and let* $c \neq 0$ *and* $m \leqslant \frac{1}{2}\gamma_\omega$ *in* (5.13). *Then* $\gamma_\chi < \gamma_\omega$.

Proof. Applying the $A(a_l, 1, 2, m)$ procedure to the space X of solutions of (4), we obtain the space $X' = \Gamma(z)X$. Lemma 3.1 and the fact that $\Gamma(z)$ is given by (3.4) imply that the matrix

$$U_l' = \Gamma(z) U_l(z)(z - a_l)^C, \tag{5.15}$$

where $C = \operatorname{diag}(m, -m)$, is holomorphically invertible at a_l. Since $\varphi_l^1 - m \geqslant \varphi_l^2 + m$, by virtue of the assumption of the lemma, the factorization (2.8) for X' at a_l has the form

$$T_l'(y) = U_l'(z)(z - a_l)^{A_l'}(y - a_l)^{E_l}, \tag{5.16}$$

where $A_l' = A_l - C$. Thus, $\gamma_\chi \leqslant \gamma_{\omega'} = \gamma_\omega - 2m$.

In some cases the Fuchsian weight of the representation (2) can be determined from the Fuchsian weight of any system with monodromy (2).

Proposition 5.2. *Let a Fuchsian system* (4) *with monodromy* (2) *be reduced to the form* (5.9)–(5.13). *If* $c = 0$ *in* (5.13), *then* $\gamma_\chi = \gamma_\omega$. *If* $c \neq 0$, *then* $\gamma_\chi = \gamma_\omega$ *if and only if* $m \geqslant \gamma_\omega$.

Proof 1. First we shall prove that if $c \neq 0$ and $m < \gamma_\omega$, then $\gamma_\chi < \gamma_\omega$. By virtue of Lemma 5.3, it is sufficient to investigate the case

$$\frac{1}{2}\gamma_\omega < m < \gamma_\omega. \tag{5.17}$$

There are two possibilities for E_l in (5.11):

a) E_l is a diagonal matrix,

b) E_l is a Jordan block.

In case a), applying the $A(a_l, 1, 2, m)$ procedure to X, we obtain a space $X' = \Gamma(z)X$ for which (5.15) and (5.16) hold, $'\varphi_l^1 = \varphi_l^1 - m$, and $'\varphi_l^2 = \varphi_l^2 + m$. By exchanging the columns of $T_l' = \Gamma T_l$, we obtain a factoriztion for X' of the form (2.8) (since E_l is a diagonal matrix) with $A_l'' = \operatorname{diag}(\varphi_l^2 + m, \varphi_l^1 - m)$ and $E_l'' = \operatorname{diag}(\rho_l^2, \rho_l^1)$. It follows from (5.17) that

$$\gamma_{\omega'} = \varphi_l^2 + m - (\varphi_l^1 - m) = 2m - \gamma_\omega < \gamma_\omega, \tag{5.18}$$

and so $\gamma_\chi < \gamma_\omega$.

Let us consider case b). If E_i is a diagonal matrix for some $i \neq l$, then exchanging the columns of $T_l(y)$ if necessary we find that (5.14) is satisfied for $U_i(z)$ from (5.9). By virtue of Lemma 5.2, if E_j is a Jordan block for any $j \neq l$, then there is also an $i \neq l$ such that (5.14) holds.

Applying the $B(a_l, a_i, 2m - \gamma_\omega, 0)$ procedure to X, we obtain $X' = \Gamma_1(z)X$. The matrix

$$U_l'(z) = \Gamma_1(z) U_l \cdot (z - a_l)^C,$$

where $C = \operatorname{diag}(2m-\gamma_\omega, 0)$, is holomorphically invertible at a_i, and its first row has the form

$$((a_l - a_i)^{2m-\gamma_\omega}, c'(z-a_l)^{\gamma_\omega - m})(1+o(1)),\ c' \neq 0.$$

The normalizations $'\varphi_l^j$ for X' at a_l satisfy the following equalities: $'\varphi_l^1 = \varphi_l^1-(2m-\gamma_\omega)$ and $'\varphi_l^2 = \varphi_l^2$. Let us note that $'\varphi_l^1 - {}'\varphi_l^2 = 2(\gamma_\omega - m)$. Thus, applying the $A(a_l, 1, 2, \gamma_\omega - m)$ procedure to X', we obtain $X'' = \Gamma_2(z)X'$. According to the proof of Lemma 5.3, the normalizations $''\varphi_l^j$ for X'' are equal:

$$''\varphi_l^1 = {}''\varphi_l^2. \tag{5.19}$$

The $A(a_l, 1, 2, \gamma_\omega - m)$ procedure preserves the form (3.5) of the first row of $\Gamma_1(z)U_i(z)$ with $t = 1$ and $k = 2m-\gamma_\omega$. Applying the $A(a_i, 1, 1, 2m-\gamma_\omega)$ procedure to X'', we obtain $\widetilde{X} = \Gamma_3(z)X''$. By virtue of Lemma 3.2 and (5.18), the normalizations $\widetilde{\varphi}_i^j$ for $\widetilde{X}$ satisfy the relations

$$\widetilde{\varphi}_i^1 = \varphi_i^1 + 2m - \gamma_\omega,\quad \widetilde{\varphi}_i^2 = \varphi_i^2. \tag{5.20}$$

For $\widetilde{X}$ we find from (5.19), (5.20), and (5.17) that

$$\gamma_{\widetilde{\omega}} = \sum_{j=1}^{n} (\widetilde{\varphi}_j^1 - \widetilde{\varphi}_j^2) = 2m - \gamma_\omega < \gamma_\omega.$$

Thus, $\gamma_\chi < \gamma_\omega$ in this case also.

2. We claim that if $c \neq 0$ and $m \geqslant \gamma_\omega$, or $c = 0$, then $\gamma_\chi = \gamma_\omega$. Suppose the contrary. Let there be a Fuchsian system (4) of Fuchsian weight $\gamma_{\omega'} < \gamma_\omega$ with the same monodromy as the original one and let X' be the space of solutions of the system. According to Corollary 5.1, we can assume that the system is reduced to (5.9)–(5.13) with a_l replaced by a_t and with γ_1 and γ_2 replaced by γ_1' and γ_2'. Since, by assumption, $\gamma_{\omega'} < \gamma_\omega$ and the parity of $\gamma_{\omega'}$ is the same as that of γ_ω (see Proposition 5.1), it follows that

$$\gamma_1' < \gamma_1,\quad \gamma_2' < \gamma_2. \tag{5.21}$$

We choose a basis (e') in X' so that the fundamental matrix $T'(y)$ constructed from this basis and the matrices $T_l(y)$ for X given by (5.10) satisfy the equality

$$(g_i^* T')(y) = (g_i^* T_l)(y),\quad i = 1, \ldots, n. \tag{5.22}$$

Let us consider $(e')|_{\widetilde{S}_t}$. There are two possibilities for the normalizations φ_t of (e'):

a) $\varphi_t(e_1') \neq \varphi_t(e_2')$;

b) $\varphi_t(e_1') = \varphi_t(e_2') = b - \gamma_2'$.

In case b) there are numbers $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$ such that $\varphi_t(\lambda_1 e_1' + \lambda_2 e_2') = b + \gamma_1'$. We pass from the basis (e) of X to $(\widetilde{e}) = (e)S$ and from the basis

e') of X' to $(\widetilde{e}') = (e')S$, where

$$S = \begin{pmatrix} 1 & \frac{\lambda_1}{\lambda_2} \\ 0 & 1 \end{pmatrix} \tag{5.23}$$

in case b), and $S = I$ in case a).

In case b), by construction, $\varphi_t(\widetilde{e}'_1) = b - \gamma'_2$, $\varphi_t(\widetilde{e}'_2) = b - \gamma'_1$, and E_t has the lower triangular form in this basis (since, by virtue of properties 1) and 2) of the normalizations, $\widetilde{e}'_2$ is an eigenvector of G_t). Therefore, $(\widetilde{e})$ is formally not an associated basis, but it is easy to see that it also satisfies Propositions 2.1–2.3.

Either case a) can be reduced to case b) if $\varphi_t(e'_1) < \varphi_t(e'_2)$, or $\varphi_t(e'_1) > \varphi_t(e'_2)$, E_t has the upper triangular form, and Propositions 2.1–2.3 hold again. Thus, for the fundamental matrix $\widetilde{T}' = T'S$, where S is given by (5.23), the factorization (2.8), (2.9) holds with

$$A'_t = \operatorname{diag}(b_{11}, b_{22}), \tag{5.24}$$

where one of the numbers b_{ij} is equal to $b + \gamma'_1$, while the other one is equal to $-b - \gamma'_2$.

For $\widetilde{T}_l = T_l \cdot S$ considered on $\widetilde{S}_l$, the factorization (5.10)–(5.13) holds again with $U_l(z)$ replaced by $\widetilde{U}_l(z)$ and with E_l replaced by $S^{-1}E_lS$, where

$$\widetilde{U}_l(z)(z - a_l)^{A_l} = U_l(z)(z - a_l)^{A_l} S = U_l(z) \widetilde{S}(z)(z - a_l)^{A_l},$$

$$\widetilde{S}(z) = \begin{pmatrix} 1 & \frac{\lambda_1}{\lambda_2}(z - a_l)^{\gamma_\omega} \\ 0 & 1 \end{pmatrix}$$

in case b), and $\widetilde{S} = I$ in case a). Thus, in case b)

$$\widetilde{U}_l(z) = \begin{pmatrix} 1 & c(z - a_l)^m + \frac{\lambda_1}{\lambda_2}(z - a_l)^{\gamma_\omega} \\ s(z - a_l)^k & 1 \end{pmatrix} (1 + o(1)).$$

Since, by assumption, either $c \neq 0$ and $m \geqslant \gamma_\omega$, or $c = 0$, it follows that $\widetilde{U}_l(z)$ again has the form (5.13) with c replaced by c' and m replaced by m', so that either $c' \neq 0$ and $m' \geqslant \gamma_\omega$, or $c' = 0$.

