Topic 9: Analysis of Riemann-Hilbert Problems

Cauchy integrals.

Definition of arcs and loops, chord-arc estimation, analyticity of Cauchy integral off contour.

Definition 1 (Arcs). *An arc C is a parametrized curve $z = z(t) = x(t)+\mathrm{i}y(t) \in \mathbb{C}$, $a \le t \le b$ for which $z'(t)$ exists as a continuous and nonvanishing function for $a < t < b$, extending continuously to $t = a$ and $t = b$, and for which $z(t)$ is one-to-one for $a \le t \le b$. Sometimes we identify the arc simply with its image in the complex plane, a simple curve with finite distinct endpoints $z(a)$ and $z(b)$ and with a continuously turning tangent at every point.*

Every arc C carries a natural orientation (increasing t) and can be reparametrized by arc length $s = s(t)$:

$$s(t) := \int_a^t \sqrt{x'(\tau)^2 + y'(\tau)^2}\,\mathrm{d}\tau = \int_a^t |z'(\tau)|\,\mathrm{d}\tau = \int_a^t |z'(\tau)\,\mathrm{d}\tau| = \int_{z(a)}^{z(t)} |\mathrm{d}z|.$$

Thus, the arc length differential is simply $\mathrm{d}s = |\mathrm{d}z|$. Since a straight line is the shortest path between two points, the arc length $s(z_1, z_2)$ between two points z_1 and z_2 lying on C satisfies

$$s(z_1, z_2) \ge |z_2 - z_1|, \quad \forall z_1, z_2 \in C. \tag{1}$$

Definition 2 (Loops). *A loop is a simple closed curve in the complex plane that is a finite collection of arcs placed end-to-end such that their orientations match and such that at each junction point the tangents of the two joining arcs make an angle $\theta \in (-\pi, \pi)$.*

Geometrically, loops are piecewise-smooth closed non-self-intersecting curves that can have corner points but not cusps. By the Jordan Theorem, each loop L divides the complex plane into a bounded set (the interior of L) and an unbounded set (the exterior of L). In considering the arc length between points z_1 and z_2 on a loop L, we will define $s(z_1, z_2)$ as the *shortest* of the two lengths between z_1 and z_2 along L. It can be shown that there exists a constant $0 < k_0 \le 1$ characteristic of each loop L such that

$$s(z_1, z_2) \ge |z_2 - z_1| \ge k_0 s(z_1, z_2) \quad \forall z_1, z_2 \in L. \tag{2}$$

From now on we will be considering matrix-valued functions on various sets in the complex plane. Consider the vector space $\mathbb{C}^{N\times N}$ of $N\times N$ complex matrices. This space can be equipped with any number of equivalent norms induced from a given norm $\|\cdot\|$ on the vector space $\mathbb{C}^N$ by the formula

$$\|\mathbf{A}\| := \max_{\|\mathbf{x}\|=1} \|\mathbf{A}\mathbf{x}\|$$

where on the right-hand side the norm is the given one on $\mathbb{C}^N$ and on the left-hand side we are defining the corresponding norm of a matrix $\mathbf{A} \in \mathbb{C}^{N\times N}$. Aside from satisfying all of the standard axioms of a norm, such a matrix norm behaves well with respect to matrix multiplication. Indeed, it follows from the definition that

$$\|\mathbf{A}\mathbf{B}\| \le \|\mathbf{A}\| \cdot \|\mathbf{B}\|, \quad \mathbf{A}, \mathbf{B} \in \mathbb{C}^{N\times N}. \tag{3}$$

Definition 3 (Cauchy integral). *Let L be a loop, and let $\mathbf{F} : L \to \mathbb{C}^{N\times N}$ be absolutely integrable with respect to arc length:*

$$\int_L \|\mathbf{F}(w)\|\,|\mathrm{d}w| < \infty.$$

For each $z \in \mathbb{C} \setminus L$, the Cauchy integral of $\mathbf{F}$ along L is

$$\mathcal{C}^L[\mathbf{F}](z) := \frac{1}{2\pi\mathrm{i}} \int_L \frac{\mathbf{F}(w)\,\mathrm{d}w}{w - z} = \frac{1}{2\pi\mathrm{i}} \sum_{j=1}^{\ell} \int_{a_j}^{b_j} \frac{\mathbf{F}(z_j(t))}{z_j(t) - z} z_j'(t)\,\mathrm{d}t,$$

where the functions $z = z_j(t)$, $j = 1, \dots, \ell$, parametrize the ℓ arcs $C_1, \dots, C_\ell$ making up L. The function $\mathbf{F} : L \to \mathbb{C}^{N\times N}$ is called the density of the Cauchy integral.

Lemma 1 (Piecewise analyticity of Cauchy integrals). *$\mathcal{C}^L[\mathbf{F}](\cdot)$ is an analytic function on the disconnected domain $\mathbb{C} \setminus L$, and $\mathcal{C}^L[\mathbf{F}](z) = O(z^{-1})$ as $z \to \infty$.*

Proof. Morera's Theorem and Fubini's Theorem to exchange integration order. Intuition: the integral is a superposition of functions $z \mapsto (w-z)^{-1}$ with pole w varying along the arcs C_j of L. Then

$$\|\mathcal{C}^L[\mathbf{F}](z)\| \le \frac{M^L(z)}{2\pi}\int_L \|\mathbf{F}(w)\|\,|\mathrm{d}w|, \quad M^L(z) := \max_{w\in L}\frac{1}{|w-z|},$$

and since L is bounded it is obvious that $M^L(z) = O(z^{-1})$ as $z \to \infty$. □

Hölder continuity, boundary values of Cauchy integrals.

Definition 4 (Hölder continuity). *Let $0 < \nu \le 1$. A matrix function $\mathbf{F}$ defined on some connected set $S \subset \mathbb{C}$ is said to be Hölder continuous on S with exponent ν if there exists a constant $K > 0$ such that $\|\mathbf{F}(z_2) - \mathbf{F}(z_1)\| \le K|z_2 - z_1|^\nu$ holds for all $z_1, z_2 \in S$.*

Thus every function that is Hölder continuous with exponent ν is continuous on S, but moreover we have a uniform estimate of the modulus of continuity in terms of a power function. The greater the value of ν, the smoother the function, although for no value of $\nu < 1$ can it be assumed that a derivative of any sort exists at any point. In the special case $\nu = 1$, Hölder continuity is sometimes called *Lipschitz continuity*, and it can be said that if S is an interval Lipschitz continuity implies the existence of a uniformly bounded derivative (Lebesgue) almost everywhere. It is an easy exercise to show that if $\mathbf{F}$ satisfies $\|\mathbf{F}(z_2) - \mathbf{F}(z_1)\| \le K|z_2 - z_1|^\nu$ for any $\nu > 1$ then $\mathbf{F} : S \to \mathbb{C}^{N\times N}$ is a constant function, which explains the restriction to $0 < \nu \le 1$. We denote by $H^\nu(S)$ the vector space of complex-valued (matrix) functions $\mathbf{F}$ that are Hölder continuous on S with exponent ν.

Let z_0 be an interior point of one of the arcs C_j of a loop L, and suppose that $\mathbf{F} : L \to \mathbb{C}^{N\times N}$ is not only absolutely integrable but also Hölder continuous with exponent ν in some neighborhood of z_0. We may write the integrand of the Cauchy integral of $\mathbf{F}$ along L in the form

$$\frac{\mathbf{F}(w)}{w-z} = \frac{\mathbf{F}(z_0)}{w-z} + \frac{\mathbf{F}(w)-\mathbf{F}(z_0)}{w-z} = \frac{\mathbf{F}(z_0)}{w-z} + \frac{\mathbf{F}(w)-\mathbf{F}(z_0)}{w-z_0} + \frac{(\mathbf{F}(w)-\mathbf{F}(z_0))(z-z_0)}{(w-z)(w-z_0)}$$

By the Residue Theorem,

$$\frac{1}{2\pi \mathrm{i}}\int_L \frac{\mathbf{F}(z_0)\,\mathrm{d}w}{w-z} = \frac{\mathbf{F}(z_0)}{2\pi\mathrm{i}}\int_L\frac{\mathrm{d}w}{w-z} = \begin{cases}\pm\mathbf{F}(z_0), & z \text{ in the interior of } L\\ 0, & z \text{ in the exterior of } L,\end{cases}$$

where the "+" sign ("−" sign) corresponds to positive (negative) orientation of the loop L. Also,

$$\left\|\int_L \frac{(\mathbf{F}(w)-\mathbf{F}(z_0))(z-z_0)}{(w-z)(w-z_0)}\,\mathrm{d}w\right\| = |z-z_0|\left\|\int_L\frac{\mathbf{F}(w)-\mathbf{F}(z_0)}{w-z_0}\frac{\mathrm{d}w}{w-z}\right\| \le |z-z_0|\int_L \frac{\|\mathbf{F}(w)-\mathbf{F}(z_0)\|}{|w-z_0|}\frac{|\mathrm{d}w|}{|w-z|}.$$

Let C_0 be the sub-arc of the arc C_j of L containing z_0 and all points of C_j of arc length at most $\delta/2$ from z_0, and suppose that δ is sufficiently small that $\mathbf{F}$ is Hölder continuous on C_0 with exponent ν. Then as $z \to z_0$ from either the interior or exterior of L, the Lebesgue Dominated Convergence Theorem implies that for each $\delta > 0$,

$$\int_{L\setminus C_0}\frac{\|\mathbf{F}(w)-\mathbf{F}(z_0)\|}{|w-z_0|}\frac{|\mathrm{d}w|}{|w-z|} \to \int_{L\setminus C_0}\frac{\|\mathbf{F}(w)-\mathbf{F}(z_0)\|}{|w-z_0|^2}\,|\mathrm{d}w| < \infty$$

because the singularity of the integrand at $w = z_0$ is bounded away from the contour of integration. Therefore, for each $\delta > 0$ and each $\epsilon > 0$ there is some $\eta = \eta(\delta,\epsilon) > 0$ such that

$$|z-z_0| < \eta(\delta,\epsilon) \quad\Longrightarrow\quad |z-z_0|\int_{L\setminus C_0}\frac{\|\mathbf{F}(w)-\mathbf{F}(z_0)\|}{|w-z_0|}\frac{|\mathrm{d}w|}{|w-z|} < \frac{\epsilon}{2}.$$

Also, using the Hölder condition satisfied by $\mathbf{F}$ on C_0,

$$|z-z_0|\int_{C_0}\frac{\|\mathbf{F}(w)-\mathbf{F}(z_0)\|}{|w-z_0|}\frac{|\mathrm{d}w|}{|w-z|} \le K|z-z_0|\int_{C_0}\frac{|\mathrm{d}w|}{|w-z_0|^{1-\nu}|w-z|} \le K\frac{|z-z_0|}{|w_*-z|}\int_{C_0}\frac{|\mathrm{d}w|}{|w-z_0|^{1-\nu}},$$

where w_* is a point of C_0 minimizing the Euclidean distance to $z \in \mathbb{C}\setminus L$. This upper bound can be made small provided that $z \to z_0$ from either the interior or exterior of L but in a *nontangential* fashion. This means that we choose once and for all some small positive angle $\theta > 0$ and insist that the vector from z_0 to z makes an angle $\phi(z)$ in the range $\theta \le |\phi(z)| \le \pi - \theta$ with the tangent vector to C_j at z_0. Consider the triangle T with vertices w_*, z_0, and z. Since w_* minimizes the distance to z, the corresponding side of

T is perpendicular to the tangent to C_j at w_*. Another side of T is the segment with endpoints w_* and z_0, a secant line to C_j through w_* that is very close to its tangent when $|z - z_0|$ is small and hence so is $|w_* - z_0|$. Therefore, when $|z - z_0|$ is small, T is approximately a right triangle with legs (w_*, z) and (w_*, z_0) and hypotenuse (z, z_0), so

$$\frac{|z - z_0|}{|w_* - z|} \approx \frac{1}{|\sin(\phi(z))|} \leq \frac{1}{\sin(\theta)}.$$

By assuming that $|z - z_0|$ is sufficiently small and that the angle of approach to z_0 is controlled by θ, we may therefore conclude that

$$\frac{|z - z_0|}{|w_* - z|} \leq \frac{2}{\sin(\theta)}.$$

For such z we then have

$$|z - z_0| \int_{C_0} \frac{\|\mathbf{F}(w) - \mathbf{F}(z_0)\|}{w - z_0} \frac{|\mathrm{d}w|}{|w - z|} \leq \frac{2K}{\sin(\theta)} \int_{C_0} \frac{|\mathrm{d}w|}{|w - z_0|^{1-\nu}}.$$

Now, let $\epsilon > 0$ be given. Since $|w - z_0|^{\nu - 1}$ is integrable for $\nu > 0$, we may choose $\delta = \delta(\epsilon)$ (the arc length of C_0) so small that

$$\frac{2K}{\sin(\theta)} \int_{C_0} \frac{|\mathrm{d}w|}{|w - z_0|^{1-\nu}} < \frac{\epsilon}{2}.$$

Then with $\delta = \delta(\epsilon)$ fixed in this way, requiring $|z - z_0| < \eta(\delta(\epsilon), \epsilon)$ gives

$$\left\| \int_L \frac{(\mathbf{F}(w) - \mathbf{F}(z_0))(z - z_0)}{(w - z)(w - z_0)} \,\mathrm{d}w \right\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

so we have proven the existence of the nontangential limit

$$\lim_{z \to z_0} \int_L \frac{(\mathbf{F}(w) - \mathbf{F}(z_0))(z - z_0)}{(w - z)(w - z_0)} \,\mathrm{d}w = 0. \tag{4}$$

Therefore, provided the density $\mathbf{F}$ satisfies a Hölder continuity condition near a non-corner point $z = z_0 \in L$, the Cauchy integral $\mathcal{C}^L[\mathbf{F}](z)$ has two (generally different) well-defined limiting values as $z \to z_0$ nontangentially from opposite sides of L:

$$\mathcal{C}^L[\mathbf{F}](z) \to \begin{cases} \pm\mathbf{F}(z_0) + \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_L \frac{\mathbf{F}(w) - \mathbf{F}(z_0)}{w - z_0} \,\mathrm{d}w, & z \to z_0 \text{ from the interior of } L \\ \dfrac{1}{2\pi \mathrm{i}} \displaystyle\int_L \frac{\mathbf{F}(w) - \mathbf{F}(z_0)}{w - z_0} \,\mathrm{d}w, & z \to z_0 \text{ from the exterior of } L \end{cases} \tag{5}$$

and the sign $\pm$ refers to the orientation of L. Note that the Hölder condition satisfied by $\mathbf{F}$ near z_0 ensures also that the integral term has an integrand that is absolutely integrable on L.