We consider the matrix-valued function

$$Y(y) = \widetilde{T}_l(y)(\widetilde{T}'(y))^{-1}. \tag{5.25}$$

It follows from (5.22) and (5.23) that $Y(y)$ is a single-valued function on CP^1. From (5.9) we find that $Y(z)$ is holomorphically invertible outside the set of points a_l, a_t, and from (5.10)–(5.13) we find that the factorization

$$Y(z) = \widetilde{U}_l(z)(z - a_l)^{A_l}(\widetilde{U}'_l)^{-1}$$

holds in a neighbourhood of a_l. Thus, the first row $y_1(z)$ of Y can be expressed as

$$y_1(z) = (\alpha\,(z - a_l)^{b+\gamma_1},\quad \beta c'\,(z - a_l)^{b+m'-\gamma_2})\ (1 + o\,(1))$$

in this neighbourhood. Hence, by virtue of the condition $m \geqslant \gamma_m$, if $c' \neq 0$ (and of course in the case $c' = 0$), we find that

$$y_1(z) = (z - a_l)^{b+\gamma_1}\, t_1(z), \tag{5.26}$$

where $t_1(z)$ is a vector-valued function holomorphic at a_l.

The matrix Y has the form

$$Y(z) = \tilde{U}_t(z)(z - a_t)^{-\tilde{A}'_t}\,(\tilde{U}'_t(z))^{-1}$$

in a neighbourhood of a_t, where $\tilde{A}'_t$ is given by (5.24). In this case we find for the row $y_1(z)$ of Y that

$$y_1(z) = \frac{1}{(z - a_t)^{b+\gamma_1}}\, t_2(z), \tag{5.27}$$

where $t_2(z)$ is a vector-valued function holomorphic at a_t.

Since $y_1(z)$ is holomorphic outside the set of points a_l, a_t, it follows from (5.21), (5.26), and (5.27) that the degree of the divisor of each of the components of $y_1(z)$ on CP^1 is strictly positive, and so $y_1(z) \equiv 0$, which contradicts the fact that $Y(z)$ is holomorphically invertible outside the set of points a_l, a_t. The contradiction means that $\gamma_\chi = \gamma_\omega$.

***Lemma* 5.4.** *Let* (3) *and* (4) *be a Fuchsian system reduced to* (5.9)–(5.13). *Then the form* ω *of the system can be written as*

$$\omega = \begin{pmatrix} \dfrac{\beta_1^1}{z - a_l} & c\,(m - \beta_1^1 + \beta_1^2)(z - a_l)^{m-1} + \varepsilon\,(z - a_l)^{\gamma_\omega - 1} \\ s\,(k + \beta_1^1 - \beta_1^2)(z - a_l)^{k-1} & \dfrac{\beta_1^2}{z - a_l} \end{pmatrix}(1 + o(1))dz\,. \tag{5.28}$$

in a neighbourhood of a_l.

Proof. We find from (5.13) that

$$U_l^{-1}(z) = \begin{pmatrix} 1 & -c\,(z - a_l)^m \\ -s\,(z - a_l)^k & 1 \end{pmatrix}(1 + o\,(1)),$$

$$\frac{dU_l}{dz} = \begin{pmatrix} \alpha\,(z - a_l)^{t_1} & cm\,(z - a_l)^{m-1} \\ sk\,(z - a_l)^{k-1} & \delta\,(z - a_l)^{t_2} \end{pmatrix}(1 + o\,(1)),$$

where $t_1 \geqslant 0$ and $t_2 \geqslant 0$. (5.28) follows now from (2.10).

Lemma 5.5. *Suppose that an element $\omega_{pq}dz$ of the matrix form ω of a Fuchsian system* (3), (4) *has a decomposition of the form*

$$\omega_{pq} = \frac{1}{z-a_l} c_{-1}^{pq} + c_0^{pq} + (z-a_l)\, c_1^{pq} + \dots \tag{5.29}$$

in a neighbourhood of a_l. Then the elements b_{pq}^i of the matrix B^i in (4) *are connected with the numbers c_k^{pq} by the following relations:*

$$\begin{cases} b_{pq}^l = c_{-1}^{pq}, \\ \displaystyle\sum_{\substack{i=1 \\ i\neq l}}^{n} b_{pq}^i \frac{1}{(a_i-a_l)^r} = -c_{r-1}^{pq}, \quad r = 1, \dots \end{cases} \tag{5.30}$$

Proof. From (5) we get

$$\omega - B^l \frac{dz}{z-a_l} = \sum_{\substack{i=1 \\ i\neq l}}^{n} B^i \frac{dz}{z-a_i},$$

which yields the proof of the lemma

Proposition 5.3. *The Fuchsian weight γ_χ of any representation* (2) *satisfies the inequality $\gamma_\chi < n-1$. If* (2) *is an irreducible representation, then*

$$\gamma_\chi \leqslant n-2. \tag{5.31}$$

Proof. Let us consider a Fuchsian system (4) with monodromy (2) and with Fuchsian weight γ_χ, reduced to the form (5.9)–(5.13). Suppose that $\gamma_\chi > n-2$. Then, according to Proposition 5.2, $m \geqslant n-1$, where m is the number in (5.13). In this case it follows from Lemma 5.4 that the element ω_{12} of the form ω of the system can be written as $\omega_{12} = o((z-a_l)^{n-3})dz$ in a neighbourhood of a_l. It also follows from (5), (2.14), and the condition $\gamma_\chi > 0$ that $b_{12}^l = 0$ and $b_{12}^1 + \dots + b_{12}^n = 0$. Therefore, we find from Lemma 5.5 that the numbers b_{12}^i for $i \neq l$ satisfy the system of $n-1$ equations (5.30)

$$\sum_{\substack{i=1 \\ i\neq l}}^{n} b_{12}^i \frac{1}{(a_i-a_l)^r} = 0, \quad r = 0, \dots, n-2,$$

whose determinant is the Vandermonde determinant of the numbers $\frac{1}{a_1-a_l}, \dots, \hat{l}, \dots, \frac{1}{a_n-a_l}$, and so it is non-zero. Thus, $b_{12}^i = 0$ for $i = 1, \dots, n$. Consequently, all the matrices B^i of (4) have the lower triangular form, and from Corollary 2.1 we find that

$$b_{11}^l = \rho_l^1 + \varphi_l^1,\ b_{22}^l = \rho_l^2 + \varphi_l^2,\ b_{11}^i = \rho_i^1,\ b_{22}^i = \rho_i^2,\ i \neq l. \tag{5.32}$$

Since (5) implies that $\sum_{i=1}^{n} \rho_i^j + \varphi_i^j = 0$ for $j = 1, 2$, and (2.5) implies that $0 \leqslant \operatorname{Re} \sum_{i=1}^{n} \rho_i^j \leqslant n - 1$ for $j = 1, 2$, it follows from the last two equalities that $\varphi_i^1 - \varphi_i^2 \leqslant n-1$, which in conjunction with the assumption that $\gamma_\chi > n-2$ yields the equality $\gamma_\chi = n-1$.

Since the matrices B^i have the lower triangular form, it follows that the representation (2) is reducible in the case in question.

***Proposition* 5.4.** *For any points $a_1, \ldots, a_n$ and any number γ that satisfies the inequalities*

$$0 < \gamma \leqslant n - 2, \tag{5.33}$$

there is an irreducible representation (2) *such that $\gamma_\chi = \gamma$ and the monodromy matrices G_i for the group* Im χ *given by* (2.5) *are non-diagonable.*

Proof. Suppose, as before, that $a_1 = 0$ and ∞ is not among the points in D. (We can always make sure that this is the case by applying a conformal transformation of the Riemann sphere.)

1. First, we shall prove the proposition in the case of an even $\gamma = 2\gamma'$. Let us consider the following two systems of equations for the unknowns $d_2, \ldots, d_n$ and $c_2, \ldots, c_n$:

$$\sum_{i=2}^{n} d_i \frac{1}{a_i^r} = \delta_{r,\gamma}, \tag{5.34}$$

and

$$\sum_{i=2}^{n} c_i \frac{1}{a_i^r} = x^2\delta_{r,n-2}, \tag{5.35}$$

where $r = 0, 1, \ldots, n$ and $\delta_{ij} = \begin{cases} 0, & i \neq j, \\ 1, & i = j. \end{cases}$

Since the determinants of (5.34) and (5.35) do not vanish, there is a j such that $d_j \neq 0$, d being the solution of (5.34). But any solution c of (5.35) has the form $c = x^2 t$ with $t_i \neq 0$ for $i = 2, \ldots, n$. Thus, we can choose the values of the roots $\sqrt{d_i t_i}$ so that $s = \sum_{i=2}^{n} \sqrt{d_i t_i} \neq 0$. Let us set $x = -\frac{\gamma}{2s} = -\frac{\gamma'}{s}$ in (5.35). Then $\sum_{i=2}^{n} \sqrt{d_i c_i} = xs = -\frac{\gamma}{2} = -\gamma'$, and so the matrices

$$B^1 = \begin{pmatrix} \gamma' & 0 \\ 0 & -\gamma' \end{pmatrix}, \quad B^i = \begin{pmatrix} \sqrt{d_i c_i} & d_i \\ -c_i & -\sqrt{d_i c_i} \end{pmatrix} \tag{5.36}$$

satisfy (5).

Let us consider a Fuchsian system (4) with (5.36). From Corollary 2.1 and the form of the matrices (5.36) we find that

$$\begin{cases} \beta_1^1 = \varphi_1^1 = \gamma', & \beta_1^2 = \varphi_1^2 = -\gamma', \\ \beta_i^j = \rho_i^j = \varphi_i^j = 0, & i \neq 1, \quad j = 1, 2 \end{cases} \tag{5.37}$$

are the exponents of the space X of solutions of the system.

Let us consider the factorization (2.8) for X at $a_1 = 0$:

$$T_1(y) = U_1(z)\, z^{A_1} y^{E_1}. \tag{5.38}$$

It follows from (5.37) that

$$A_1 = \begin{pmatrix} \gamma' & 0 \\ 0 & -\gamma' \end{pmatrix}, \quad E_1 = \begin{pmatrix} 0 & \alpha \\ 0 & 0 \end{pmatrix}. \tag{5.39}$$

From (5.36) and (2.14) we find that

$$\begin{pmatrix} \gamma' & 0 \\ 0 & -\gamma' \end{pmatrix} = U_1(0) \begin{pmatrix} \gamma' & 0 \\ 0 & -\gamma' \end{pmatrix} U_1^{-1}(0),$$

and so

$$U_1(0) = \begin{pmatrix} u_{11} & 0 \\ 0 & u_{22} \end{pmatrix}.$$

We change the basis of $X|_{\tilde{S}_1}$: $e_1' = \frac{e_1}{u_{11}}$ and $e_2' = \frac{e_2}{u_{22}}$. In the new basis, which we call (e) as before, the matrices in (5.38) have the form (5.13) and (5.39) with a_l replaced by $a_1 = 0$ and with α replaced by $\varepsilon = \alpha \frac{u_{11}}{u_{22}}$. From (5.34) and Lemma 5.5 for the matrix form ω of the constructed system (3), (4), (5.36), we get

$$\omega_{12} = (z^{\gamma-1} + o(z^{\gamma-1}))\, dz. \tag{5.40}$$

It follows from (5.40) and Lemma 5.4 that the number m for $U_1(z)$ in (5.13) satisfies the inequality $m \geqslant \gamma$. From this equality and Proposition 5.2 we find that $\gamma_\omega = \gamma_\chi = \gamma$ for the constructed system.

It remains to prove that the monodromy matricres $G_1, \ldots, G_n$ of the constructed system (4), (5.36) are non-diagonable and the monodromy representation of the system is irreducible.

By virtue of (2.5), to prove that $G_1, \ldots, G_n$ are non-diagonable it suffices to show that so are $E_1, \ldots, E_n$. It follows from (5.37) and (2.11) that for $i \neq 1$ the matrices L_i in (2.14) coincide with E_i. Thus, by virtue of (2.14), the matrices E_i are non-diagonable since the matrices B_i in (5.6) are non-diagonable for $i \neq 1$.

We claim that E_i is also non-diagonable. We observe that if $c \neq 0$, then $m = \gamma$, and if $c = 0$, then the expression $c(m - \beta_1^1 + \beta_1^2)$ in (5.28) vanishes by virtue of (5.37). Therefore, if $m \geqslant \gamma$, then the element ω_{12} in (5.28) is of the form

$$\omega_{12} = (\varepsilon z^{\gamma_\omega - 1} + o\,(z^{\gamma_\omega - 1}))\,dz.$$

Comparing the last equality with (5.40) and using the equality $\gamma_\omega = \gamma_\chi = \gamma$, which was proved earlier, we find that $\varepsilon = 1$ and $\alpha = \frac{u_{22}}{u_{11}} \neq 0$, which means that E_1 is a non-diagonable matrix.

However, if the monodromy representation of the constructed system (4), (5.36) were reducible, then the fact that $G_1, \ldots, G_n$ are non-diagonable would imply that the representation is commutative. In this case we would find from Proposition 5.1 that $\gamma_\chi = 0$, contrary to the equality $\gamma_\chi = \gamma > 0$ already proved.

2. We consider the case when $\gamma = 2\gamma'' + 1$ is odd. In this case the proof can be carried out exactly as in the case of even γ but with the matrices B^i in (5.36) replaced by

$$B^1 = \begin{pmatrix} -n+2+\gamma'' & 0 \\ 0 & -n+1-\gamma'' \end{pmatrix}, \tag{5.36'}$$

$$B^i = \begin{pmatrix} \sqrt{d_i c_i} + \frac{2n-3}{2n-2} & d_i \\ -c_i & -\sqrt{d_i c_i} + \frac{2n-3}{2n-2} \end{pmatrix}, \quad i \neq 1,$$

with A_1 replaced by

$$A_1 = \begin{pmatrix} -n+2+\gamma'' & 0 \\ 0 & -n+1-\gamma'' \end{pmatrix},$$

and with the exponents in (5.37) replaced by

$$\begin{cases} \beta_1^1 = \varphi_1^1 = -n+2+\gamma'', \ \beta_1^2 = \varphi_1^2 = -n+1-\gamma'', \\ \beta_i^j = \rho_i^j = \frac{2n-3}{2n-2}, \quad i \neq 1, \ j = 1, 2. \end{cases} \tag{5.37'}$$

Remark 5.1. Since T_1 in (5.38) has the form (5.10)–(5.13), it follows that

$$\varphi_1(e_{11}^1) = \gamma', \quad \varphi_1(e_{12}^1) \geqslant 0, \quad \varphi_1(e_{21}^1) \geqslant \gamma', \quad \varphi_1(e_{22}^1) = -\gamma'$$

for the elements e_{ij}^1 of T_1.

Example 5.1. The monodromy representation of the Fuchsian system

$$df = \left[\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\frac{dz}{z} + \frac{1}{6}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\frac{dz}{z+1} + \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\frac{ldz}{z-1} + \frac{1}{3}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\frac{dz}{z-\frac{1}{2}}\right]. \tag{5.41}$$

with four singular points $a_1 = 0$, $a_2 = -1$, $a_3 = 1$, $a_4 = 1/2$ is of Fuchsian weight $\gamma_\chi = 2$ and the monodromy matrices $G_1, \ldots, G_4$ are non-diagonable.

§2. We denote by F_n a group with n generators $h_1, \ldots, h_n$ satisfying the identity relation $h_1 \cdot \ldots \cdot h_n = e$, and we denote by $\varkappa_a$ the isomorphism

$$\varkappa_a\colon \pi_1 (CP^1 \setminus D, z_0) \to F_n, \tag{5.42}$$

such that $\varkappa_a (g_i) = h_i$ for the generators g_i defined at the beginning of §2.

Any representation (2) can be written in the form $\chi = \chi' \circ \varkappa_a$, where

$$\chi'\colon F_n \to GL (p; C) \tag{5.43}$$

is a representation of the group F_n. In what follows we shall denote χ by $\chi(a)$, where $a = (a_1, \ldots, a_n)$.

The proof of the following proposition is straightforward.

Proposition **5.5.** *If for two sequences of points $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ there is a conformal transformation $r : CP^1 \to CP^1$ such that*

$$r(a_i) = b_i,\ i = 1, \ldots, n;\ r(\gamma_i^a) = \gamma_i^b,\ i = 1, \ldots, n,$$

for the paths γ_i defined at the beginning of §2, then $\gamma_{\chi(a)} = \gamma_{\chi(b)}$.

Proposition **5.6.** *Let the Fuchsian weight $\gamma_{\chi(a)}$ of a representation $\chi(a)$ with non-diagonable monodromy matrices $\chi'(h_1), \ldots, \chi'(h_n)$ be greater than one. Then there is an $\varepsilon > 0$ and an index l such that the inequality*

$$\gamma_{\chi(a')} < \gamma_{\chi(a)} - 1$$

holds for any sequence of points $a' = (a_1, \ldots, a_{l-1}, a_l + t, a_{l+1}, \ldots, a_n)$ with $0 < |t| < \varepsilon$.

Proof. Let us consider a Fuchsian system (4) with monodromy (2) and with Fuchsian weight $\gamma = \gamma_{\chi(a)}$ reduced to the form (5.9)–(5.13) for some point a_l. It follows from Proposition 5.4 that in (5.13) either $c \neq 0$ and $m \geqslant \gamma$, or $c = 0$. We use a conformal map to transform a_l into 0 and ∞ into ∞. We denote the resulting sequence again by $a_1, \ldots, a_n$.

We consider the isomonodromic deformation

$$df = \left[\left(\frac{B^1(t)}{z-t} + \sum_{i=2}^{n} \frac{B^i(t)}{z-a_i}\right) dz\right] f \tag{5.44}$$

of the constructed system, where

$$\begin{cases} \dot{B}^i(t) = \dfrac{1}{a_i - t} [B^i(t), B^1(t)], \\ B^i(0) = B^i,\ \sum\limits_{i=1}^{n} B^i(t) = 0, \end{cases} \tag{5.45}$$

and t varies over a small neighbourhood of 0. For sufficiently small t, (5.44) has the same monodromy as the original system (4) (see [23]).

From (5.12), (5.13), and (2.14) we get

$$B^1(0)=\begin{pmatrix} b+\gamma_1 & 0 \\ 0 & b-\gamma_2\end{pmatrix}.$$

Hence, it follows from (5.45) that

$$\frac{dB^i(0)}{dt}=\frac{1}{a_i}\begin{pmatrix} 0 & -\gamma b^i_{12} \\ \gamma b^i_{21} & 0\end{pmatrix}$$

and

$$B^i(t)=\begin{pmatrix} b^i_{11}+o(t) & b^i_{12}-\frac{\gamma}{a_i}b^i_{12}t+o(t) \\ b^i_{21}+\frac{\gamma}{a_i}b^i_{21}t+o(t) & b^i_{22}+o(t)\end{pmatrix}. \tag{5.46}$$

From this and again from (5.45), we have

$$B^1(t)=-\sum_{i=2}^{n}B^i(t)=\begin{pmatrix} b+\gamma_1+o(t) & o(t) \\ o(1) & b-\gamma_2+o(t)\end{pmatrix} \tag{5.47}$$

since, by Lemmas 5.5, 5.4 and Proposition 5.3, the equality

$$\sum_{i=2}^{n}\frac{b^i_{12}}{a_i^r}=h\delta_{r,\gamma},\ h\neq 0,\ r=0,\dots,\gamma,\ 2\leqslant\gamma\leqslant n-2 \tag{5.48}$$

holds.