This calculation also goes through in the case that z tends to a corner point z_0 of L, provided that $\mathbf{F}$ is Hölder continuous with exponent ν on a neighborhood of the corner point and that (as is part of the definition of a loop) the corner has a nonzero interior angle (i.e., it's not a cusp). The meaning of nontangential approach of $z \to z_0$ is then that z cannot approach z_0 tangentially to either arc joining at z_0.

Plemelj formula, Plemelj-Privalov theorem, elementary properties of boundary operators.

Definition 5. *Let $\mathbf{F} : L \to \mathbb{C}^{N \times N}$ be Hölder continuous near a point $z_0 \in L$, with exponent $\nu \in (0, 1]$. Then the nontangential boundary values taken by $\mathcal{C}^L[\mathbf{F}](z)$ on L as $z \to z_0$ from the left (right) of L according to its orientation are denoted $\mathcal{C}_+^L[\mathbf{F}](z_0)$ ($\mathcal{C}_-^L[\mathbf{F}](z_0)$).*

An immediate corollary of the formula (5) is the following.

Proposition 1 (Plemelj formula)**.** *Suppose that $\mathbf{F}$ is Hölder continuous with exponent ν near a point z_0 of a loop L. Then,*

$$\mathcal{C}_+^L[\mathbf{F}](z_0) - \mathcal{C}_-^L[\mathbf{F}](z_0) = \mathbf{F}(z_0).$$

The vector space of Hölder continuous matrix functions on a loop L, $H^\nu(L)$, can be given a norm. Indeed, the inequality $\|\mathbf{F}(z_2)-\mathbf{F}(z_1)\| \le K|z_2-z_1|^\nu$ holding for all $z_1, z_2 \in L$ is equivalent to the condition $h_\nu(\mathbf{F}) < \infty$, where

$$h_\nu(\mathbf{F}) := \sup_{\substack{z_1,z_2\in L\\ z_1\neq z_2}} \frac{\|\mathbf{F}(z_2)-\mathbf{F}(z_1)\|}{|z_2-z_1|^\nu}.$$

The quantity h_ν satisfies the triangle inequality and is homogeneous with respect to scalar multiplication, but it is not a norm because $h_\nu(\mathbf{F}) = 0$ for all constant functions $\mathbf{F} \in H^\nu(L)$ (and constant functions indeed are contained in $H^\nu(L)$ for all $0 < \nu \le 1$). Therefore, it is necessary to add another term to distinguish the constants. Observe that if $h_\nu(\mathbf{F}) < \infty$ then in particular $\mathbf{F} : L \to \mathbb{C}^{N\times N}$ is continuous on L; therefore as L is a compact subset of $\mathbb{C}$ it follows also that $\|\mathbf{F}\|_\infty < \infty$, where

$$\|\mathbf{F}\|_\infty := \max_{z\in L} \|\mathbf{F}(z)\|.$$

The norm on $H^\nu(L)$ is defined by:

$$\|\mathbf{F}\|_\nu := \|\mathbf{F}\|_\infty + h_\nu(\mathbf{F}).$$

It still holds that $\mathbf{F} \in H^\nu(L)$ if and only if $\|\mathbf{F}\|_\nu < \infty$. We do not prove that this definition satisfies all of the axioms of a norm, but it is the case. It is also the case that $H^\nu(L)$ is complete with respect to convergence in this norm, so it makes $H^\nu(L)$ into a Banach space of matrix-valued functions on the loop L. Observe that $H^\nu(L)$ is also closed under pointwise multiplication: if $\mathbf{A}$ and $\mathbf{B}$ are in $H^\nu(L)$, then so is $\mathbf{AB}$ with the definition $\mathbf{AB}(z) := \mathbf{A}(z)\mathbf{B}(z)$ for $z \in L$. Moreover, we have

(6)

$$\begin{aligned}
\|\mathbf{AB}\|_\nu =&\max_{z\in L}\|\mathbf{A}(z)\mathbf{B}(z)\| + \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)\mathbf{B}(z_2)-\mathbf{A}(z_1)\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu}\\
=&\max_{z\in L}\|\mathbf{A}(z)\mathbf{B}(z)\| + \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)\mathbf{B}(z_2)-\mathbf{A}(z_2)\mathbf{B}(z_1)+\mathbf{A}(z_2)\mathbf{B}(z_1)-\mathbf{A}(z_1)\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu}\\
\le&\max_{z\in L}\|\mathbf{A}(z)\mathbf{B}(z)\| + \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)\mathbf{B}(z_2)-\mathbf{A}(z_2)\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu} + \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)\mathbf{B}(z_1)-\mathbf{A}(z_1)\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu}\\
\le&\max_{z\in L}\|\mathbf{A}(z)\|\cdot\|\mathbf{B}(z)\| + \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)\|\cdot\|\mathbf{B}(z_2)-\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu} + \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)-\mathbf{A}(z_1)\|\cdot\|\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu}\\
\le&\max_{z\in L}\|\mathbf{A}(z)\|\cdot\max_{z\in L}\|\mathbf{B}(z)\| + \max_{z\in L}\|\mathbf{A}(z)\| \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{B}(z_2)-\mathbf{B}(z_1)\|}{|z_2-z_1|^\nu}\\
&\qquad + \max_{z\in L}\|\mathbf{B}(z)\| \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{A}(z_2)-\mathbf{A}(z_1)\|}{|z_2-z_1|^\nu}\\
=&\|\mathbf{A}\|_\infty\|\mathbf{B}\|_\infty + \|\mathbf{A}\|_\infty h_\nu(\mathbf{B}) + \|\mathbf{B}\|_\infty h_\nu(\mathbf{A})\\
\le&\|\mathbf{A}\|_\infty\|\mathbf{B}\|_\infty + \|\mathbf{A}\|_\infty h_\nu(\mathbf{B}) + \|\mathbf{B}\|_\infty h_\nu(\mathbf{A}) + h_\nu(\mathbf{A})h_\nu(\mathbf{B})\\
=&\|\mathbf{A}\|_\nu\|\mathbf{B}\|_\nu,
\end{aligned}$$

where to get the fourth line we used (3). Therefore a natural analogue of the matrix norm inequality (3) holds also for the norm on $H^\nu(L)$.

Theorem 1 (Plemelj-Privalov)**.** *Suppose that $\mathbf{F} \in H^\nu(L)$, and that $0 < \nu < 1$. Then the boundary values $\mathcal{C}^L_\pm[\mathbf{F}](z_0)$ considered as functions of $z_0 \in L$ are also in $H^\nu(L)$, and moreover there is a constant $M > 0$ depending only on the geometry of L such that*

$$\|\mathcal{C}^L_\pm[\mathbf{F}]\|_\nu \le M\|\mathbf{F}\|_\nu, \quad \forall \mathbf{F} \in H^\nu(L).$$

Note that the theorem is false for $\nu = 1$, but if $\mathbf{F} \in H^1(L)$, then for each $\epsilon > 0$, the boundary values $\mathcal{C}_\pm^L[\mathbf{F}](\cdot)$ are in $H^{1-\epsilon}(L)$.

Proof. According to the formula (5), it is enough to show that for $\mathbf{G} : L \to \mathbb{C}^{N\times N}$ defined by

$$\mathbf{G}(z) := \frac{1}{2\pi \mathrm{i}} \int_L \frac{\mathbf{F}(w) - \mathbf{F}(z)}{w - z} \, \mathrm{d}w, \quad z \in L,$$

$\|\mathbf{G}\|_\nu \leq K \|\mathbf{F}\|_\nu$ holds for some overall constant K. First, observe that for $z \in L$,

$$\|\mathbf{G}(z)\| \leq \frac{1}{2\pi} \int_L \frac{\|\mathbf{F}(w) - \mathbf{F}(z)\|}{|w - z|} |\mathrm{d}w| \leq \frac{h_\nu(\mathbf{F})}{2\pi} \int_L \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}} \leq \frac{\|\mathbf{F}\|_\nu}{2\pi} \int_L \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}}.$$

Because $\nu > 0$, the latter integral is finite for every $z \in L$, and it is even a continuous function of $z \in L$. Therefore, taking the maximum over $z \in L$,

$$\|\mathbf{G}\|_\infty \leq K_1 \|\mathbf{F}\|_\nu, \quad K_1 := \frac{1}{2\pi} \max_{z \in L} \int_L \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}}. \tag{7}$$

Now consider $\mathbf{G}(z) - \mathbf{G}(z_0)$ for two points $z, z_0 \in L$:

$$\mathbf{G}(z) - \mathbf{G}(z_0) = \frac{1}{2\pi \mathrm{i}} \int_L \left[\frac{\mathbf{F}(w) - \mathbf{F}(z)}{w - z} - \frac{\mathbf{F}(w) - \mathbf{F}(z_0)}{w - z_0} \right] \mathrm{d}w.$$

One crude estimate of the difference is simply:

$$\begin{aligned} \|\mathbf{G}(z) - \mathbf{G}(z_0)\| &\leq \frac{1}{2\pi} \int_L \frac{\|\mathbf{F}(w) - \mathbf{F}(z)\|}{|w - z|} |\mathrm{d}w| + \frac{1}{2\pi} \int_L \frac{\|\mathbf{F}(w) - \mathbf{F}(z_0)\|}{|w - z_0|} |\mathrm{d}w| \\ &\leq \frac{h_\nu(\mathbf{F})}{2\pi} \left[\int_L \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}} + \int_L \frac{|\mathrm{d}w|}{|w - z_0|^{1-\nu}} \right] \\ &\leq \frac{\|\mathbf{F}\|_\nu}{2\pi} \left[\int_L \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}} + \int_L \frac{|\mathrm{d}w|}{|w - z_0|^{1-\nu}} \right] \\ &\leq 2K_1 \|\mathbf{F}\|_\nu. \end{aligned}$$

Recall that $s(z, z_0)$ denotes the shortest arc length along L between z and z_0. Using (2),

$$\frac{\|\mathbf{G}(z) - \mathbf{G}(z_0)\|}{|z - z_0|^\nu} \leq \frac{2K_1}{k_0^\nu s(z, z_0)^\nu} \|\mathbf{F}\|_\nu \leq \frac{2K_1}{k_0^\nu} \left(\frac{4}{s(L)} \right)^\nu \|\mathbf{F}\|_\nu, \quad \text{whenever} \quad s(z, z_0) \geq \frac{1}{4} s(L), \tag{8}$$

where $s(L)$ denotes the total arc length of L. Now, suppose $s(z, z_0) < \frac{1}{4} s(L)$, and let L_0 denote the part of L consisting of points w with $s(w, z_0) < 2s(z, z_0)$, a strict subset of L under the inequality in force on $s(z, z_0)$. L_0 contains both z and z_0. Then

$$\begin{aligned} \|\mathbf{G}(z) - \mathbf{G}(z_0)\| \leq \frac{1}{2\pi} &\int_{L_0} \left\| \frac{\mathbf{F}(w) - \mathbf{F}(z)}{w - z} - \frac{\mathbf{F}(w) - \mathbf{F}(z_0)}{w - z_0} \right\| |\mathrm{d}w| \\ &+ \frac{1}{2\pi} \left\| \int_{L \setminus L_0} \left[\frac{\mathbf{F}(w) - \mathbf{F}(z)}{w - z} - \frac{\mathbf{F}(w) - \mathbf{F}(z_0)}{w - z_0} \right] \mathrm{d}w \right\|. \end{aligned} \tag{9}$$

Now,

$$\begin{aligned} \int_{L_0} \left\| \frac{\mathbf{F}(w) - \mathbf{F}(z)}{w - z} - \frac{\mathbf{F}(w) - \mathbf{F}(z_0)}{w - z_0} \right\| |\mathrm{d}w| &\leq \int_{L_0} \frac{\|\mathbf{F}(w) - \mathbf{F}(z)\|}{|w - z|} |\mathrm{d}w| + \int_{L_0} \frac{\|\mathbf{F}(w) - \mathbf{F}(z_0)\|}{|w - z_0|} |\mathrm{d}w| \\ &\leq h_\nu(\mathbf{F}) \left[\int_{L_0} \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}} + \int_{L_0} \frac{|\mathrm{d}w|}{|w - z_0|^{1-\nu}} \right] \\ &\leq \|\mathbf{F}\|_\nu \cdot \left[\int_{L_0} \frac{|\mathrm{d}w|}{|w - z|^{1-\nu}} + \int_{L_0} \frac{|\mathrm{d}w|}{|w - z_0|^{1-\nu}} \right] \\ &\leq \frac{\|\mathbf{F}\|_\nu}{k_0^{1-\nu}} \left[\int_{L_0} \frac{|\mathrm{d}w|}{s(w, z)^{1-\nu}} + \int_{L_0} \frac{|\mathrm{d}w|}{s(w, z_0)^{1-\nu}} \right], \end{aligned}$$

where we used (2). By the reverse triangle inequality, $s(w,z) \geq |s(w,z_0) - s(z,z_0)|$. Therefore, recalling that $|\mathrm{d}w|$ denotes the arc length differential $\mathrm{d}s$,

$$\begin{aligned}
\int_{L_0} \left\| \frac{\mathbf{F}(w)-\mathbf{F}(z)}{w-z} - \frac{\mathbf{F}(w)-\mathbf{F}(z_0)}{w-z_0} \right\| |\mathrm{d}w| &\leq \frac{\|\mathbf{F}\|_\nu}{k_0^{1-\nu}} \left[\int_{-2s(z,z_0)}^{2s(z,z_0)} \frac{\mathrm{d}s}{||s|-s(z,z_0)|^{1-\nu}} + \int_{-2s(z,z_0)}^{2s(z,z_0)} \frac{\mathrm{d}s}{|s|^{1-\nu}} \right] \\
&= \frac{\|\mathbf{F}\|_\nu}{k_0^{1-\nu}} s(z,z_0)^\nu \left[\int_{-2}^{2} \frac{\mathrm{d}m}{||m|-1|^{1-\nu}} + \int_{-2}^{2} \frac{\mathrm{d}m}{|m|^{1-\nu}} \right] \\
&\leq K_2 \|\mathbf{F}\|_\nu |z-z_0|^\nu,
\end{aligned} \tag{10}$$

where

$$K_2 := \frac{1}{k_0} \int_{-2}^{2} \frac{\mathrm{d}m}{||m|-1|^{1-\nu}} + \frac{1}{k_0} \int_{-2}^{2} \frac{\mathrm{d}m}{|m|^{1-\nu}} < \infty.$$

To get the second line in (10) we used the rescaling substitution $s = s(z,z_0)m$, and to get the third line we again used (2). Next, consider the integral over $L \setminus L_0$, in which we split up the integrand as follows:

$$\frac{\mathbf{F}(w)-\mathbf{F}(z)}{w-z} - \frac{\mathbf{F}(w)-\mathbf{F}(z_0)}{w-z_0} = (z-z_0)\frac{\mathbf{F}(w)-\mathbf{F}(z)}{(w-z)(w-z_0)} - \frac{\mathbf{F}(z)-\mathbf{F}(z_0)}{w-z_0}$$

By exact integration,

$$\int_{L\setminus L_0} \frac{\mathbf{F}(z)-\mathbf{F}(z_0)}{w-z_0}\,\mathrm{d}w = (\mathbf{F}(z)-\mathbf{F}(z_0)) \int_{L\setminus L_0} \frac{\mathrm{d}w}{w-z_0} = (\mathbf{F}(z)-\mathbf{F}(z_0)) \left[\log(z'-z_0) - \log(z''-z_0)\right],$$

for some branches of the complex logarithm, where the initial endpoint of L_0 is z' and the terminal endpoint of L_0 is z''. Of course the imaginary parts of the logarithms are bounded, and

$$\mathrm{Re}\left[\log(z'-z_0) - \log(z''-z_0)\right] = \ln\left|\frac{z'-z_0}{z''-z_0}\right|.$$

But, using (2), we get

$$\ln(k_0) = \ln\left(\frac{k_0 s(z',z_0)}{s(z'',z_0)}\right) \leq \ln\left|\frac{z'-z_0}{z''-z_0}\right| \leq \ln\left(\frac{s(z',z_0)}{k_0 s(z'',z_0)}\right) = -\ln(k_0),$$

because both points z' and z'' have exactly the same arc length distance of $2s(z,z_0)$ from z_0. Since $0 < k_0 \leq 1$, from this it follows that the difference in logarithms is uniformly bounded independently of z_0 and z in L. Therefore, there is some constant $K_3 > 0$ such that

$$\left\| \int_{L\setminus L_0} \frac{\mathbf{F}(z)-\mathbf{F}(z_0)}{w-z_0}\,\mathrm{d}w \right\| \leq K_3 \|\mathbf{F}(z)-\mathbf{F}(z_0)\| \leq K_3 h_\nu(\mathbf{F})|z-z_0|^\nu \leq K_3 \|\mathbf{F}\|_\nu |z-z_0|^\nu. \tag{11}$$

Also, again using (2) and the reverse triangle inequality,

$$\begin{aligned}
\left\| \int_{L\setminus L_0} (z-z_0)\frac{\mathbf{F}(w)-\mathbf{F}(z)}{(w-z)(w-z_0)}\,\mathrm{d}w \right\| &\leq h_\nu(\mathbf{F})|z-z_0| \int_{L\setminus L_0} \frac{|\mathrm{d}w|}{|w-z|^{1-\nu}|w-z_0|} \\
&\leq \|\mathbf{F}\|_\nu |z-z_0| \int_{L\setminus L_0} \frac{|\mathrm{d}w|}{|w-z|^{1-\nu}|w-z_0|} \\
&\leq \frac{\|\mathbf{F}\|_\nu |z-z_0|}{k_0^{2-\nu}} \left[\int_{-s(L)/2}^{-2s(z,z_0)} + \int_{2s(z,z_0)}^{s(L)/2} \right] \frac{\mathrm{d}s}{||s|-s(z,z_0)|^{1-\nu}|s|}.
\end{aligned}$$

But it is easy to see that over the whole range of integration $2s(z,z_0) \leq |s| \leq \frac{1}{2}s(L)$,

$$\frac{|s|-s(z,z_0)}{|s|} = 1 - \frac{s(z,z_0)}{|s|} \geq 1 - \frac{1}{2} = \frac{1}{2},$$

so

$$\left\| \int_{L\setminus L_0} (z-z_0)\frac{\mathbf{F}(w)-\mathbf{F}(z)}{(w-z)(w-z_0)}\,\mathrm{d}w \right\| \le \frac{2^{1-\nu}\|\mathbf{F}\|_\nu |z-z_0|}{k_0^{2-\nu}}\left[\int_{-s(L)/2}^{-2s(z,z_0)} + \int_{2s(z,z_0)}^{s(L)/2}\right]\frac{\mathrm{d}s}{|s|^{2-\nu}}$$

$$= \left(\frac{2}{k_0}\right)^{2-\nu}\frac{\|\mathbf{F}\|_\nu |z-z_0|}{1-\nu}\left[(2s(z,z_0))^{\nu-1} - \left(\frac{1}{2}s(L)\right)^{\nu-1}\right]$$

$$\le \frac{2\|\mathbf{F}\|_\nu}{k_0^{2-\nu}(1-\nu)}\left[1-\left(\frac{1}{4}s(L)\right)^{\nu-1}|z-z_0|^{1-\nu}\right]|z-z_0|^\nu$$

where in the last line we used $s(z,z_0)\ge |z-z_0|$ and hence $s(z,z_0)^{\nu-1}\le |z-z_0|^{\nu-1}$ because $\nu<1$. Since $|z-z_0|^{1-\nu}$ is uniformly bounded as z,z_0 range over the bounded set L, there is evidently a constant $K_4>0$ such that

$$\left\| \int_{L\setminus L_0} (z-z_0)\frac{\mathbf{F}(w)-\mathbf{F}(z)}{(w-z)(w-z_0)}\,\mathrm{d}w \right\| \le K_4\|\mathbf{F}\|_\nu |z-z_0|^\nu. \tag{12}$$

Using (10), (11), and (12) in (9) shows that

$$\frac{\|\mathbf{G}(z)-\mathbf{G}(z_0)\|}{|z-z_0|^\nu} \le (K_2+K_3+K_4)\,\|\mathbf{F}(\cdot)\| \quad \text{whenever} \quad s(z,z_0)<\frac{1}{4}s(L). \tag{13}$$

Combining (8) with (13) and taking the supremum over $z,z_0\in L$ with $z\ne z_0$ gives

$$h_\nu(\mathbf{G})\le K_5\|\mathbf{F}\|_\nu, \quad K_5 := \max\left\{\frac{2K_1}{k_0^\nu}\left(\frac{4}{s(L)}\right)^\nu, K_2+K_3+K_4\right\}.$$

Finally, combining this with (7) gives

$$\|\mathbf{G}\|_\nu = \|\mathbf{G}\|_\infty + h_\nu(\mathbf{G}) \le K\|\mathbf{F}\|_\nu, \quad K := K_1+K_5,$$

which completes the proof. □

Our proof mimics that given in [3, §19]. The Plemelj-Privalov Theorem asserts that $\mathcal{C}_\pm^L$ may be interpreted as bounded linear operators on the Banach space $H^\nu(L)$ when $0<\nu<1$.

It can be further shown that (see [3, §22]) the Cauchy integral $\mathcal{C}^L[\mathbf{F}](z)$ of a function $\mathbf{F}\in H^\nu(L)$ with $0<\nu<1$ is Hölder continuous in both the closure of the interior of L (taking the boundary value from within) and also in the closure of the exterior of L (taking the boundary value from outside). This implies that, after the fact, we may dispense with the device of taking boundary-values in a strictly non-tangential fashion. Because Hölder continuity implies mere continuity, we then have the following, in which for operators $\mathcal{A}$, $\mathcal{B}$ acting on a space, the composition $\mathcal{A}\circ\mathcal{B}$ denotes the action $\mathcal{A}(\mathcal{B}\mathbf{F})$.

Proposition 2. *The bounded operators $\mathcal{C}_\pm^L$ acting on $H^\nu(L)$ for $0<\nu<1$ satisfy the identities*

$$\mathcal{C}_+^L\circ\mathcal{C}_-^L = \mathcal{C}_-^L\circ\mathcal{C}_+^L = 0.$$

Proof. Let $\mathbf{F}\in H^\nu(L)$, and consider $\mathcal{C}^L[\mathcal{C}_-^L[\mathbf{F}]](z)$, for z on the "+" side of L. Since $\mathcal{C}_-^L[\mathbf{F}](\cdot)$ is the boundary value of a function analytic on the "−" side of L and continuous up to L from that side, and since z is on the other side of L, the contour of integration in the outer integral may be deformed into the region on the "−" side of L by the Generalized Cauchy Integral Theorem[1]. If this region is the interior of L, then $\mathcal{C}^L[\mathcal{C}_-^L[\mathbf{F}]](z)=0$ for all such z, while if this region is the exterior of L, then the same holds true because the integrand decays like w^{-2} as $w\to\infty$. Letting z tend to L again from the "+" side yields the identity $\mathcal{C}_+^L\circ\mathcal{C}_-^L=0$. The proof that $\mathcal{C}_-^L\circ\mathcal{C}_+^L=0$ is similar. □

Corollary 1. *The bounded operators $\mathcal{C}_\pm^L$ acting on $H^\nu(L)$ for $0<\nu<1$ satisfy the identities*

$$\mathcal{C}_+^L\circ\mathcal{C}_+^L=\mathcal{C}_+^L \quad \textit{and} \quad (-\mathcal{C}_-^L)\circ(-\mathcal{C}_-^L) = -\mathcal{C}_-^L.$$

[1] See, e.g., [5, pg. 60]. The Generalized Cauchy Integral Theorem is just like the Cauchy Integral Theorem except that it only requires continuity of the function integrated up to the curve (plus, of course, analyticity in the open interior).

Proof. Combine Proposition 2 with Proposition 1 (which has the operator interpretation of $\mathcal{C}_+^L - \mathcal{C}_-^L = \mathcal{I}$). $\square$

These results show that the operators $\pm\mathcal{C}_\pm^L$ are *complementary projections* on the Banach space $H^\nu(L)$ for $0 < \nu < 1$.

Examples. Let L be the unit circle in the complex plane: $|z| = 1$. We can think of L as arising from two arcs, namely the upper semicircle parametrized by $z = \mathrm{e}^{\mathrm{i}t}$, $0 \le t \le \pi$, and the lower semicircle parametrized by $z = \mathrm{e}^{\mathrm{i}t}$, $-\pi \le t \le 0$, placed end-to-end. The orientation of L is counter-clockwise. By simple trigonometry (the law of cosines), if $w, z \in L$ are separated by a shortest arc length of $s(w,z) \le \pi$, the corresponding chord length is $|w - z| = \sqrt{2(1-\cos(s(w,z)))}$. Therefore, the minimum value of $|w-z|/s(w,z)$ is, by calculus,

$$k_0 := \inf_{0<s<\pi} \frac{\sqrt{2(1-\cos(s))}}{s} = \frac{2}{\pi}$$

and hence for points on L we have the sharp bounds $\frac{2}{\pi}s(w,z) \le |w-z| \le s(w,z)$.

Consider the scalar function (or 1×1 matrix function) defined on L by $f(z(t)) := |\sin(t)|^\nu$, for $-\pi \le t \le \pi$, where $\nu > 0$ is a parameter. We claim that $f \in H^\nu(L)$ provided $\nu \le 1$, and that $f \in H^1(L)$ for $\nu > 1$. It is not hard to see this; indeed the "roughest" points for f on L are the points $z = \pm 1$ where f fails to be differentiable with respect to arc length t for $\nu \le 1$. But near these points we have $|\sin(t)|^\nu \approx |t|^\nu$ or $|\sin(t)|^\nu \approx |\pm\pi - t|^\nu$.