The factorization (2.8) for the space X of solutions of (5.44) at t is of the form

$$T_1(y,t)=U_1(z,t)(z-t)^{A_1}(y-t)^{E_1}.$$

It follows from (2.14) that $U_1(t,t)B^1(0)=B^1(t)U_1(t,t)$. Thus

$$U_1(t,t)=\begin{pmatrix} 1+x_1\cdot t+o(t) & o(t) \\ o(1) & 1+x_2\cdot t+o(t)\end{pmatrix}.$$

We shall reduce the system (5.44) to the form (5.9)–(5.13). With this end in view we pass from the space X of solutions of the system to $X'=U^{-1}(t,t)X$. The matrices of coefficients of the new system have the form

$$\begin{cases} \tilde{B}^1(t)=\begin{pmatrix} b+\gamma_1 & 0 \\ 0 & b-\gamma_2\end{pmatrix}, \\ \tilde{B}^i(t)=U_1^{-1}(t,t)B^i(t)U_1(t,t)= \\ \quad =\begin{pmatrix} * & b^i_{12}-(x_1-x_2)b^i_{12}t-\gamma\frac{b^i_{12}}{a_i}t+o(t) \\ * & * \end{pmatrix}.\end{cases} \tag{5.49}$$

From (5.49) and (5.46) we get

$$\sum_{i=2}^{n} \frac{\widehat{b}^{i}_{12}(t)}{a_i^r} = o(t), \ 0 \leqslant r \leqslant \gamma - 2,$$

$$\sum_{i=2}^{n} \frac{\widetilde{b}^{i}_{12}(t)}{a_i^{\gamma-1}} = -\gamma h t + o(t).$$

Thus, according to Lemma 5.5, we find that for any sufficiently small t, $^{12}_{\gamma-2}(t) \neq 0$ in formula (5.29) for the form $\widetilde{\omega}$ of the new system in a neighbourhood of $z = t$. Hence, using (5.28) and the equality $\gamma_\omega = \gamma$, we find that $c \neq 0$ and $m \leqslant \gamma - 1$ in the factorization (5.13) for the new system. From Proposition 5.2 we deduce that $\gamma_{\chi(a')} < \gamma_{\chi(a)}$. Proposition 5.1 implies that the parity of $\gamma_{\chi(a)}$ is the same as that of $\gamma_{\chi(a')}$. Therefore, $\gamma_{\chi(a')} < \gamma_{\chi(a)} - 1$, where $a' = (a_1 + t, a_2, \ldots, a_n)$ and t is sufficiently small. To complete the proof of the proposition, we only need to apply the conformal transformation inverse to the conformal transformation introduced at the beginning of the proof and use Proposition 5.5.

§6. The Riemann–Hilbert problem for a system of three equations

In this section we shall describe all representations (2) of dimension $p = 3$ for which the Riemann–Hilbert problem has a negative solution.

***Proposition* 6.1.** *Suppose that a representation* (2) *of dimension* $p = 3$ *is reducible and each generator* G_i *of the group* Im χ *defined by* (2.5) *can be reduced to a Jordan block. The Riemann–Hilbert problem for the representation* (2) *is soluble if and only if the equality*

$$\gamma_{\chi_2} = 0 \tag{6.1}$$

holds for the corresponding two-dimensional subrepresentation or quotient representation χ_2 *of* (2).

Proof. Necessity. 1. Let us consider the system (4) with monodromy (2). Let X_i be an invariant subspace of dimension $i = 1$ or $i = 2$ of the space X of solutions of the system with respect to the action of the monodromy operators. Since the matrices E_j in (2.8) have the form of Jordan blocks, it follows that the first i elements of the associated basis at a_j constitute a basis of $X_i|_{\tilde{S}_j}$. It follows from property b) of the associated basis that $\varphi_j^1 \geqslant \varphi_j^2 \geqslant \varphi_j^3$. Thus,

$$\operatorname{Re} \beta_j^1 \geqslant \operatorname{Re} \beta_j^2 \geqslant \operatorname{Re} \beta_j^3, \ j = 1, \ldots, n, \tag{6.2}$$

where $\beta_j^1, \ldots, \beta_j^i$ are the exponents of X_i at a_j. From condition (2.16) for Fuchsian systems and (6.2) we find that

$$s_i = \sum_{j=1}^{n} \sum_{l=1}^{i} \beta_j^l \geqslant 0,$$

which, together with (3.11), yields

$$s_i = 0. \tag{6.3}$$

Hence, by virtue of (2.16) and (6.2), we conclude that

$$\operatorname{Re}\beta_j^1 = \operatorname{Re}\beta_j^2 = \operatorname{Re}\beta_j^3, \quad \varphi_j^1 = \varphi_j^2 = \varphi_j^3, \quad j = 1, \dots, n. \tag{6.4}$$

2. Let us consider the fundamental matrix $T(y)$ of X such that the first i of its columns form a basis of X_i. Then $T'(y) = U_1^{-1}(a_1)T(y)$ satisfies (3.14):

$$T'(y) = \begin{pmatrix} \tilde{T}(y) & * \\ & * \\ 0\,0 & e_{33} \end{pmatrix}, \quad T'(y) = \begin{pmatrix} e_{11} & ** \\ 0 & \\ 0 & \tilde{T}(y) \end{pmatrix}. \tag{6.5}$$

a) $i = 2$ b) $i = 1$

(Otherwise, according to the proof of Lemma 3.7, the sum in (6.3) could be increased, which is impossible by virtue of (3.11).) The matrix $\tilde{T}$ in (6.5) is the fundamental matrix of the space $\tilde{X}$ of solutions of (4) with the monodromy χ_2. Thus, we find from (6.4) and (6.5) a) that $\tilde{\varphi}_j^1 = \varphi_j^1$, $\tilde{\varphi}_j^2 = \varphi_j^2$, and $\gamma_{\chi_2} = 0$ for $i = 2$, where $\tilde{\varphi}_j^k$ are the normalizations for $\tilde{X}$.

According to Definition 2.2 of the normalizations, if (6.5) b) holds, then $\tilde{\varphi}_j^1 \geqslant \varphi_j^2$ and $\tilde{\varphi}_j^2 \geqslant \varphi_j^3$. Hence, by virtue of (6.3) and conditon (2.16) for Fuchsian systems, we find again that $\tilde{\varphi}_j^1 = \varphi_j^2$, $\tilde{\varphi}_j^2 = \varphi_j^3$, and $\gamma_{\chi_2} = 0$ for $i = 1$.

Sufficiency. 1. We consider a system (4) with regular singular points $a_1, \dots, a_n$ that is a Fuchsian system at $a_2, \dots, a_n$ with monodromy (2) such that the fundamental matrices T of the space of solutions X are of the form (6.5). (The existence of such a system is proved in Lemma 3.7.) We find from [6] and Corollary 5.1 that there is a meromorphic matrix-valued function $\tilde{\Gamma}(z)$ on CP^1 that is holomorphically invertible outside the set of points $a_1, \dots, a_n$ and such that for the space $\tilde{X}'$ with the fundamental matrix $\tilde{T}' = \tilde{\Gamma}(z)\tilde{T}$ the factorization (5.9)–(5.13) holds, where $\gamma_1 = \gamma_2 = 0$ by virtue of the condition $\gamma_{\chi_2} = 0$.

We denote by c_j the number $\varphi_j(e_{33})$ in case (6.5) a), and $\varphi_j(e_{11})$ in case (6.5) b). We pass from X to $X' = \Gamma(z)X$, where

$$\Gamma(z) = \begin{pmatrix} \Gamma_1(z) & 0 \\ & 0 \\ 0 \quad 0 & 1 \end{pmatrix}, \quad \Gamma(z) = \begin{pmatrix} 1 & 0 \quad 0 \\ 0 & \\ 0 & \Gamma_1(z) \end{pmatrix}, \tag{6.6}$$

a) $i = 2$ b) $i = 1$

$$\Gamma_1(z) = \prod_{j=1,\, j\neq l}^{n} \left(\frac{z-a_j}{z-a_l}\right)^{c_j} \tilde{\Gamma}(z).$$

The fundamental matrix $T' = \Gamma(z)T$ of X' is of the form (6.5), and the normalizations $'\varphi_j^k$ of X' satisfy the equalities

$$\text{(6.7)} \qquad \begin{cases} '\varphi_j^1 = '\varphi_j^2 = \varphi_j(e_{33}) = c_j, & j = 1, \ldots, n, \text{ in case a}), \\ c_j = '\varphi_j^1 = \varphi_j(\tilde{e}_1) = \varphi_j(\tilde{e}_2), & j = 1, \ldots, n, \text{ in case b}), \end{cases}$$

where $\tilde{e}_1$ and $\tilde{e}_2$ are the rows of $\tilde{T}$. For $j \neq l$ the equalities (6.7) follow from the form of $\Gamma_1(z)$ and (5.9). We shall prove the equalities for $j = l$. We denote by ρ_j the eigenvalues of the Jordan blocks E_j. For $e_{11}(y)$ and $e_{33}(y)$ we have

$$\text{(6.8)} \qquad \sum_{j=1}^{n} (\rho_j + c_j) = 0.$$

On the other hand, by virtue of (2.16), we find that

$$\sum_{j=1}^{n} (2\rho_j + '\tilde{\varphi}_j^1 + '\tilde{\varphi}_j^2) = 0$$

for $\tilde{X}'$. Since $'\tilde{\varphi}_j^1 = '\tilde{\varphi}_j^2$, it follows from this equality and (6.8) that

$$\sum_{j=1}^{n} c_j = \sum_{j=1}^{n} '\tilde{\varphi}_j^1 = \sum_{j=1}^{n} '\tilde{\varphi}_j^2.$$

The case $j = l$ now follows from (6.7) with $j \neq l$.