With this example, we can also show how the Cauchy boundary-value operators $\mathcal{C}_\pm^L$ fail to map the Lipschitz space $H^1(L)$ to itself. We first calculate $\mathcal{C}_\pm^L[f](z)$ for $f(z(t)) = |\sin(t)|$, which gives $f \in H^1(L)$. Noting that $f(z) = (z - z^{-1})/(2\mathrm{i})$ on the upper semicircle of L and that $f(z) = -(z-z^{-1})/(2\mathrm{i})$ on the lower semicircle of L, we get

$$\mathcal{C}^L[f](z) = \frac{1}{2\pi\mathrm{i}}\int_1^{-1} \frac{1}{2\mathrm{i}}\frac{w-w^{-1}}{w-z}\,\mathrm{d}w - \frac{1}{2\pi\mathrm{i}}\int_{-1}^{1}\frac{1}{2\mathrm{i}}\frac{w-w^{-1}}{w-z}\,\mathrm{d}w$$

where the first (second) integral is taken over the upper (lower) semicircle. Since

$$\frac{w-w^{-1}}{w-z} = 1 + \frac{z-z^{-1}}{w-z} + \frac{z^{-1}}{w},$$

we get

$$\mathcal{C}^L[f](z) = \frac{1}{\pi} - \frac{z-z^{-1}}{4\pi}\int_1^{-1}\frac{\mathrm{d}w}{w-z} + \frac{z-z^{-1}}{4\pi}\int_{-1}^{1}\frac{\mathrm{d}w}{w-z}$$

where again the first (second) integral is over the upper (lower) semicircle. Now observe that if $\log(\cdot)$ refers to the principal branch, $\log(\mathrm{i}(w-z))$ ($\log(-\mathrm{i}(w-z))$) is an antiderivative of $(w-z)^{-1}$ analytic except on a vertical branch cut emanating upwards (downwards) from z. Therefore,

$$\mathcal{C}^L[f](z) = \frac{1}{\pi} - \frac{z-z^{-1}}{4\pi}[\log(-\mathrm{i}(-1-z)) - \log(-\mathrm{i}(1-z))] + \frac{z-z^{-1}}{4\pi}[\log(\mathrm{i}(1-z)) - \log(\mathrm{i}(-1-z))], \quad |z| < 1,$$

while, noting that $\log(\mathrm{e}^{-\mathrm{i}\arg(z)}(z-w))$ is an antiderivative of $(w-z)^{-1}$ analytic except on a branch cut emanating radially from z away from the origin,

$$\mathcal{C}^L[f](z) = \frac{1}{\pi} - \frac{z-z^{-1}}{2\pi}[\log(\mathrm{e}^{-\mathrm{i}\arg(z)}(z+1)) - \log(\mathrm{e}^{-\mathrm{i}\arg(z)}(z-1))], \quad |z| > 1.$$

Therefore, letting z tend to L from inside and outside of the unit circle, on which $z = \mathrm{e}^{\mathrm{i}t}$,

$$\mathcal{C}_+^L[f](z(t)) = \frac{1}{\pi} - \frac{\mathrm{i}\sin(t)}{2\pi}[\log(-\mathrm{i}(-1-\mathrm{e}^{\mathrm{i}t})) - \log(-\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t})) + \log(\mathrm{i}(-1-\mathrm{e}^{\mathrm{i}t})) - \log(\mathrm{i}(1-\mathrm{e}^{\mathrm{i}t}))], \quad -\pi \le t \le \pi$$

and

$$\mathcal{C}_-^L[f](z(t)) = \frac{1}{\pi} - \frac{\mathrm{i}\sin(t)}{\pi}[\log(1+\mathrm{e}^{-\mathrm{i}t}) - \log(1-\mathrm{e}^{-\mathrm{i}t})], \quad -\pi \le t \le \pi.$$

These boundary values *do not* lie in $H^1(L)$. Indeed, near $t = 0$ corresponding to $z = 1$ both boundary values have a dominant term proportional to $|t|\ln(|t|^{-1})$ which while continuous through $t = 0$ fails to be Lipschitz (its derivative blows up like $\ln(|t|^{-1})$). A similar phenomenon occurs near $z = -1$. The difference of the boundary values is, of course, Lipschitz, being as from the Plemelj formula we have $\mathcal{C}_+^L[f](z(t)) - \mathcal{C}_-^L[f](z(t)) = |\sin(t)|$.

Additional technical properties of the spaces $H^\nu(L)$. Another kind of bounded operator on $H^\nu(L)$ is that obtained by multiplying on the right or left by a fixed function in the same space:

Definition 6 (Multiplication operators)**.** *Let $\mathbf{M} \in H^\nu(L)$ be a fixed function. The operator of left-multiplication by $\mathbf{M}$ is denoted $\mathcal{L}_\mathbf{M}$ and has the action $\mathcal{L}_\mathbf{M}\mathbf{F}(z) := \mathbf{M}(z)\mathbf{F}(z)$. The operator of right-multiplication by $\mathbf{M}$ is denoted $\mathcal{R}_\mathbf{M}$ and has the action $\mathcal{R}_\mathbf{M}\mathbf{F}(z) := \mathbf{F}(z)\mathbf{M}(z)$.*

It follows easily from (6) that $\mathcal{L}_\mathbf{M}$ and $\mathcal{R}_\mathbf{M}$ are both bounded operators on $H^\nu(L)$ with norm less than or equal to $\|\mathbf{M}\|_\nu$:

$$\|\mathcal{L}_\mathbf{M}\mathbf{F}\|_\nu \le \|\mathbf{M}\|_\nu\|\mathbf{F}\|_\nu \quad \text{and} \quad \|\mathcal{R}_\mathbf{M}\mathbf{F}\|_\nu \le \|\mathbf{M}\|_\nu\|\mathbf{F}\|_\nu.$$

Since $H^\mu(L) \subset H^\nu(L)$ whenever $\mu > \nu$, we may consider multiplication operators with $\mathbf{M} \in H^\mu(L)$ for $\mu > \nu$ as a special case. The action of these multiplication operators can be composed with the action of the operators $\mathcal{C}^L_\pm$. Of particular interest are the commutators $[\mathcal{C}^L_\pm, \mathcal{L}_\mathbf{M}] = \mathcal{C}^L_\pm \circ \mathcal{L}_\mathbf{M} - \mathcal{L}_\mathbf{M} \circ \mathcal{C}^L_\pm$ and $[\mathcal{C}^L_\pm, \mathcal{R}_\mathbf{M}] = \mathcal{C}^L_\pm \circ \mathcal{R}_\mathbf{M} - \mathcal{R}_\mathbf{M} \circ \mathcal{C}^L_\pm$. Written out in terms of their action on a function $\mathbf{F} \in H^\nu(L)$, we have

$$\begin{aligned}[\mathcal{C}^L_\pm, \mathcal{L}_\mathbf{M}]\mathbf{F}(z) &= \frac{1}{2\pi \mathrm{i}}\int_L \frac{\mathbf{M}(w)\mathbf{F}(w) - \mathbf{M}(z)\mathbf{F}(z)}{w-z}\,\mathrm{d}w - \mathbf{M}(z)\frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathbf{F}(w)-\mathbf{F}(z)}{w-z}\,\mathrm{d}w \\ &= \frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathbf{M}(w)-\mathbf{M}(z)}{w-z}\mathbf{F}(w)\,\mathrm{d}w, \quad z \in L,\end{aligned}$$

and

$$\begin{aligned}[\mathcal{C}^L_\pm, \mathcal{R}_\mathbf{M}]\mathbf{F}(z) &= \frac{1}{2\pi \mathrm{i}}\int_L \frac{\mathbf{F}(w)\mathbf{M}(w) - \mathbf{F}(z)\mathbf{M}(z)}{w-z}\mathrm{d}w - \frac{1}{2\pi\mathrm{i}}\int_L \frac{\mathbf{F}(w)-\mathbf{F}(z)}{w-z}\,\mathrm{d}w\cdot\mathbf{M}(z) \\ &= \frac{1}{2\pi\mathrm{i}}\int_L \mathbf{F}(w)\frac{\mathbf{M}(w)-\mathbf{M}(z)}{w-z}\,\mathrm{d}w, \quad z \in L.\end{aligned}$$

The subscript "±" on the left-hand side is not reflected on the right-hand side, because according to the Plemelj formula (Proposition 1) the difference between $\mathcal{C}^L_+$ and $\mathcal{C}^L_-$ is the identity operator, which commutes with everything. Notice also that on the right-hand side in these formulae, the variable z appears in the argument of $\mathbf{M}$ and no longer in the argument of $\mathbf{F}$. This suggests that it will be smoothness properties of $\mathbf{M}$ rather than $\mathbf{F}$ that determine those of $[\mathcal{C}^L_\pm, \mathcal{L}_\mathbf{M}]\mathbf{F}$ and $[\mathcal{C}^L_\pm, \mathcal{R}_\mathbf{M}]\mathbf{F}$. Indeed, we have the following result:

Proposition 3 (Mapping properties of commutators)**.** *Suppose that $0 < \mu, \nu < 1$ and that $\mathbf{M} \in H^\mu(L)$. Then $[\mathcal{C}^L_\pm, \mathcal{L}_\mathbf{M}]$ and $[\mathcal{C}^L_\pm, \mathcal{R}_\mathbf{M}]$ are bounded linear operators from $H^\nu(L)$ to $H^\mu(L)$.*

Proof. This is virtually the same proof as that of the Plemelj-Privalov theorem (Theorem 1), except that we estimate the $H^\mu(L)$ norm and find an upper bound proportional to $\|\mathbf{F}\|_\infty$ which in turn is bounded by $\|\mathbf{F}\|_\nu$ (but observe that the quantity $h_\nu(\mathbf{F})$ never appears in the estimates). □

We have remarked that $H^\mu(L) \subset H^\nu(L)$ whenever $\mu > \nu$. Indeed, suppose that $\mu > \nu$ and that $\mathbf{F} \in H^\mu(L)$. Then, it is easy to see that

$$h_\nu(\mathbf{F}) = \sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}} \frac{\|\mathbf{F}(z_2)-\mathbf{F}(z_1)\|}{|z_2-z_1|^\nu} \le \sup_{z_1,z_2\in L}|z_2-z_1|^{\mu-\nu}\sup_{\substack{z_1,z_2\in L\\ z_2\neq z_1}}\frac{\|\mathbf{F}(z_2)-\mathbf{F}(z_1)\|}{|z_2-z_1|^\mu} = \sup_{z_1,z_2\in L}|z_2-z_1|^{\mu-\nu}\cdot h_\mu(\mathbf{F}).$$

But since $\mu > \nu$, $|z_2 - z_1|^{\mu-\nu}$ is uniformly bounded on the (bounded) loop L, so there is a constant $C > 0$ such that $h_\nu(\mathbf{F}) \le Ch_\mu(\mathbf{F})$ whenever $\mu \ge \nu$. It follows easily that $\mu \ge \nu$ and $\|\mathbf{F}\|_\mu < \infty$ implies that also $\|\mathbf{F}\|_\nu < \infty$, i.e., $H^\mu(L) \subset H^\nu(L)$.

Definition 7 (Inclusion map)**.** *Let $0 < \nu \le \mu \le 1$. The inclusion map $\mathcal{I}_{\mu\to\nu}$ is the linear operator from $H^\mu(L)$ into $H^\nu(L)$ that simply acts as the identity.*

Another important fact in the theory is the following. The notion of a compact operator is introduced, for example, in [1, §8.5].

Proposition 4 (Compactness of inclusion map)**.** *Suppose that $0 < \nu < \mu \le 1$. Then $\mathcal{I}_{\mu\to\nu} : H^\mu(L) \to H^\nu(L)$ is compact.*

Proof. We need to show that the image of every bounded set in $H^\mu(L)$ is precompact in the $H^\nu(L)$ topology, i.e., that if $B \subset H^\mu(L)$ consists of infinitely many $\mathbf{F}$ with $\|\mathbf{F}\|_\mu \leq C$ for some constant C, then there exists a sequence $\{\mathbf{F}_n\}_{n=1}^\infty \subset B$ that is convergent in the $H^\nu(L)$ norm.

The condition $\|\mathbf{F}\|_\mu \leq C$ implies that $\|\mathbf{F}\|_\infty \leq C$, i.e., that the family B of functions $\mathbf{F}$ is uniformly bounded, and also that $h_\mu(\mathbf{F}) \leq C$, which in turn implies that $\|\mathbf{F}(z) - \mathbf{F}(w)\| \leq C|z-w|^\mu$ for all $z, w \in L$. But this further implies that the family B of functions $\mathbf{F}$ is *equicontinuous*, i.e., the modulus of continuity is independent of $\mathbf{F}$. By the Arzelá-Ascoli Theorem[2], there exists a sequence $\{\mathbf{F}_n\}_{n=1}^\infty \subset B$ that is uniformly convergent to some limit function $\mathbf{F}_0$, i.e., $\|\mathbf{F}_n - \mathbf{F}_0\|_\infty \to 0$ as $n \to \infty$. The limit function obviously[3] also satisfies $\|\mathbf{F}_0\|_\mu \leq C$ and hence lies in $H^\mu(L) \subset H^\nu(L)$.

Next consider the quantity

$$\mathbf{H}_n(z,w) := \frac{\mathbf{F}_n(z) - \mathbf{F}_n(w)}{|z-w|^\nu}, \quad z, w \in L, \quad z \neq w, \quad n = 1, 2, 3, \dots.$$

In fact, since $\mathbf{F}_n \in H^\mu(L)$ and $\mu > \nu$, it follows that $\mathbf{H}_n(z,w) \to \mathbf{0}$ as $z \to w$, so we simply define it naturally on the diagonal as $\mathbf{H}_n(z,z) = 0$, $z \in L$. It can be shown that this function is Hölder continuous with exponent $\mu - \nu > 0$ on the Cartesian product $L \times L$ in the sense that [3, pg. 15]

$$\|\mathbf{H}_n(z,w) - \mathbf{H}_n(z',w')\| \leq \|\mathbf{F}_n\|_\mu \left[|z-z'|^{\mu-\nu} + |w-w'|^{\mu-\nu}\right], \quad z, z', w, w' \in L, \quad n = 1, 2, 3, \dots.$$

But since $\|\mathbf{F}_n\|_\mu \leq C$, we now observe uniform boundedness and equicontinuity of the sequence of two-variable functions $\{\mathbf{H}_n\}_{n=1}^\infty$ on $L \times L$, and so again by the Arzelá-Ascoli Theorem there is a subsequence $\{\mathbf{H}_{n_k}\}_{k=1}^\infty$ that is uniformly convergent on $L \times L$ to some limit function; moreover we may identify this limit function as

$$\mathbf{H}_0(z,w) := \frac{\mathbf{F}_0(z) - \mathbf{F}_0(w)}{|z-w|^\nu}$$

because clearly $\mathbf{H}_{n_k}(z,w) \to \mathbf{H}_0(z,w)$ in the sense of pointwise convergence since $\mathbf{F}_{n_k}$ converges uniformly (and hence also pointwise) on L to $\mathbf{F}_0$.

It therefore follows that the sequence $\{\mathbf{F}_{n_k}\}_{k=1}^\infty \subset B \subset H^\mu(L)$, which is also a sequence in $H^\nu(L)$ by the inclusion map $\mathcal{I}_{\mu\to\nu}$, satisfies

$$\|\mathbf{F}_{n_k} - \mathbf{F}_0\|_\nu = \|\mathbf{F}_{n_k} - \mathbf{F}_0\|_\infty + h_\nu(\mathbf{F}_{n_k} - \mathbf{F}_0) = \|\mathbf{F}_{n,k} - \mathbf{F}_0\|_\infty + \sup_{z,w\in L} \|\mathbf{H}_{n,k}(z,w) - \mathbf{H}_0(z,w)\|$$

which tends to zero as $k \to \infty$ by uniform convergence of $\mathbf{F}_{n,k}$ to $\mathbf{F}_0$ on L and by uniform convergence of $\mathbf{H}_{n,k}$ to $\mathbf{H}_0$ on $L \times L$. □

The above proof is adapted from [4, pgs. 102–103]. The main application of this result is the following:

Corollary 2. *Let $0 < \nu < \mu < 1$, and suppose that $\mathbf{M} \in H^\mu(L)$. The commutators $[\mathcal{C}_\pm^L, \mathcal{L}_\mathbf{M}]$ and $[\mathcal{C}_\pm^L, \mathcal{R}_\mathbf{M}]$ are compact operators on $H^\nu(L)$.*

Proof. The commutators are bounded from $H^\nu(L)$ into $H^\mu(L)$, so composition with the compact inclusion map $\mathcal{I}_{\mu\to\nu}$ yields a compact operator in each case. □

The purpose in considering compactness properties of commutators is that we will use them later to prove that a certain singular integral operator is Fredholm, an important step toward developing a notion of the Fredholm alternative for solving Riemann-Hilbert problems.

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[2]See [1, pg. 156]. The version formulated there involves functions on an interval $[a,b]$ but easily generalizes to the present context.

[3]The argument goes as follows. Let n be arbitrary, and consider any $(x,y,z) \in L^3$ with $x \neq y$. By using $\mathbf{F}_0(\cdot) = \mathbf{F}_n(\cdot) + (\mathbf{F}_0(\cdot) - \mathbf{F}_n(\cdot))$, applying the triangle inequality and using $\|\mathbf{F}_n\|_\mu \leq C$, we can easily derive the inequality

$$\|\mathbf{F}_0(z)\| + \frac{\|\mathbf{F}_0(x) - \mathbf{F}_0(y)\|}{|x-y|^\mu} \leq C + \|\mathbf{F}_0(z) - \mathbf{F}_n(z)\| + \frac{\|\mathbf{F}_0(x) - \mathbf{F}_n(x)\| + \|\mathbf{F}_0(y) - \mathbf{F}_n(y)\|}{|x-y|^\mu}.$$

By uniform convergence of $\mathbf{F}_n$ to $\mathbf{F}_0$ on L, there exists $n_0(\epsilon)$ for each $\epsilon > 0$ such that $n \geq n_0(\epsilon)$ guarantees that $\|\mathbf{F}_0(x) - \mathbf{F}_n(x)\| < \epsilon$ holds for all $x \in L$. Given $z \in L$ and $x \neq y$, choose $n \geq \max(n_0(\frac{1}{3}\epsilon), n_0(\frac{1}{3}|x-y|^\mu\epsilon))$. Then we have

$$\|\mathbf{F}_0(z)\| + \frac{\|\mathbf{F}_0(x) - \mathbf{F}_0(y)\|}{|x-y|^\mu} \leq C + \epsilon$$

holding for all $z \in L$ and $x \neq y$. Hence taking the supremum over $z \in L$ and $x \neq y$ in L gives $\|\mathbf{F}_0\|_\mu \leq C + \epsilon$. But this inequality holds for all $\epsilon > 0$, and therefore $\|\mathbf{F}_0\|_\mu \leq C$.

Riemann-Hilbert problems on bounded contours.

Definition 8 (Admissible contours)**.** *An admissible contour Σ is a finite union of arcs such that for each point $z_0 \in \mathbb{C}$ belonging to two or more arcs, the corresponding tangent lines to the arcs at z_0 are distinct.*

Every admissible contour Σ has a finite set X of *exceptional points*, namely the endpoints of the arcs and any points belonging to two or more arcs. Each component of $\Sigma^\circ := \Sigma \setminus X$ is an oriented arc without endpoints. See Figure 1 for an example. We denote by $H^\mu_\circ(\Sigma)$ the space of matrix functions $\mathbf{F} : \Sigma^\circ \to \mathbb{C}^{N\times N}$

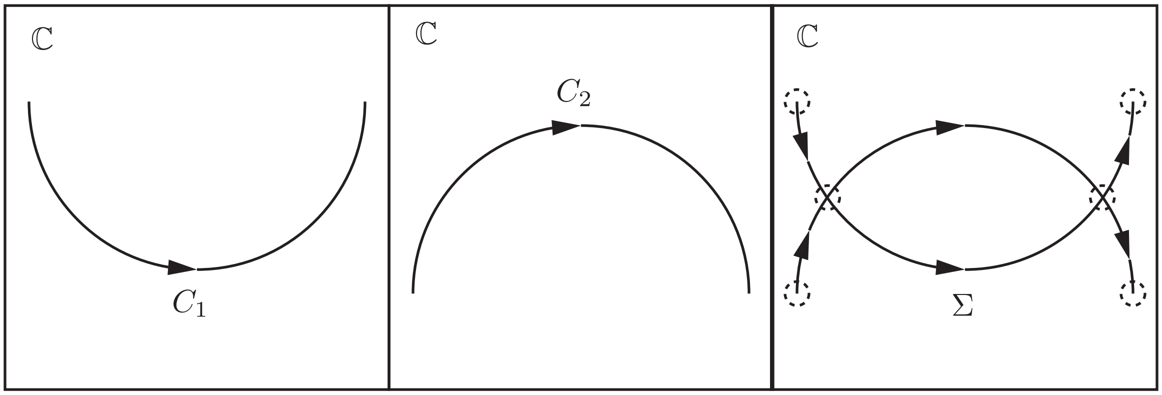

FIGURE 1. Left two panels: semicircular arcs C_1 and C_2. Right panel: an admissible contour Σ composed of C_1 and C_2. The six exceptional points in the set X are circled, and the six components of Σ° are oriented circular arcs without endpoints. Note the transversal intersections of C_1 and C_2.

with the property that on each component Σ_j of Σ°, $\mathbf{F}$ has a continuous extension to the initial and terminal endpoints that lies in $H^\mu(\overline{\Sigma}_j)$. The norm of $\mathbf{F} \in H^\mu_\circ(\Sigma)$ is

$$\|\mathbf{F}\|^\circ_\mu := \sum_j \|\mathbf{F}|_{\Sigma_j}\|_\mu$$

where on the right-hand side we have the $H^\mu(\overline{\Sigma}_j)$ norm.

Definition 9 (Admissible jump matrices)**.** *Let Σ be an admissible contour. A mapping $\mathbf{V} : \Sigma^\circ \to \mathbb{C}^{N\times N}$ is called an admissible jump matrix on Σ with Hölder exponent μ if $\mathbf{V} \in H^\mu_\circ(\Sigma)$ and the following properties hold.*

- $\det(\mathbf{V}(z)) = 1$ *for all $z \in \Sigma^\circ$. (Thus $\mathbf{V}(z) \in \mathrm{SL}(N,\mathbb{C}) \subset \mathbb{C}^{N\times N}$.)*
- *Let $z_0 \in X$ be an exceptional point of Σ, let the $K \geq 1$ components of Σ° that join at z_0 be labeled in counter-clockwise order about z_0 as $\Sigma_1, \dots, \Sigma_K$, denote the orientation of each Σ_j by $\sigma_j = 1$ ($\sigma_j = -1$) if Σ_j is oriented away from (toward) z_0, and let*

$$\mathbf{V}_j := \lim_{\substack{z\to z_0 \\ z\in\Sigma_j}} \mathbf{V}(z).$$

Then

$$\mathbf{V}_1^{\sigma_1}\mathbf{V}_2^{\sigma_2}\cdots\mathbf{V}_K^{\sigma_K} = \mathbb{I} \tag{14}$$

holds for each $z_0 \in X$.

Note that if X contains any arc endpoints z_0 that belong to only one arc, then for an admissible jump matrix we have $\mathbf{V}(z) \to \mathbb{I}$ as $z \to z_0$ within Σ°, while $\mathbf{V}$ generally need not extend continuously to other points of X (because it can have different limits from different components of Σ°, although these limits must be related by (14)). Now we can formulate a class of matrix Riemann-Hilbert problems. Recall that we use the subscript "+" (resp., "−") to refer to the left (resp., right) side of an arc according to its orientation.

Riemann-Hilbert Problem 1. *Let Σ be an admissible contour and let $\mathbf{V} : \Sigma^\circ \to \mathbb{C}^{N\times N}$ be an admissible jump matrix with Hölder exponent μ. Seek a matrix-valued function $\mathbf{M} : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{N\times N}$ with the following properties:*

> ***Analyticity:*** $\mathbf{M}$ *is analytic (i.e., its N^2 matrix elements are all analytic) in the domain $\mathbb{C} \setminus \Sigma$.*
> ***Jump condition:*** $\mathbf{M}$ *extends continuously to the boundary Σ from each component of $\mathbb{C} \setminus \Sigma$, taking boundary values $\mathbf{M}_\pm(z)$ on each component Σ_j of Σ° defined by*
>
> $$\mathbf{M}_\pm(z) := \lim_{\substack{w\to z \\ w \text{ on the } \pm \text{ side of } \Sigma_j}} \mathbf{M}(w), \quad z \in \Sigma_j,$$
>
> *and the boundary values satisfy the jump condition*
>
> $$\mathbf{M}_+(z) = \mathbf{M}_-(z)\mathbf{V}(z), \quad z \in \Sigma^\circ.$$
>
> ***Normalization:*** $\mathbf{M}(z) \to \mathbb{I}$ *as $z \to \infty$.*

Note that since Σ is a bounded set, in fact any solution of the Riemann-Hilbert problem has a convergent Laurent expansion for sufficiently large $|z|$:

$$\mathbf{M}(z) = \mathbb{I} + \sum_{n=1}^{\infty} \mathbf{M}_n z^{-n}, \quad |z| > \sup_{w\in\Sigma} |w|.$$

The given *data* of the Riemann-Hilbert problem is the pair $(\Sigma, \mathbf{V})$. We can get some results about this problem right away.

Proposition 5 (Unimodularity and uniqueness)**.** *Suppose Riemann-Hilbert Problem 1 has a solution $\mathbf{M}(z)$. Then $\det(\mathbf{M}(z)) = 1$ holds for all $z \in \mathbb{C} \setminus \Sigma$, and there are no other solutions.*

Proof. The matrix function $\mathbf{M}(z)$ is analytic for $z \in \mathbb{C} \setminus \Sigma$ and continuous up to the boundary Σ, so the same holds for the scalar function $f(z) := \det(\mathbf{M}(z))$ as a polynomial in the matrix entries of $\mathbf{M}(z)$. By taking determinants in the jump condition, using the fact that $\det(\mathbf{V}(z)) = 1$ for all $z \in \Sigma^\circ$ by definition of an admissible jump matrix, we see that at each point of Σ°, the boundary values of $f(z)$ agree: $f_+(z) = f_-(z)$. Therefore $f(z)$ may be defined for $z \in \Sigma$ by continuity to be a function continuous in the whole complex plane and analytic for $z \in \mathbb{C} \setminus \Sigma$. It follows from Morera's Theorem combined with the Generalized Cauchy Integral Theorem that f is an entire function of z. Since $\mathbf{M}(z) \to \mathbb{I}$ as $z \to \infty$, $f(z) \to 1$ in the same limit, so by Liouville's Theorem, $f(z) = \det(\mathbf{M}(z)) = 1$ for all $z \in \mathbb{C} \setminus \Sigma$. This proves the first statement.