2. The remaining part of the proof breaks up into two cases: a) $i = 2$, b) $i = 1$.

First we consider case b). It follows from (6.7) and the construction of $\tilde{T}'$ that in a neighbourhood of a_j the matrix $T'(y)$ can be represented in the form

$$\text{(6.9)} \qquad T'(y) = U_j(z)(z - a_j)^{c_j I}(y - a_j)^{E_j},$$

where

$$U_j(z) = \begin{pmatrix} u_{11} & u_{12} & u_{13} \\ 0 & & \\ 0 & \multicolumn{2}{c}{\tilde{U}_j(z)} \end{pmatrix}$$

and $u_{11}(a_j) \neq 0$. $\tilde{U}_j(z)$ is holomorphically invertible at a_j. Generally speaking, $u_{12}(z)$ and $u_{13}(z)$ have poles of order p and q at a_j, respectively.

We denote the rows of $\tilde{U}_j(z)$ by $u_2(z)$ and $u_3(z)$. Since $u_2(a_j)$ and $u_3(a_j)$ are linearly independent, these are polynomials $Q_2\left(\frac{1}{z - a_j}\right)$ and $Q_3\left(\frac{1}{z - a_j}\right)$ of order not greater than $\max(p, q)$ such that the vector $t(z) = Q_2 u_2 + Q_3 u_3 + (u_{12}, u_{13})$ is holomorphic at a_j. (The construction of Q_2 and Q_3 is carried out in the same way as the construction of Q in Lemma 3.1.) We pass from X' to ${}^1X' = \Gamma_2(z)X'$, where

$$\Gamma_2(z) = \begin{pmatrix} 1 & Q_2 & Q_3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The matrix ${}^1U_j' = \Gamma_2(z)U_j$ is now holomorphically invertible at a_j. Thus, (6.7) b) takes the form $'\varphi_j^1 = {}'\varphi_j^2 = {}'\varphi_j^3 = c_j$.

By applying the procedure described above for all $j = 1, ..., n$, we obtain a space ${}^nX'$. Since the matrices ${}^nU_j'$ are holomorphically invertible, it follows from (6.9) and (2.10) that ${}^nX'$ is the space of solutions of a Fuchsian system (4) with the given monodromy (2).

Let us consider case a). By applying the $A(a_j, 3, 3, c_j)$ procedure to the matrix $T'(y)(y-a_j)^{-E_j}$, we obtain a space ${}^1X'$ such that the matrix ${}^1U_j'(z)$ in (6.9) is holomorphically invertible. We perform the procedure for all $j = 1, ..., n$. The remaining part of the proof is carried out in exactly the same way as in case b).

Remark 6.1. Suppose that the representation (2) is reducible and each of the generators G_i can be reduced to a Jordan block. If there is a Fuchsian system (4) with monodromy (2), then the exponents β_i^j for the space of solutions of the system satisfy the relations

$$\beta_i^1 = \ldots = \beta_i^p, \quad i = 1, \ldots, n.$$

The proof can be carried out by analogy with the first part of the proof of the necessity of the condition in Proposition 6.1.

Remark 6.2. The proof of the sufficiency in Proposition 6.1 is also valid if the assumption that each of the matrices G_i can be reduced to a Jordan block is replaced by the following: each of the matrices $G_i = \chi_2(g_i)$ can be reduced to the form $\begin{pmatrix} \lambda_i & * \\ 0 & \lambda_i \end{pmatrix}$.

Proof. If each of the matrices G_i has an eigenvalue of multiplicity three, the proof is the same as for Proposition 6.1. But if $\rho_l^1 \neq \rho_l^2$ or $\rho_l^2 \neq \rho_l^3$ for some l, then the proof is the same as that of the sufficiency of the condition in Proposition 6.1 up to formula (6.7) (which holds for $j \neq l$) and from part 2 to the end. Moreover, the procedure described in part 2 leads to the equalities $'\varphi_j^1 = {}'\varphi_j^2 = {}'\varphi_j^3 = c_j$ for $j \neq l$, $'\varphi_l^2 = {}'\varphi_l^3$ and, generally speaking, $'\varphi_l^1 \neq {}'\varphi_l^2$. (In particular, it may happen that $'\varphi_l^1 < {}'\varphi_l^2$.) Since E_l has the form

$$E_l = \begin{pmatrix} \rho_l^1 & 0 & 0 \\ 0 & & \\ 0 & & E_l' \end{pmatrix}$$

under the assumption $\rho_l^1 \neq \rho_l^2$, it follows again from (2.10) that the constructed system is Fuchsian.

***Proposition* 6.2.** *For any representation* (2) *of dimension* $p = 3$ *whose image is contained in the set of upper triangular matrices, the Riemann–Hilbert problem is soluble.*

Proof. We consider a system (3) with regular singular points $a_1, \ldots, a_n$ and with the given monodromy (2) such that the fundamental matrix $T(y)$ of the space X of solutions of the system has the form

$$T(y) = \begin{pmatrix} e_{11} & * & * \\ 0 & e_{22} & * \\ 0 & 0 & e_{33} \end{pmatrix}. \tag{6.10}$$

The existence of such a system was proved in part 2a) of Lemma 3.7. We set $c_j^i = \varphi_i(e_{jj})$ for the normalization of $e_{jj}(y)$ at a_i. We can assume that

$$c_i^1 = c_i^2 = c_i^3 = 0, \ i = 1, \ldots, \hat{l}, \ldots, n, \tag{6.11}$$

where l is any index belonging to $\{1, \ldots, n\}$. (Otherwise, applying the $\prod_{i \neq l} B(a_i, a_l, c_i^1, c_i^2, c_i^3)$ transformation to X, we would obtain a fundamental matrix of the form (6.10) with the condition (6.11).) The matrices G_i of the monodromy operators g_i^* in the basis formed by the columns of $T(y)$ have the upper triangular form. Hence, the same is true for the matrices E_i given by (2.5).

The remaining part of the proof can be reduced to the analysis of the following cases:

1) one of the matrices E_i is diagonable;
2) $\rho_i^1 = \rho_i^2$ for all i or $\rho_i^2 = \rho_i^3$ for all i;
3) $\rho_i^1 = \rho_i^3$ for all i and $\rho_l^1 \neq \rho_l^2$ for some l;
4) $\rho_l^1 \neq \rho_l^3$ for some l.

Case 1) is analysed in [7]. Case 2) follows from Remark 6.2 and Proposition 5.1. Let us consider case 3). We note that in this case $\varphi_l(e_{11}) = \varphi_l(e_{33}) = c_l$ by virtue of (6.11) and equality (2.16) for $e_{11}(y)$ and $e_{33}(y)$. We pass from the fundamental matrix $T(y)$ to $T_s(y) = T(y)S$, where S is an upper triangular matrix such that

$$S^{-1}E_lS = E_l^s = \begin{pmatrix} \rho_l^1 & 0 & 1 \\ 0 & \rho_l^2 & 0 \\ 0 & 0 & \rho_l^1 \end{pmatrix}. \tag{6.12}$$

We apply the procedure

$$A(a_l, 2, 2, \varphi_l(e_{22})) \cdot A(a_l, 3, 3, \varphi_l(e_{33})) \cdot \prod_{i \neq l} A(a_i, 2, 2, 0) \cdot A(a_i, 3, 3, 0) \tag{6.13}$$

to $T_s \cdot (y - a_l)^{-E_l^s}$. In this way, we find a space X' in which the factorization (2.8), (2.9) holds for any a_i with $i \neq l$, and $T_s(y)$ can be represented as

$$T_s(y) = U_l(z)(z - a_l)^{A_l}(y - a_l)^{E_l^s}, \tag{6.14}$$

$$A_l = \operatorname{diag}(c_l, \ \varphi_l(e_{22}), \ c_l),$$

at a_l, where $U_l(z)$ is holomorphically invertible at a_l. Proposition 2.1 holds for the matrices E_l^s and A_l. Thus, we find from (2.10) that X' is the space of solutions of a system (4) that is Fuchsian at a_l.

In case 4) the local representation χ_l given in (2.3) can be decomposed into a direct sum of a one-dimensional representation χ_l^1 and a two-dimensional representation so that the eigenvector e of χ_l^1 is either an eigenvector for all G_i (if $\rho_l^1 \neq \rho_l^2$), or it defines an eigenvector of the quotient representation χ_1 of χ (if $\rho_l^2 \neq \rho_l^3$). The proof of case 4) of Proposition 6.2 follows from the proof of Proposition 6.3 below.

***Proposition* 6.3.** *Let a representation* (2) *of dimension* $p = 3$ *be such that one of the local representations* χ_l *in* (2.3) *can be decomposed into a direct sum of a one-dimensional representation* χ_l^1 *and a two-dimensional representation. Then the Riemann–Hilbert problem for* (2) *is soluble.*

Proof. We consider a system (3) with regular singular points $a_1, \ldots, a_n$ that is Fuchsian everywhere except at a_l with the given monodromy representation (2). By virtue of (2.15) and (2.16), to prove the proposition it suffices to show that the sum s of the exponents of the space X of solutions of the system can always be increased.

We consider the factorization (2.8) for X in a neighbourhood of a_l with an associated basis chosen in such a way that E_l has the form

$$E_l = \begin{pmatrix} \rho_l^1 & 0 & 0 \\ 0 & \rho_l^2 & \varepsilon \\ 0 & 0 & \rho_l^3 \end{pmatrix}. \tag{6.15}$$

By Lemma 4.1, we can assume that in this case $U_l(z)$ has the form (4.1). For definiteness, let $c_{33} > 0$ in (4.1). The remaining part of the proof can be reduced to the analysis of the following cases:

1) $\varphi_l(e_{31}) \leqslant \varphi_l(e_{32})$ and $u_{31}^l(z) \not\equiv 0$;
2) $\varphi_l(e_{31}) > \varphi_l(e_{32})$;
3) $u_{31}^l(z) \equiv u_{32}^l(z) \equiv 0$, $u_{12}^l(a_l) = 0$, and $u_{12}^l(z) \not\equiv 0$;
4) $u_{31}^l(z) \equiv u_{32}^l(z) \equiv u_{12}^l(z) \equiv 0$.

In case 1) we apply the $A(a_l, 3, 1, \varphi_l(e_{31}))$ procedure to X, in case 2) we apply the $A(a_l, 3, 2, \varphi_l(e_{32}))$ procedure, and in case 3) we apply both the $A(a_l, 3, 3, \varphi_l(e_{33}))$ and the $A(a_l, 1, 2, \varphi_l(e_{12}))$ procedures. In each of the three cases we find as a consequence of Remark 3.1 that $s' > s$, where s' is the sum of the exponents of $X' = \Gamma(z)X$.

But if $u_{31}^l(z) \equiv u_{32}^l(z) \equiv u_{12}^l(z) \equiv 0$ (case 4)), then we find from (6.15) that for the factorization (2.8) at a_l the identity $e_{31}(y) \equiv e_{32}(y) \equiv e_{12}(y) \equiv 0$ holds, that is, the representation χ has the upper triangular form and the eigenvector e of the local representation χ_l^1 is not an eigenvector of χ and it does not define an eigenvector of the corresponding one-dimensional quotient representation. In this case Proposition 6.3 follows from parts 1)–3) of Proposition 6.2, which have already been proved.

From Theorem 1 and Propositions 6.1 – 6.3 we get the following result.

Corollary 6.1. *The Riemann – Hilbert problem for a representation* (2) *of dimension* $p = 3$ *has a negative solution if and only if the following three conditons hold*:

a) *each generator* G_i *of the group* $\operatorname{Im} \chi$ *can be reduced to a Jordan block*;

b) *the representation* (2) *is reducible*;

c) *the corresponding two-dimensional subrepresentation or quotient representation* χ_2 *is irreducible and* $\gamma_{\chi_2} > 0$.

We proceed to the proofs of Theorems 2 and 3, which were stated in the introduction.

Proof of Theorem 2. Let us consider a reducible representation (2) of dimension $p = 3$ such that each of the matrices G_i can be reduced to a Jordan block and the corresponding two-dimensional subrepresentation or quotient representation χ_2 is reducible. Since (2) is reducible, it follows that for the corresponding one-dimensional subrepresentation or quotient representation the number

$$r = \sum_{i=1}^{3} \rho_i$$

is an integer, ρ_i being the eigenvalues of the Jordan blocks E_i. Therefore, by Proposition 5.1, we conclude that γ_{χ_2} is an integral number. Since χ_2 is irreducible, it follows from Proposition 5.3 for $n = 3$ that $\gamma_{\chi_2} \leqslant 1$, and so $\gamma_{\chi_2} = 0$. Now Theorem 2 follows from Corollary 6.1.

Proof of Theorem 3.

1. First we shall prove the theorem for $p = 3$, for which it is sufficient to construct a representation (2) that satisfies the conditions of Corollary 6.1.

Let us consider the system ((4), (5.36)). Since the number of points n is greater than three, there are 2-vectors $b_2, \ldots, b_n$ such that $b_j \neq 0$ for $j = 2, \ldots, n$, and

$$\sum_{j=2}^{n} b_j = 0, \quad \operatorname{rank} {}'B^j = 2, \; j = 2, \ldots, n, \tag{6.16}$$

where we set

$${}'B^1 = \begin{pmatrix} 0 & z^{-\gamma'} & 0 \\ 0 & & \\ 0 & & B^1 \end{pmatrix}, \quad {}'B^i = \begin{pmatrix} 0 & b_i \\ 0 & \\ 0 & B^i \end{pmatrix}. \tag{6.17}$$

Here B^i are the matrices given by (5.36), and $\gamma' = \gamma_\chi/2$, where γ_χ is the Fuchsian weight of the monodromy representation (2) of ((4), (5.36)), and $\gamma_\chi > 0$.

Let us consider the system ((4), (6.17)). By virtue of (6.16), the system satisfies condition (5). We claim that the monodromy representation of the system satisfies the conditions of Corollary 6.1.

It follows from the construction of ((4), (6.17)) that conditions b) and c) of Corollary 6.1 are satisfied.

Since for $i > 1$ the matrices $'B^i$ given by (6.17) can be reduced to Jordan blocks, we find from (2.11) and (2.14) that the same is true for the matrices E_i, and so for $G_i = \exp(2\pi i E_i)$.

We consider the space of solutions of ((4), (6.17)) in a neighbourhood of $a_1 = 0$. Let $T_1(y)$ be the fundamental matrix of ((4), (5.36)) given by (5.38). Then

$$T = \begin{pmatrix} 1 & e_{12} & e_{13} \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{T_1}} \\ 0 & & \end{pmatrix} \tag{6.18}$$

is the fundamental matrix for ((4), (6.17)). Substituting $T(y)$ into this system, we get

$$\begin{cases} \dfrac{de_{12}}{dz} = z^{-\gamma'-1}e_{22} + \displaystyle\sum_{i=2}^{n} \frac{1}{z-a_i}(b_{12}^i e_{22} + b_{13}^i e_{32}), \\ \dfrac{de_{13}}{dz} = z^{-\gamma'-1}e_{23} + \displaystyle\sum_{i=2}^{n} \frac{1}{z-a_i}(b_{12}^i e_{23} + b_{13}^i e_{33}). \end{cases} \tag{6.19}$$

Since it follows from Remark 5.1 that

$$\varphi_1(e_{22}) = \gamma', \ \varphi_1(e_{23}) \geqslant 0, \ \varphi_1(e_{33}) = -\gamma', \ \varphi_1(e_{32}) \geqslant \gamma',$$

where $\varphi_1(f)$ is the normalization of f at $a_1 = 0$, we find from (6.19) that

$$\varphi_1\left(\frac{de_{12}}{dz}\right) = -1, \quad \varphi_1\left(\frac{de_{13}}{dz}\right) = -\gamma' - 1. \tag{6.20}$$

It follows from (6.20) that $a_1 = 0$ is a regular singular point for ((4), (6.17)).

The matrix E_1 in the factorization (2.8) of $T(y)$ has the form

$$E_1 = \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}. \tag{6.21}$$

If the equality $\alpha = 0$ were satisfied, y^{E_1} would be of the form

$$y^{E_1} = \begin{pmatrix} 1 & 0 & \beta \ln y \\ 0 & 1 & \ln y \\ 0 & 0 & 1 \end{pmatrix}$$

and e_{12} in (6.18) would be a single-valued meromorphic function in a neighbourhood of $a_1 = 0$. But then $\varphi_1\left(\frac{de_{12}}{dz}\right)$ would be non-negative if $e_{12}(z)$

olomorphic and it would be less than -1 if e_{12} had a pole at $a_1 = 0$. ve obtained a contradiction with the first equality in (6.20). This means $\neq 0$ in (6.21) and E_1 can be reduced to a Jordan block. Therefore, G_1 e same property. Condition a) of Corollary 6.1 is satisfied.

s, the Riemann–Hilbert problem for the monodromy representation (2) (6.17)) has a negative solution.

t us consider the matrices

$$(6.22) \qquad {}_p'B^i = \begin{pmatrix} \overbrace{0 \quad \cdots \quad 0}^{p} & \\ \vdots \qquad\quad \vdots & \\ \vdots \qquad\quad \vdots & \\ \qquad\qquad 0 & \\ 0 \quad \cdots \quad 0 & 'B^i \end{pmatrix}.$$

the monodromy representation (2) of ((4), (6.22)) there are no Fuchsian ems that realize the representation. Theorem 3 is proved.

mple 6.1. For the monodromy representation (2) of the system

$$(6.23) \qquad df = \left(\begin{pmatrix} 0 & 1 & 0 \\ 0 & z & 0 \\ 0 & 0 & -z \end{pmatrix} \frac{dz}{z^2} + \frac{1}{6} \begin{pmatrix} 0 & 6 & 0 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \frac{dz}{z+1} + \right.$$

$$\left. + \frac{1}{2} \begin{pmatrix} 0 & 0 & 2 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{pmatrix} \frac{dz}{z-1} + \frac{1}{3} \begin{pmatrix} 0 & -3 & -3 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \frac{dz}{z - \frac{1}{2}} \right) f$$

h regular singular points $0, -1, 1, 1/2$, there are no Fuchsian systems that alize the representation.

By virtue of (2.10), from Corollary 5.1 and from the proof of Theorem 3, obtain the following result.

rollary **6.2.** *Let a representation* (2) *of dimension* $p = 3$ *satisfy conditons* a),), c) *of Corollary* 6.1. *Then there is a system* (3) *with monodromy* (2) *and ith regular singular points* $a_1, \ldots, a_n$ *that is Fuchsian at all the points except ne, where the order of the pole of the matrix of coefficients is equal to* $\frac{\gamma_{\chi_2}}{2} + 1$, *nd there are no such systems with a pole of order less than* $\frac{\gamma_{\chi_2}}{2} + 1$.