For the second part, suppose $\tilde{\mathbf{M}}(z)$ is a second solution of the same problem. By the first part, we have $\det(\tilde{\mathbf{M}}(z)) = 1$ so the inverse matrix $\tilde{\mathbf{M}}(z)^{-1}$ is also an analytic function of $z \in \mathbb{C} \setminus \Sigma$, because the entries of the inverse matrix can be expressed by Cramer's rule as ratios of determinants — polynomials in the analytic matrix entries of $\tilde{\mathbf{M}}(z)$ — and the denominator is always $\det(\tilde{\mathbf{M}}(z)) = 1$. Therefore, the product $\mathbf{P}(z) := \mathbf{M}(z)\tilde{\mathbf{M}}(z)^{-1}$ is a matrix that is analytic for $z \in \mathbb{C} \setminus \Sigma$. From the jump condition satisfied by both $\mathbf{M}(z)$ and $\tilde{\mathbf{M}}(z)$, we calculate that

$$\mathbf{P}_+(z) = \mathbf{M}_+(z)\tilde{\mathbf{M}}_+(z)^{-1} = \mathbf{M}_-(z)\mathbf{V}(z)[\tilde{\mathbf{M}}_-(z)\mathbf{V}(z)]^{-1} = \mathbf{M}_-(z)\tilde{\mathbf{M}}_-(z)^{-1} = \mathbf{P}_-(z), \quad z \in \Sigma^\circ,$$

so $\mathbf{P}(z)$ may be considered as a matrix function continuous for $z \in \mathbb{C}$ and analytic for $z \in \mathbb{C} \setminus \Sigma$. Also,

$$\lim_{z\to\infty} \mathbf{P}(z) = \lim_{z\to\infty} \mathbf{M}(z)\tilde{\mathbf{M}}(z)^{-1} = \mathbb{I} \cdot \mathbb{I} = \mathbb{I}.$$

Therefore, by the same argument as worked for the determinant, the matrix $\mathbf{P}(z)$ is entire and equal to its constant limit at $z = \infty$: $\mathbf{P}(z) = \mathbb{I}$ for all $z \in \mathbb{C} \setminus \Sigma$, in other words $\tilde{\mathbf{M}}(z) = \mathbf{M}(z)$ for all $z \in \mathbb{C} \setminus \Sigma$. This proves the second statement. □

The fact that $\det(\mathbf{M}(z)) = 1$ for the solution of Riemann-Hilbert Problem 1, assuming a solution exists, explains the necessity of the condition (14) in the definition of an admissible jump matrix. Indeed, considering a neighborhood of an exceptional point $z_0 \in X$ about which there are components Ω_j, $j = 1, \ldots, K$, of $\mathbb{C} \setminus \Sigma$ in counter-clockwise order, the solution $\mathbf{M}(z)$ is required to have a well-defined limit $\mathbf{M}_j$ as $z \to z_0$ from Ω_j. But these limiting values are related to those in neighboring sectors by the jump conditions across the arcs

Σ_j meeting at z_0, in particular by their limiting forms as $z \to z_0$, which involve the matrices $\mathbf{V}_j$. Assuming that Ω_j comes just before Σ_j in counter-clockwise order, we have the relations

$$\mathbf{M}_2 = \mathbf{M}_1\mathbf{V}_1^{\sigma_1}, \quad \mathbf{M}_3 = \mathbf{M}_2\mathbf{V}_2^{\sigma_2}, \quad \dots \quad \mathbf{M}_K = \mathbf{M}_{K-1}\mathbf{V}_{K-1}^{\sigma_{K-1}}$$

and also $\mathbf{M}_1 = \mathbf{M}_K\mathbf{V}_K^{\sigma_K}$. But these together imply that

$$\mathbf{M}_1 = \mathbf{M}_1\mathbf{V}_1^{\sigma_1}\mathbf{V}_2^{\sigma_2}\cdots\mathbf{V}_K^{\sigma_K},$$

so since $\mathbf{M}_1$ is invertible, we see that (14) is a necessary condition for existence of a solution to Riemann-Hilbert Problem 1.

Example. Consider the special case that $N = 2$ (i.e., we are dealing with 2×2 matrices) and let $\mathbf{V} : \Sigma^\circ \to \mathbb{C}^{2\times 2}$ be an admissible jump matrix on Σ of the form

$$\mathbf{V}(z) = \begin{bmatrix} 1 & v(z) \\ 0 & 1 \end{bmatrix}, \quad z \in \Sigma^\circ, \tag{15}$$

where v is a scalar function in $H^\mu_\circ(\Sigma)$ that, by the admissibility criterion, satisfies

$$\sum_j \sigma_j v_j = 0, \tag{16}$$

for limits v_j as $z \to z_0$ along arcs of Σ meeting at z_0 with orientation indices σ_j. We can seek a solution $\mathbf{M}(z)$ of Riemann-Hilbert Problem 1 also in the form of an upper triangular matrix:

$$\mathbf{M}(z) = \begin{bmatrix} 1 & m(z) \\ 0 & 1 \end{bmatrix}, \quad z \in \mathbb{C} \setminus \Sigma \tag{17}$$

for some scalar function m analytic in the indicated domain. By the normalization condition we require $m(z) \to 0$ as $z \to \infty$, and noting that the generally noncommutative multiplicative jump condition on $\mathbf{M}(z)$ becomes, for jump matrices of the form (15) and an ansatz of the form (17), simply the additive jump condition

$$m_+(z) = m_-(z) + v(z), \quad \text{or} \quad m_+(z) - m_-(z) = v(z), \quad z \in \Sigma^\circ.$$

This type of Riemann-Hilbert problem is solved by appealing to the Plemelj formula. Indeed,

$$m(z) = \mathcal{C}^\Sigma[v](z) = \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{v(w)\,\mathrm{d}w}{w - z}. \tag{18}$$

Combining this explicit formula with (17) gives a candidate solution of Riemann-Hilbert Problem 1. The only thing that remains to be confirmed is that $m(z)$ as given by (18) is continuous up to Σ from each component of $\mathbb{C}\setminus\Sigma$. But this is clear by Hölder continuity of v for all non-exceptional points $z \in \Sigma^\circ$. It is an exercise to use the condition (16) to check that $m(z)$ has a limit as $z \to z_0 \in X$ from each local component of $\mathbb{C}\setminus\Sigma$. Therefore, $\mathbf{M}(z)$ given by (17)–(18) solves Riemann-Hilbert Problem 1. According to Proposition 5 it is the only solution. Clearly, $\det(\mathbf{M}(z)) = 1$ holds.

Hölder function spaces adapted to bounded contours with self-intersection points.

Definition 10 (Complete admissible contours)**.** *An admissible contour Σ is said to be complete if it divides the complex plane into two disjoint regions, Ω^+ and Ω^- (i.e., $\mathbb{C}$ is the disjoint union of Ω^+, Ω^-, and Σ) and if Σ may simultaneously be considered as a collection of loops $\{L_j^+\}_{j=1}^{N_+}$ each of which is the positively-oriented boundary of a component Ω_j^+ of Ω^+ and as a collection of loops $\{L_j^-\}_{j=1}^{N_-}$ each of which is the negatively-oriented boundary of a component Ω_j^- of Ω^-.*

Without any loss of generality, we may assume that Riemann-Hilbert Problem 1 is formulated relative to a complete admissible contour Σ. Indeed, given an arbitrary admissible contour Σ_0 we can produce a complete admissible contour Σ simply by (i) including a finite number of additional arcs and (ii) reversing the orientation of some of the components of Σ°. When we replace Σ_0 with its "completion" Σ, we also have to modify the given admissible jump matrix $\mathbf{V}_0$ as follows: (i) on each arc added to Σ_0 to produce Σ we define $\mathbf{V}(z) = \mathbb{I}$, and (ii) on the remaining components of Σ° we either define $\mathbf{V}(z) = \mathbf{V}_0(z)^{-1}$ or $\mathbf{V}(z) = \mathbf{V}_0(z)$, depending upon whether the orientation of the component had to be changed or not. It is easy to check that $\mathbf{V}$ is still an admissible jump matrix now relative to the complete contour Σ, and that the jump conditions have not been changed in essence. Thus the Riemann-Hilbert problem with data $(\Sigma_0, \mathbf{V}_0)$ has exactly the same solutions as does that with data $(\Sigma, \mathbf{V})$. We assume from now on that the admissible contour Σ is complete.

Exactly one of the domains $\{\Omega_j^\pm\}_{j=1}^{N_\pm}$ is unbounded. Note that at each exceptional point z_0 of Σ, an even number of components of Σ° meet with alternating orientation going around the point z_0 in the positive direction.

Example. Let the admissible contour from Figure 1 be renamed Σ_0. This contour can be completed by adding two arcs and re-orienting some arcs, as illustrated in Figure 2. Let the four endpoints of Σ_0 be denoted $\pm w$ and $\pm w^*$, and consider the 2×2 admissible jump matrix $\mathbf{V}_0$ on Σ_0 given by $\mathbf{V}_0(z) = \exp[(z^2 - w^2)(z^2 - w^{*2})\sigma_3]$. It is easy to confirm that $\mathbf{V}_0(\pm w) = \mathbf{V}_0(\pm w^*) = \mathbb{I}$, and that $\mathbf{V}_0(z)$ has well-defined limits as z tends to either of the self-intersection points of Σ_0 along any of the four arcs meeting at those points. Thus (taking into account the orientation indices σ_j of the arcs meeting at each such point, two of which are $+1$ and two of which are -1) $\mathbf{V}_0(z)$ satisfies the condition (14) at all six exceptional points of Σ_0. Furthermore $\mathbf{V}_0(z)$ is clearly analytic on each of the six components of Σ_0° and hence satisfies the required Hölder continuity condition for any positive exponent $\mu \le 1$. Finally, $\det(\mathbf{V}_0(z)) = 1$. In going from the admissible contour Σ_0 to its completion Σ, we must also modify the jump matrix as follows. On the three components of Σ_0° in the upper half-plane, we leave $\mathbf{V}_0(z)$ alone by defining there $\mathbf{V}(z) := \mathbf{V}_0(z)$, while as the orientation of the components of Σ_0° in the lower half-plane are reversed in the completion process, we define on these components $\mathbf{V}(z) := \mathbf{V}_0(z)^{-1}$. Finally, on the two new arcs added to complete the contour (dashed curves in the left-hand panel of Figure 2) we simply define $\mathbf{V}(z) := \mathbb{I}$. It is an exercise to confirm that $\mathbf{V}(z)$ is then an admissible jump matrix on the completion Σ, and that the jump condition in Riemann-Hilbert Problem 1 is unchanged modulo re-labeling of the boundary values.

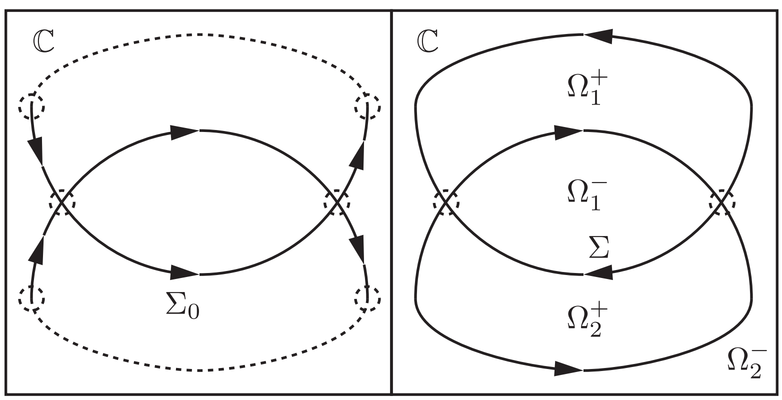

Figure 2. The process of completing an admissible contour. Left panel: the contour from Figure 1 renamed Σ_0 can be completed by adding two arcs (dashed) and re-orienting some arcs. Right-panel: the resulting complete admissible contour Σ divides the complex plane into Ω^+ on its left and Ω^- on its right. There are now only two exceptional points.

Hölder spaces for complete contours.

Definition 11. *We denote by $H^\nu_\pm(\Sigma)$ the Banach space of $N_\pm$-tuples of matrix functions $(\mathbf{F}_1, \dots, \mathbf{F}_{N_\pm})$ such that $\mathbf{F}_j \in H^\nu(L_j^\pm)$, equipped with the norm*

$$\|(\mathbf{F}_1, \dots, \mathbf{F}_{N_\pm})\|_\nu^\pm := \|\mathbf{F}_1\|_\nu + \cdots + \|\mathbf{F}_{N_\pm}\|_\nu$$

with the norms on the right-hand side being taken over the corresponding loops.

Each element of $H^\nu_\pm(\Sigma)$ may be regarded as a matrix function on Σ, provided one allows the function to take multiple values at exceptional points common to multiple loops $L_j^\pm$. If such a function in $H^\nu_+(\Sigma)$ agrees with a function in $H^\nu_-(\Sigma)$ away from the exceptional points, it admits a single-valued Hölder-continuous extension to all of Σ and hence can be identified with an element of $H^\nu(\Sigma)$. Conversely, it is easy to see that every $\mathbf{F} \in H^\nu(\Sigma)$ can be viewed simultaneously as an element of $H^\nu_+(\Sigma)$ and of $H^\nu_-(\Sigma)$. Thus $H^\nu_+(\Sigma) \cap H^\nu_-(\Sigma) = H^\nu(\Sigma)$.

We have the following analogues of the basic results for Hölder-continuous functions on loops. In all of these statements, Σ is a complete admissible contour.

Proposition 6. *Let $0 < \nu \le \mu \le 1$. The inclusion map $\mathcal{I}_{\mu\to\nu}$ can be defined from $H^\mu_\pm(\Sigma)$ to $H^\nu_\pm(\Sigma)$ or from $H^\mu(\Sigma)$ to $H^\nu(\Sigma)$. It is compact whenever $\nu < \mu$.*

Proposition 7. *If $\mathbf{M} \in H^\nu_\pm(\Sigma)$, then $\mathcal{L}_\mathbf{M}$ and $\mathcal{R}_\mathbf{M}$ are bounded on $H^\nu_\pm(\Sigma)$ and from $H^\nu(\Sigma)$ to $H^\nu_\pm(\Sigma)$. If $\mathbf{M} \in H^\nu(\Sigma)$, then $\mathcal{L}_\mathbf{M}$ and $\mathcal{R}_\mathbf{M}$ are bounded on $H^\nu(\Sigma)$, $H^\nu_+(\Sigma)$, and $H^\nu_-(\Sigma)$.*

Proposition 8 (Generalized Plemelj-Privalov Theorem)**.** *Suppose that $0 < \nu < 1$. The Cauchy operators $\mathcal{C}^\Sigma_\pm$ are bounded on $H^\nu_\pm(\Sigma)$ and from $H^\nu_\mp(\Sigma)$ to $H^\nu(\Sigma)$. On the space $H^\nu_+(\Sigma) \cup H^\nu_-(\Sigma)$, the following operator identities hold:*

$$\mathcal{C}^\Sigma_+ - \mathcal{C}^\Sigma_- = \mathcal{I}, \quad \mathcal{C}^\Sigma_\pm \circ \mathcal{C}^\Sigma_\mp = 0, \quad (\pm\mathcal{C}^\Sigma_\pm)^2 = \pm\mathcal{C}^\Sigma_\pm.$$

Proposition 9. *Suppose that $0 < \mu, \nu < 1$. If $\mathbf{M} \in H^\mu_+(\Sigma)$ (respectively, if $\mathbf{M} \in H^\mu_-(\Sigma)$ or $H^\mu(\Sigma)$), then the commutators $[\mathcal{C}^\Sigma_\pm, \mathcal{L}_\mathbf{M}]$ and $[\mathcal{C}^\Sigma_\pm, \mathcal{R}_\mathbf{M}]$ are bounded from $H^\nu_+(\Sigma)$, $H^\nu_-(\Sigma)$, or $H^\nu(\Sigma)$ to $H^\mu_+(\Sigma)$ (respectively, to $H^\mu_-(\Sigma)$ or $H^\mu(\Sigma)$). If $\nu < \mu$ and the commutators are followed by the inclusion map $\mathcal{I}_{\mu\to\nu}$, they all become compact.*

The proofs of Propositions 6–9 are similar to those for loops, but additional care must be taken in adding up the contributions to Cauchy integrals from various loops. One further useful fact is the following.