From Proposition 5.3 and Corollary 6.2 we obtain the following result.

Corollary **6.3.** *For any representation* (2) *of dimension* $p = 3$ *there is a system* (3) *with regular singular points* $a_1, \ldots, a_n$ *and with monodromy* (2) *that is Fuchsian at all the points except perhaps one, where the order of the pole of the matrix of coefficients does not exceed* $\left[\frac{n}{2}\right]$, *where* $[x]$ *is the integral part of* x.

We write (2) in the form $\chi = \chi(a)$ (see (5.42) and (5.43)). From Proposition 5.6 and Corollary 6.1 we obtain the following result.

***Corollary* 6.4.** *For any sequence of points $a = (a_1, \ldots, a_n)$, any representation (5.43) of dimension $p = 3$, and any $\varepsilon > 0$, there is a sequence of points $a' = (a'_1, \ldots, a'_n)$ such that $|a'_i - a_i| < \varepsilon$ and the Riemann–Hilbert problem for $\chi(a')$ is soluble.*

Corollary 6.4 means that there are no sequences of (3×3)-matrices $G_1, \ldots, G_n$ with $G_1 \cdot \ldots \cdot G_n = I$ such that the Riemann–Hilbert problem for the representation $\chi(a)$ has a negative solution for all sequences of points $a = (a_1, \ldots, a_n)$.

§7. The Fuchsian weight and two-dimensional holomorphic vector bundles on the Riemann sphere

For any complex vector bundle with constant transition functions one can introduce an integrable holomorphic connection (see [24]). Let $(e_1^i, \ldots, e_p^i)$ be an arbitrary basis of sections of the bundle $F'|_{U_i}$, where F' denotes the vector bundle associated with the bundle constructed in part 1.3 of §1. The holomorphic connection on F' will be denoted by ∇', and the matrix of this connection in the basis $(e_1^i, \ldots, e_p^i)$ by ω^i. Let f denote the coordinates of a section s in (e^i). Then

$$\nabla' f = (d - \omega^i) f.$$

Thus, ∇' defines a family of local systems of differential equations on $CP^1 \backslash D$:

$$(d - \omega^i) f = 0. \tag{7.1}$$

In a basis $(s_1, \ldots, s_p)$ consisting of p global holomorphic sections that are linearly independent at any point, which exists because F' is a holomorphically trivial section, (7.1) defines a system of linear differential equations (3) on the entire set $CP^1 \backslash D$. It follows from the construction of the system that its monodromy coincides with (2). If we pass from $(s_1, \ldots, s_p)$ to $(s'_1, \ldots, s'_p) = (s_1, \ldots, s_p)\Gamma^{-1}(z)$, then the system (3) constructed from ∇' will turn into

$$df = \omega' f,$$

where ω' is given by (1.5). Thus, any connection defined on the trivial bundle is equivalent to a system of linear differential equations (to within scale transformations (1.5)).

We extend (F', ∇') to a bundle with a connection (F^0, ∇^0) on the entire set CP^1 in the following way. Let O_i be a neighbourhood of a_i that has simply-connected intersections with neighbourhoods $U_{\lambda_1}, \ldots, U_{\lambda_k}$ belonging to a covering $\{U_\lambda\}$. (See Fig. 2, where $k = 3$.)

The bundle F' in part 1.3 of §1 is constructed in such a way that only one of the transition functions $g_{kl} : U_{\lambda_k} \cap U_{\lambda_l} \to GL(p; C)$ differs from I. Let g_{1k} be this function; then $g_{1k} = G_i$. We choose the branch of $(y - a_i)^{E_i}$ on

U_{λ_i} with E_i given by (2.5) that corresponds to the sheet of the covering $O_i \cap U_{\lambda_1}$ that contains the point $y_i \in \widetilde{S}_i$ mentioned earlier. We set

$$g_{01} = (z - a_i)^{E_i}, \qquad (7.2)$$

$(z-a_i)^{E_i}$ is the chosen branch. We denote by g_{02} the analytic ation of g_{01} in $O_i \cap U_{\lambda_2}$ along a path around a_i oriented in the ockwise direction, and so on. Then we obtain

$$g_{0k} = g_{01} \cdot G_i = g_{01} \cdot g_{1k} \qquad (7.3)$$

$\cap U_{\lambda_k} \cap U_{\lambda_1}$. The extension of F' to all $a_i \in D$ as described above e denoted by F^0. If $(e_1^k, \ldots, e_p^k)$ is a basis of horizontal sections in $F'|_{U_{\lambda_k}}$ h means that the matrix ω' of ∇' does not vanish in this basis), then it ws from (7.2) and (7.3) that $(\xi_1, \ldots, \xi_p) = (e_1^k, \ldots, e_p^k) g_{0k}^{-1}$ is a basis in $\{a_i\}$. According to (1.5), the matrix ω_0^k of ∇' in the basis $(\xi_1, \ldots, \xi_p)|_{O_i \cap U_{\lambda_k}}$ ual to

$$\omega_0^k = dg_{0k} \cdot g_{0k}^{-1} = E_i \frac{dz}{z - a_i}. \qquad (7.4)$$

llows from (7.4) that ∇' can be extended to a connection ∇^0 on the entire dle F^0 that is holomorphic on F^0 except at the points belonging to D, re ∇^0 has poles of the first order.

n [20] the constructed extension (F^0, ∇^0) is called the Manin extension of ∇'). (In [20] the extension is constructed for an n-dimensional manifold a divisor D with normal intersections.) The connection ∇^0 has the given odromy (2). Moreover, if F^0 were holomorphically trivial, then the mann–Hilbert problem for (2) would be soluble. But, as a rule, F^0 is ologically non-trivial, because its first Chern class c_1 is equal to

$$c_1 = \sum_{i=1}^{n} \operatorname{Sp} E_i. \qquad (7.5)$$

e assume that the identification $H^2(CP^1, \mathbb{Z}) \cong \mathbb{Z}$ is given.)(1)

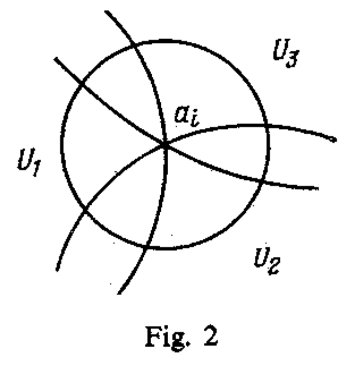


Fig. 2

For the principal bundle there is always a meromorphic section that is holomorphically vertible outside the set of points $a_1, \ldots, a_n$, independently of whether the associated ndle F^0 is trivial or not. Using this section rather than that of F' given in part 1.3, we btain a system with the given monodromy and with regular singular points $a_1, \ldots, a_n$.

Let us consider all possible extensions $(F_\lambda, \nabla_\lambda)$ of (F', ∇') onto the entire set CP^1 such that the extended connection ∇_λ has simple poles at the points belonging to D. All such extensions at a_i can be described with the aid of Proposition 2.3. In fact, let E_i be reduced to the Jordan normal form. We consider a block matrix A_i such that each A_i corresponds to a Jordan block E_i^m of E_i and $A_i^m = \mathrm{diag}(b_1^m, \dots, b_{k_m}^m)$, where k_m is the dimension of block, $b_i^m \in \mathbb{Z}$, and $b_1^m \geqslant \dots \geqslant b_{k_m}^m$. We replace the transition function g_{01} in (7.2) by

$$g_{01}^{\lambda_i} = (z - a_i)^{A_i^{\lambda_i}} (z - a_i)^{E_i}, \tag{7.6}$$

where $A_i^{\lambda_i}$ is an arbitrary matrix described above. It follows from Propositions 2.1–2.3 that by extending the bundle (F', ∇') to all points a_i with the aid of (7.6), we obtain a bundle $(F_\lambda, \nabla_\lambda)$ with $\lambda = (\lambda_1, \dots, \lambda_n)$ such that the connection ∇_λ has singularities of Fuchsian type at the points $a_i \in D$.

***Proposition* 7.1.** *The Riemann–Hilbert problem for the reprsentation (2) is soluble if and only if at least one of the bundles F_λ is holomorphically trivial.*

The proof follows from Proposition 2.3.

By the Birkhoff–Grothendieck theorem [5], any holomorphic vector bundle F on CP^1 can be represented as a direct sum of one-dimensional bundles:

$$F \cong \mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_p), \tag{7.7}$$

where $\mathcal{O}(l)$ is a bundle defined by two neighbourhoods $U_0 = CP^1\backslash 0$ and $U_1 = CP^1\backslash\infty$ and the transition function $g_{01} = z^l$, where $l \in \mathbb{Z}$.

The sequence of numbers $k_1 \geqslant \dots \geqslant k_p$ in (7.7), which we shall call the *Grothendieck indices*, uniquely defines the holomorphic type of F, while the topological type of F is completely defined by the first Chern class $c_1 = k_1 + \dots + k_p$ of the bundle.

***Proposition* 7.2.** *Let (F^0, ∇^0) be a bundle constructed from a representation (2) of dimension $p = 2$ and of Fuchsian weight γ_χ. Then*

$$k_1 - k_2 = \gamma_\chi, \tag{7.8}$$

where k_1, k_2 are the Grothendieck indices of F^0.

Proof. Let us consider the space X of solutions of a Fuchsian system (4) with the given monodromy (2) and with Fuchsian weight γ_χ. According to Corollary 5.1, we can assume that the system is of the form (5.9)–(5.13), where, according to Proposition 5.2, the number m in (5.13) satisfies the inequality $m \geqslant \gamma_\chi$.

Let us consider the bundle F_l defined by the neighbourhoods $U_1 = CP^1\backslash a$ and $U_0 = O_l$ and the transition function $g'_{10} = U_l(z)(z-a_l)^{A_l}$, where $U_l(z)$ and A_l are given by (5.12) and (5.13). It is not difficult to show that the bundles d F^0 are holomorphically equivalent. It follows from (5.13) and (5.11)

$$U_l(z)(z-a_l)^{A_l} = (z-a_l)^{A_l} U'_l(z), \tag{7.9}$$

e

$$U'_l(z) = \begin{pmatrix} 1 & c(z-a_l)^{m-\gamma_\chi} \\ s(z-a_l)^{k+\gamma_\chi} & 1 \end{pmatrix}(1+o(1))$$

olomorphically invertible in O_l by virtue of the condition $m \geqslant \gamma_\chi$. Thus, s holomorphically equivalent to the bundle

$$\mathcal{O}(-b+\gamma_2) \oplus \mathcal{O}(-b-\gamma_1),$$

so

$$k_1 = -b+\gamma_2,\ k_2 = -b-\gamma_1,\ k_1-k_2 = \gamma_\chi. \tag{7.10}$$

ark 7.1. From (7.5) and the fact that $k_1+k_2 = c_1$, where c_1 is the first ern class of F^0, it follows that

$$k_1 = \frac{1}{2}\gamma_\chi + \frac{1}{2}\sum_{i=1}^{n} \operatorname{Sp} E_i, \quad k_2 = -\frac{1}{2}\gamma_\chi + \frac{1}{2}\sum_{i=1}^{n} \operatorname{Sp} E_i. \tag{7.11}$$

rollary **7.1.** *Any two-dimensional holomorphic vector bundle on CP^1 is lomorphically equivalent to a bundle F that is a continuation of the Manin ndle constructed from an irreducible representation* (2).

The proof follows from (7.7) and Propositions 5.4 and 7.2.

It follows that for any two-dimensional holomorphic vector bundle E on P^1 we can introduce an irreducible connection ∇_λ that is holomorphic in the mplement of a finite set of points, where it has singularities of the type of mple poles. We shall call such a connection an *irreducible Fuchsian nnection.*

We denote by n_λ the number of singular points of an irreducible Fuchsian onnection ∇_λ. let us consider the set Λ of all such connections and put

$$n = \min_{\Lambda} n_\lambda.$$

Corollary **7.2.** *The number n is connected with the Grothendieck indices k_1 and k_2 of E by the following relation, which holds for $k_1 > k_2$:*

$$n-2 = k_1-k_2. \tag{7.12}$$

The proof follows from Propositions 5.4 and 7.2 ($k_1 > k_2$).

Therefore, the holomorphic type (k_1, k_2) of a bundle on CP^1 is completely determined by the Chern class k_1+k_2 (which is a topological invariant) and the least number $n = k_1-k_2+2$ of singular points for all irreducible Fuchsian connections of the given bundle (which is an analytic invariant).

We consider the bundle (F_0^l, ∇_0^l) constructed by means of (7.6) for $A_l = \mathrm{diag}(-k_2, -k_1)$ and $A_1 = \ldots, \hat{l}, \ldots = A_n = 0$. From Proposition 7.1 and Corollary 5.1 we obtain the following corollary.

***Corollary* 7.3.** *F_0^l is a holomorphically trivial bundle for some l.*

We consider a finite covering $\mathfrak{U}$ of the Riemann sphere CP^1. Let $\mathfrak{W}$ be a covering such that $\mathfrak{W} \ll \mathfrak{U}$ (for the notation see [30], Russian ed., p. 117). The holomorphic vector bundle E on CP^1 is defined by a cocycle

$$f_E \in \mathbb{Z}^1(\mathfrak{U}; \mathcal{O}^{GL(p;C)}).$$

Propositions 7.2 and 5.6 yield the following result.

***Proposition* 7.3.** *For any two-dimensional holomorphic vector bundle E on CP^1 with the first Chern class c_1 and for any $\varepsilon > 0$ there is a holomorphic vector bundle E' such that $\|f_E - f_{E'}\|_{C(\mathfrak{W})} < \varepsilon$, and the Grothendieck indices k_1' and k_2' of E' are equal to*

$$k_1' = \left[\frac{c_1+1}{2}\right], \quad k_2' = c_1 - k_1',$$

where $[x]$ is the integral part of the number x.

From Proposition 7.2 we obtain the following formulation of Corollary 6.1.

***Proposition* 7.4.** *The Riemann–Hilbert problem for a representation* (2) *of dimension $p = 3$ has a negative solution if and only if the following three conditions are satisfied:*

1) *all local representations* (2.3) *constructed from* (2) *are irreducible;*

2) *the representation* (2) *is reducible and the corresponding two-dimensional subrepresentation or quotient representation χ_2 is irreducible;*

3) *the Grothendieck indices k_1 and k_2 of the Manin extension of the bundle constructed from χ_2 are distinct.*

Conclusion

1. Several formulations of the problem of reconstructing a Fuchsian equation from a representation (2) can be extracted from Riemann's work [25]. All of the formulations can be reduced to the following four:

1) the 21-st Hilbert problem formulated in the class of Fuchsian linear differential equations of order p (in what follows we denote it by H.);

2) the 21-st Hilbert problem in the class of Fuchsian linear systems (the R.–H. problem);

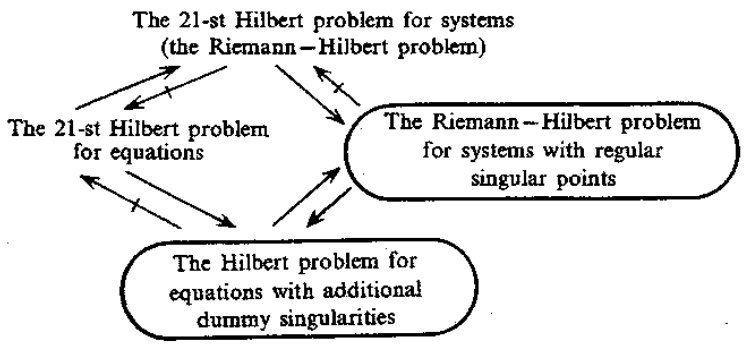
3) the analogue of the H. problem formulated in the class of Fuchsian linear differential equations with additional "dummy" singularities (that is, singularities that do not contribute to the monodromy); we denote the problem by H.D.;

the analogue of the R. – H. problem formulated in the class of systems egular singular points (the R.S. problem).

or example, the problem of defining a system of p functions that are tic outside the set of points $a_1, \dots, a_n$, have finite poles at $a_1, \dots, a_n$, and rgo the given transformations for any path around these points, which posed by Riemann in 1857, can be reduced in the obvious way to the problem (see [25]). The R. – H. problem and the R.S. problem with tional dummy singularities can easily be reduced to the R.S. problem. obtain such a reduction, one should use Lemma 4.1 and the scale nsformation $\Gamma_i(z) = \left(\frac{z - a_1}{z - b_i}\right)^{C_i}$ for all dummy singular points b_i.)

t follows from Plemelj's results [7] that the R.S. problem is soluble. gne's results [20] and the paper [26] by Nastold yield the solubility of the . problem. (See [27] and [28] for information on the number of additional mmy singular points arising for the solution of the H.D. problem.)

Since the H. problem for the representation (2) is soluble, the solubility of analogous R. – H. problem follows (see [28]). Finally, the solubility of the S. problem does not imply that the R. – H. problem is soluble (Example 6.1), d the solubility of the R. – H. problem for some representation (2) does not ply that the H. problem for this representation is soluble [28]. Everything id above can be reduced to the following diagram. (An oval shape means at the corresponding problem is soluble for all representations. An arrow eans that the solubility of one of the problems for a given representation) implies the solubility of the other.)



2. The classical Riemann – Hilbert problem admits several generaliztions. An analogous problem on a Riemann surface was considered in Röhrl's papers [17] and [29], already mentioned in the Introduction, and in [30] – [33].

The articles by Röhrl [17] and Deligne [20] gave an impetus to the formulation and the study of the multidimensional Riemann – Hilbert problem, which consists in studying the question of the existence of a completely integrable Pfaffian system of Fuchsian type on a complex analytic manifold M^n with a given divisor of singularities D and with a given monodromy group.

The first results obtained in this direction were concerned with the analysis of some trivial cases: a contractible Stein manifold (in this case the solubility of the Riemann–Hilbert problem was proved by Gerard in [34]) and an n-dimensional complex projective space for a commutative monodromy [35]–[37]. In [35] and [38] the first examples of negative solutions of the multidimensional Riemann–Hilbert problem were obtained.

It was also proved that the multidimensional Riemann–Hilbert problem on a projective manifold and on a Stein manifold is soluble in the class of systems with additional dummy singularities [39]. Moreover, it was proved in [40] that the problem is soluble on a connected Stein manifold of dimension two with the condition $H^2(M^n, \mathbb{Z}) = 0$. Necessary conditions for the Riemann–Hilbert problem to be soluble, expressed in terms of a representation of the fundamental group, were obtained in [41]. One of the obstacles in the path of the solution of the multidimensional Riemann–Hilbert problem is that the problem concerning the description of the local space of solutions of a Pfaffian system of Fuchsian type has not been solved so far. (The construction of such a space for a divisor D with a normal intersection was studied in [42]–[44].

3. The Riemann–Hilbert problem and its analogues have many applications in various areas of mathematics, physics, and mechanics. Among these applications we should like to mention the interesting applications to quantum field theory [45]–[49], the Wiener-Hopf theory [50], and mechancs [51]. For more detailed information about these applications, see [52].

Various aspects of the Riemann–Hilbert problem are presented in [53]–[59]. (The last two of these articles are connected with the multidimensional Riemann–Hilbert problem.)

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