Proposition 10. *Suppose that $\mathbf{F}, \mathbf{G} \in H^\nu_+(\Sigma) \cup H^\nu_-(\Sigma)$. Let $\mathbf{H}_\pm(z)$ denote the pointwise matrix product $\mathbf{H}_\pm(z) := \mathcal{C}^\Sigma_\pm[\mathbf{F}](z)\mathcal{C}^\Sigma_\pm[\mathbf{G}](z)$. Then $\mathcal{C}^\Sigma_\mp[\mathbf{H}_\pm](z) = \mathbf{0}$.*

Proof. The function $\mathbf{H}_\pm(z)$ is the product of boundary values of functions analytic in $\Omega^\pm$ and decaying like $O(z^{-1})$ as $z \to \infty$ in any unbounded component of the latter domain. Therefore $\mathbf{H}_\pm(z)$ is also such an analytic function, but decaying like $O(z^{-2})$ as $z \to \infty$. In calculating $\mathcal{C}^\Sigma_\mp[\mathbf{H}_\pm](z)$, we first let $z \in \Omega^\mp$ and apply the Generalized Cauchy Integral Theorem to deduce (by deforming the path of integration into $\Omega^\pm_j$ from each loop $L^\pm_j$) that $\mathcal{C}^\Sigma[\mathbf{H}_\pm](z) = \mathbf{0}$ for $z \in \Omega^\mp$. The result follows by taking the limit $z \to \Sigma$. □

Example. The various spaces associated to a complete admissible contour Σ and the way the Cauchy boundary operators $\mathcal{C}^\Sigma_\pm$ relate these spaces can be illustrated by the following simple example. Take the contour Σ illustrated in the right-hand panel of Figure 2. Consider the function $f : \Sigma^\circ \to \mathbb{C}$ defined as $f(z) := 1$ ($f(z) := -1$) on the boundary of the component Ω^+_1 (Ω^+_2) of Ω^+. This function is (may be identified with) an element of the space $H^\nu_+(\Sigma)$, because its restriction to each loop L^+_1, L^+_2 separately is a constant. However, it is not in $H^\nu_-(\Sigma)$, because there are jump discontinuities along the boundary of Ω^-, which consists of two loops L^-_1, L^-_2. The Cauchy integral of f along Σ is easy to calculate by residues by first splitting the integral into two integrals over the two loops L^+_1, L^+_2:

$$\mathcal{C}^\Sigma[f](z) = \begin{cases} 1, & z \in \Omega^+_1 \\ -1, & z \in \Omega^+_2 \\ 0, & z \in \Omega^- = \Omega^-_1 \cup \Omega^-_2. \end{cases}$$

Thus, letting z tend to Σ° from Ω^+ yields the result $\mathcal{C}^\Sigma_+[f](z) = f(z)$, so $\mathcal{C}^\Sigma_+[f] \in H^\nu_+(\Sigma)$ as guaranteed by Proposition 8. Letting z tend to Σ° from Ω^- yields the even simpler result $\mathcal{C}^\Sigma_-[f](z) = 0$, a constant function that is obviously in $H^\nu(\Sigma)$ (and obviously in both $H^\nu_+(\Sigma)$ and $H^\nu_-(\Sigma)$). Therefore $\mathcal{C}^\Sigma_-[f] \in H^\nu(\Sigma)$ for $f \in H^\nu_+(\Sigma)$ again as guaranteed in general by Proposition 8.

Equivalent integral equation. We begin by defining an appropriate sort of algebraic factorization of the jump matrix from Riemann-Hilbert Problem 1.

Definition 12 (Admissible factorization)**.** *Let Σ be a complete admissible contour and let $\mathbf{V}$ be a corresponding admissible jump matrix of Hölder exponent $\mu < 1$. Then $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$ is an admissible factorization of the Riemann-Hilbert data $(\Sigma, \mathbf{V})$ provided that*

$$\mathbf{V}(z) = \mathbf{B}^-(z)^{-1}\mathbf{B}^+(z), \quad z \in \Sigma^\circ \tag{19}$$

where $\mathbf{B}^\pm(z)$ are unimodular matrices and $\mathbf{B}^\pm \in H^\mu_\pm(\Sigma)$.

The condition $\det(\mathbf{B}^\pm(z)) = 1$ is not essential (invertibility is enough) but is convenient given that $\det(\mathbf{V}(z)) = 1$. There are many different admissible factorizations corresponding to fixed Riemann-Hilbert data $(\Sigma, \mathbf{V})$. Indeed, suppose $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$ is an admissible factorization of $(\Sigma, \mathbf{V})$. Then so is $(\Sigma, \tilde{\mathbf{B}}^+, \tilde{\mathbf{B}}^-)$ where, given any $\mathbf{Y} \in H^\mu(\Sigma)$ with $\det(\mathbf{Y}(z)) = 1$,

$$\tilde{\mathbf{B}}^+(z) = \mathbf{Y}(z)\mathbf{B}^+(z) \quad \text{and} \quad \tilde{\mathbf{B}}^-(z) = \mathbf{Y}(z)\mathbf{B}^-(z). \tag{20}$$

We show that there exists at least one admissible factorization by a direct construction, which relies on the following technical lemma.

Lemma 2 (Unimodular interpolation)**.** *Let $\mathbf{A}$ and $\mathbf{B}$ be matrices with unit determinant. Then there exists a matrix function $\mathbf{F} : [0,1] \to \mathbb{C}^{N\times N}$ that is of class $C^\infty([0,1])$ and satisfies $\mathbf{F}(0) = \mathbf{A}$ and $\mathbf{F}(1) = \mathbf{B}$, and $\det(\mathbf{F}(t)) = 1$ for all $t \in [0,1]$.*

Proof. Write the unit determinant matrix $\mathbf{A}^{-1}\mathbf{B}$ in Jordan canonical form as

$$\mathbf{A}^{-1}\mathbf{B} = \mathbf{S}(\mathbf{D} + \mathbf{N})\mathbf{S}^{-1},$$

where $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_N)$ is a diagonal matrix of eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$, and where $\mathbf{N}$ is the corresponding upper triangular nilpotent part of the Jordan form. Let $\ell_1, \dots, \ell_N$ be complex numbers (logarithms) such that $e^{\ell_j} = d_j \neq 0$, and such that $\ell_1 + \cdots + \ell_N = 0$ (this is possible by choice of branches of the complex logarithm because $d_1 \cdots d_N = 1$). Then set $\mathbf{L} := \operatorname{diag}(\ell_1, \dots, \ell_N)$, and define

$$\mathbf{F}(t) := \mathbf{A}\mathbf{S}\left[\exp(t\mathbf{L}) + t\mathbf{N}\right]\mathbf{S}^{-1}. \tag{21}$$

Clearly we get $\mathbf{F}(0) = \mathbf{A}$ and $\mathbf{F}(1) = \mathbf{B}$. It is also obvious that $\mathbf{F}$ is infinitely differentiable. Finally, $\det(\mathbf{F}(t)) = \det(\exp(t\mathbf{L}) + t\mathbf{N}) = e^{t\ell_1} \cdots e^{t\ell_N} = e^{t(\ell_1 + \cdots + \ell_N)} = 1$. □

To describe a systematic construction of an admissible factorization of Riemann-Hilbert data $(\Sigma, \mathbf{V})$, we first let $X_0 \subset X$ denote those (exceptional) self-intersection points of Σ at which at least one of the jump matrix limits $\mathbf{V}_j$ at z_0 is not the identity matrix $\mathbb{I}$ (and therefore by (14) at least two of the $\mathbf{V}_j$ are not identity matrices). We isolate each point $z_0 \in X_0$ by letting $D(z_0)$ be a disk centered at z_0 of sufficiently small radius that it contains only z_0 and K arcs joining z_0 to the boundary $\partial D(z_0)$ (and in particular $D(z_0)$ does not contain any other points of X_0). Set

$$\mathbf{B}^+(z) := \mathbf{V}(z) \quad \text{and} \quad \mathbf{B}^-(z) := \mathbb{I}, \quad z \in \Sigma \setminus \bigcup_{z_0 \in X_0} D(z_0),$$

that is, we take a trivial factorization of $\mathbf{V}(z)$ away from all self-intersection points of Σ at which we do not consistently have $\mathbf{V}(z_0) = \mathbb{I}$. Now for each $z_0 \in X_0$, $\Sigma \cap D(z_0)$ consists of an even number of arcs joining the boundary of $D(z_0)$ with z_0 with alternating in/out orientations; we order them in counter-clockwise order about z_0 as $C_1, C_2, \dots, C_K$, K even, and we assume that C_1 is oriented toward z_0. Take a smooth parametrization $z = z_k(t)$, $0 \le t \le 1$ for each arc C_k as a constant multiple of arc length. Observe that $z_k(0)$ lies on $\partial D(z_0)$ and $z_k(1) = z_0$ for k odd while $z_k(0) = z_0$ and $z_k(1)$ lies on $\partial D(z_0)$ for k even. To define $\mathbf{B}^\pm(z)$ on the arcs $C_1, \dots, C_K$ within $D(z_0)$, we first set, by natural continuation of the definitions outside of $D(z_0)$,

$$\mathbf{B}^+(z) := \mathbf{V}(z) \quad \text{and} \quad \mathbf{B}^-(z) := \mathbb{I}, \quad z \in C_1.$$

Suppose that $\mathbf{B}^\pm(z)$ have been defined on C_{k-1}, and we will now explain how to choose $\mathbf{B}^\pm(z)$ on C_k so that they are locally continuous along $\partial\Omega^\pm$. If k is even, note that between C_k and C_{k-1} lies a component of Ω^- so we will require continuity of $\mathbf{B}^-(z)$ along the boundary of this component by setting $\mathbf{B}^-(z_k(0)) = \mathbf{B}^-(z_{k-1}(1))$. Then, we invoke Lemma 2 to obtain $\mathbf{B}^-(z)$ on C_k as a matrix with unit determinant satisfying

$\mathbf{B}^-(z_k(1)) = \mathbb{I}$. This function is Lipschitz as a function of scaled arc length t, but this implies that it is also in $H^1(C_k)$ by the standard upper bound of arc length in terms of straight-line distance. With $\mathbf{B}^-(z)$ defined on C_k, we set $\mathbf{B}^+(z) := \mathbf{B}^-(z)\mathbf{V}(z)$ for $z \in C_k$. On the other hand, if k is odd, then between C_k and C_{k-1} lies a component of Ω^+ so we require continuity of $\mathbf{B}^+(z)$ along the boundary of this component. Thus we define $\mathbf{B}^+(z_k(t))$ for $0 \le t \le 1$ by again invoking Lemma 2 using $\mathbf{B}^+(z_k(0)) = \mathbf{V}(z_k(0))$ and $\mathbf{B}^+(z_k(1)) = \mathbf{B}^+(z_{k-1}(0))$, and then obtain $\mathbf{B}^-(z)$ on C_k by $\mathbf{B}^-(z) := \mathbf{B}^+(z)\mathbf{V}(z)^{-1}$.

The fact that this factorization $\mathbf{V}(z) = \mathbf{B}^-(z)^{-1}\mathbf{B}^+(z)$ is admissible clearly boils down to showing that $\mathbf{B}^\pm \in H^\mu_\pm(\Sigma)$, i.e., that $\mathbf{B}^\pm(z)$ is Hölder continuous around each loop $L_j^\pm$, even where several loops meet at an exceptional point $z_0 \in X$. This continuity follows from the above construction due to the identity (14) satisfied by the limiting jump matrices at each self-intersection point z_0 of Σ. In the case of self-intersection points $z_0 \in X \setminus X_0$, the limit of $\mathbf{B}^\pm(z)$ as $z \to z_0$ is $\mathbb{I}$ for each arc meeting z_0, so the desired continuity holds automatically. We call an admissible factorization $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$ of $(\Sigma, \mathbf{V})$ of the type just constructed a *standard factorization*.

In many situations one can show that a standard factorization has the property that if $\|\mathbf{V} - \mathbb{I}\|^\circ_\mu$ is small, then so are $\|\mathbf{B}^\pm - \mathbb{I}\|^\pm_\mu$. In fact, given concrete choices of the various parameters (e.g. disk radii, interpolating function $\mathbf{F}$) of a standard factorization, one can easily prove an estimate of the form

$$\|\mathbf{B}^\pm - \mathbb{I}\|^\pm_\mu \le K_\Sigma \|\mathbf{V} - \mathbb{I}\|^\circ_\mu \tag{22}$$

for some constant K_Σ independent of the admissible jump matrix as long as X_0 is fixed. Another property that will be useful in applications is:

Proposition 11. *Let Σ be a complete admissible contour and let $\mathbf{V}$ be a corresponding admissible jump matrix of Hölder exponent $\mu < 1$ with the property that for some point $z_0 \in X$, $\mathbf{V}(z) = \mathbb{I} + O(|z - z_0|^p)$. Then for a standard factorization, it also holds that $\mathbf{B}^\pm(z) = \mathbb{I} + O(|z - z_0|^p)$.*

Proof. A standard factorization sets $\mathbf{B}^+(z) = \mathbf{V}(z)$ and $\mathbf{B}^-(z) = \mathbb{I}$ on the part of Σ in a neighborhood of each such point z_0 (because $z_0 \in X \setminus X_0$). □

Now for any $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$ admissible and associated with Riemann-Hilbert data $(\Sigma, \mathbf{V})$, define

$$\mathbf{W}^+(z) := \mathbf{B}^+(z) - \mathbb{I} \in H^\mu_+(\Sigma) \quad \text{and} \quad \mathbf{W}^-(z) := \mathbb{I} - \mathbf{B}^-(z) \in H^\mu_-(\Sigma). \tag{23}$$

The key operator in studying Riemann-Hilbert Problem 1 is then the following:

$$\mathcal{C}_\mathbf{W} := \mathcal{C}^\Sigma_+ \circ \mathcal{R}_{\mathbf{W}^-} + \mathcal{C}^\Sigma_- \circ \mathcal{R}_{\mathbf{W}^+}.$$

In more concrete terms, the action of $\mathcal{C}_\mathbf{W}$ on a function $\mathbf{F} : \Sigma^\circ \to \mathbb{C}^{N\times N}$ is

$$\mathcal{C}_\mathbf{W}[\mathbf{F}](z) = \mathcal{C}^\Sigma_+[\mathbf{F}\mathbf{W}^-](z) + \mathcal{C}^\Sigma_-[\mathbf{F}\mathbf{W}^+](z), \quad z \in \Sigma.$$

Proposition 12. *$\mathcal{C}_\mathbf{W}$ is a bounded linear operator on $H^\nu(\Sigma)$ whenever $\nu \le \mu < 1$.*

Proof. Since $H^\mu_-(\Sigma) \subset H^\nu_-(\Sigma)$ for $\nu \le \mu$, $\mathcal{R}_{\mathbf{W}^-}$ is bounded from $H^\nu(\Sigma)$ to $H^\nu_-(\Sigma)$, on which $\mathcal{C}^\Sigma_+$ is bounded with range $H^\nu(\Sigma)$. Likewise $\mathcal{R}_{\mathbf{W}^+}$ is bounded from $H^\nu(\Sigma)$ to $H^\nu_+(\Sigma)$, on which $\mathcal{C}^\Sigma_-$ is bounded with range $H^\nu(\Sigma)$. □

Theorem 2 (Singular integral equation for the Riemann-Hilbert problem)**.** *Let $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$ be an admissible factorization of Riemann-Hilbert data $(\Sigma, \mathbf{V})$ of Hölder exponent $\mu < 1$, and let $\nu \le \mu$. Every solution $\mathbf{X} \in H^\nu(\Sigma)$ of the singular integral equation*

$$(\mathcal{I} - \mathcal{C}_\mathbf{W})\mathbf{X} = \mathbb{I} \in H^\nu(\Sigma) \tag{24}$$

gives a solution of Riemann-Hilbert Problem 1 via the formula

$$\mathbf{M}(z) = \mathbb{I} + \mathcal{C}^\Sigma[\mathbf{X}(\mathbf{W}^+ + \mathbf{W}^-)](z) := \mathbb{I} + \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\mathbf{X}(w)(\mathbf{W}^+(w) + \mathbf{W}^-(w))}{w - z}\, \mathrm{d}w, \quad z \in \mathbb{C} \setminus \Sigma. \tag{25}$$

Proof. Suppose that $\mathbf{X}$ is a solution of (24). The formula (25) then defines $\mathbf{M}(z)$ as an analytic function of z in the domain $\mathbb{C} \setminus \Sigma$ that is Hölder continuous up to Σ with exponent ν and that satisfies the normalization condition $\mathbf{M}(z) \to \mathbb{I}$ as $z \to \infty$. We now show that $\mathbf{M}(z)$ given by (25) satisfies the jump condition

$\mathbf{M}_+(z) = \mathbf{M}_-(z)\mathbf{V}(z)$, which in view of the factorization (19) and the definition (23) can be written in the form

$$\mathbf{M}_+(z)(\mathbb{I} + \mathbf{W}^+(z))^{-1} = \mathbf{M}_-(z)(\mathbb{I} - \mathbf{W}^-(z))^{-1}.$$

Substituting from (25) yields

$$(\mathbb{I} + \mathcal{C}_+^\Sigma[\mathbf{X}\mathbf{W}^+](z) + \mathcal{C}_+^\Sigma[\mathbf{X}\mathbf{W}^-](z))(\mathbb{I} + \mathbf{W}^+(z))^{-1} = (\mathbb{I} + \mathcal{C}_-^\Sigma[\mathbf{X}\mathbf{W}^+](z) + \mathcal{C}_-^\Sigma[\mathbf{X}\mathbf{W}^-](z))(\mathbb{I} - \mathbf{W}^-(z))^{-1}.$$

Then, using the integral equation (24) satisfied by $\mathbf{X}$, this becomes

$$(\mathbf{X}(z)+\mathcal{C}_+^\Sigma[\mathbf{X}\mathbf{W}^+](z)-\mathcal{C}_-^\Sigma[\mathbf{X}\mathbf{W}^+](z))(\mathbb{I}+\mathbf{W}^+(z))^{-1} = (\mathbf{X}(z)-\mathcal{C}_+^\Sigma[\mathbf{X}\mathbf{W}^-](z)+\mathcal{C}_-^\Sigma[\mathbf{X}\mathbf{W}^-](z))(\mathbb{I}-\mathbf{W}^-(z))^{-1},$$

which after using the Plemelj formula $\mathcal{C}_+^\Sigma - \mathcal{C}_-^\Sigma = \mathcal{I}$ becomes an identity $\mathbf{X}(z) = \mathbf{X}(z)$ for $z \in \Sigma^\circ$. □

Proposition 13. *Given two admissible factorizations* $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$ *and* $(\Sigma, \tilde{\mathbf{B}}^+, \tilde{\mathbf{B}}^-)$ *of the same Riemann-Hilbert data* $(\Sigma, \mathbf{V})$ *and related by* (20)*, by analogy with* (23) *set* $\tilde{\mathbf{W}}^\pm(z) := \pm(\tilde{\mathbf{B}}^\pm(z) - \mathbb{I})$*. Then, the associated operators* $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ *and* $\mathcal{I} - \mathcal{C}_{\tilde{\mathbf{W}}}$ *are related explicitly by*

$$\mathcal{I} - \mathcal{C}_{\tilde{\mathbf{W}}} = (\mathcal{I} - \mathcal{C}_{\mathbf{W}}) \circ \mathcal{R}_{\mathbf{Y}}.$$

Hence $\mathcal{I} - \mathcal{C}_{\tilde{\mathbf{W}}}$ *is invertible on* $H^\nu(\Sigma)$ *if and only if* $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ *is.*

Proof. Observe first that from (20) and (23),

$$\tilde{\mathbf{W}}^\pm(z) = \pm(\mathbf{Y}(z)\mathbf{B}^\pm(z) - \mathbb{I}) = \pm(\mathbf{Y}(z) - \mathbb{I} \pm \mathbf{Y}(z)\mathbf{W}^\pm(z)) = \pm(\mathbf{Y}(z) - \mathbb{I}) + \mathbf{Y}(z)\mathbf{W}^\pm(z).$$

Therefore, by definition of $\mathcal{C}_{\tilde{\mathbf{W}}}$, we have

$$\begin{aligned}(\mathcal{I} - \mathcal{C}_{\tilde{\mathbf{W}}})[\mathbf{X}](z) &:= \mathbf{X}(z) - \mathcal{C}_+^\Sigma[\mathbf{X}\tilde{\mathbf{W}}^-](z) - \mathcal{C}_-^\Sigma[\mathbf{X}\tilde{\mathbf{W}}^+](z) \\ &= \mathbf{X}(z) + \mathcal{C}_+^\Sigma[\mathbf{X}(\mathbf{Y} - \mathbb{I})](z) - \mathcal{C}_-^\Sigma[\mathbf{X}(\mathbf{Y} - \mathbb{I})](z) - \mathcal{C}_+^\Sigma[\mathbf{X}\mathbf{Y}\mathbf{W}^-](z) - \mathcal{C}_-^\Sigma[\mathbf{X}\mathbf{Y}\mathbf{W}^+](z).\end{aligned}$$

Finally, using the Plemelj formula on the second and third terms on the right-hand side, we obtain

$$\begin{aligned}(\mathcal{I} - \mathcal{C}_{\tilde{\mathbf{W}}})[\mathbf{X}](z) &= \mathbf{X}(z)\mathbf{Y}(z) - \mathcal{C}_+^\Sigma[\mathbf{X}\mathbf{Y}\mathbf{W}^-](z) - \mathcal{C}_-^\Sigma[\mathbf{X}\mathbf{Y}\mathbf{W}^+](z) \\ &= (\mathcal{I} - \mathcal{C}_{\mathbf{W}})[\mathbf{X}\mathbf{Y}](z) \\ &= ((\mathcal{I} - \mathcal{C}_{\mathbf{W}}) \circ \mathcal{R}_{\mathbf{Y}})[\mathbf{X}](z).\end{aligned}$$

Since right-multiplication by the invertible $\mathbf{Y} \in H^\nu(\Sigma)$ is an isomorphism of $H^\nu(\Sigma)$, the proof is finished. □

For applications, the following result will be useful.

Proposition 14. *Suppose that* (24) *has a solution* $\mathbf{X} \in H^\nu(\Sigma)$ *and that* $z_0 \in \Sigma$ *is a point at which* $\mathbf{W}^\pm(z)$ *vanish to all orders as* $z \to z_0$ *along each component of* Σ° *meeting* z_0*. Then the corresponding (unique) solution* $\mathbf{M}(z)$ *of Riemann-Hilbert Problem 1 given by* (25) *has an asymptotic expansion in powers of* $z - z_0$*:*

$$\mathbf{M}(z) - \mathbb{I} \sim \sum_{n=0}^{\infty} \mathbf{M}_n (z - z_0)^n, \quad z \to z_0, \quad z \in \mathbb{C} \setminus \Sigma, \tag{26}$$

with coefficients

$$\mathbf{M}_n := \frac{1}{2\pi \mathrm{i}} \int_\Sigma \frac{\mathbf{X}(w)(\mathbf{W}^+(w) + \mathbf{W}^-(w))}{(w - z_0)^{n+1}} \, \mathrm{d}w, \quad n = 0, 1, 2, \ldots. \tag{27}$$

Note that these integrals are all absolutely convergent, and that according to Proposition 11 the hypotheses are guaranteed with the use of a standard factorization provided $\mathbf{V}(z) - \mathbb{I}$ *vanishes to all orders as* $z \to z_0$*.*

Proof. For any $M = 0, 1, 2, \ldots$, we can write the Cauchy kernel $(w - z)^{-1}$ in the form

$$\frac{1}{w - z} = \frac{1}{(w - z_0) - (z - z_0)} = \frac{1}{w - z_0} \cdot \frac{1}{1 - \rho} = \frac{1}{w - z_0} \left[\sum_{n=0}^{M} \rho^n + \frac{\rho^{M+1}}{1 - \rho} \right], \quad \rho := \frac{z - z_0}{w - z_0},$$

or,

$$\frac{1}{w - z} = \sum_{n=0}^{M} \frac{(z - z_0)^n}{(w - z_0)^{n+1}} + \frac{(z - z_0)^{M+1}}{(w - z_0)^{M+1}(w - z)}.$$

Therefore, from (25) we have

$$\mathbf{M}(z) - \mathbb{I} - \sum_{n=0}^{M} \mathbf{M}_n (z - z_0)^n = (z - z_0)^{M+1} \mathcal{C}^{\Sigma}[\mathbf{F}_M](z), \quad \mathbf{F}_M(w) := \frac{\mathbf{X}(w)(\mathbf{W}^+(w) + \mathbf{W}^-(w))}{(w - z_0)^{M+1}}.$$

But the hypotheses on $\mathbf{W}^{\pm}$ imply that $\mathbf{F}_M \in H^{\nu}(\Sigma^{\circ})$ for all $M = 0, 1, 2, \dots$, so $\mathcal{C}^{\Sigma}[\mathbf{F}_M](z)$ is Hölder continuous with exponent ν up to Σ, from which it follows that the right-hand side is $O((z - z_0)^{M+1})$ as $z \to z_0$. □

Observe that the asymptotic expansion (26) is generally either divergent, or if convergent its sum does not equal $\mathbf{M}(z) - \mathbb{I}$. Indeed, if the series were to converge, its sum would have to be analytic at z_0. However, while by hypothesis the jump matrices approach the identity matrix as $z \to z_0$ faster than any power of $z - z_0$, they need not *equal* the identity matrix locally, and therefore $\mathbf{M}(z) - \mathbb{I}$ need not even be continuous in a neighborhood of z_0, for the result to hold.

Small-norm problems. The easiest way to solve the integral equation (24), and hence Riemann-Hilbert Problem 1, is to apply iteration, representing the inverse operator $(\mathcal{I} - \mathcal{C}_{\mathbf{W}})^{-1}$ on $H^{\nu}(\Sigma)$ by its Neumann series

$$(\mathcal{I} - \mathcal{C}_{\mathbf{W}})^{-1} = \mathcal{I} + \mathcal{C}_{\mathbf{W}} + \mathcal{C}_{\mathbf{W}} \circ \mathcal{C}_{\mathbf{W}} + \cdots = \sum_{n=0}^{\infty} \mathcal{C}_{\mathbf{W}}^n.$$

The Neumann series converges if $\|\mathcal{C}_{\mathbf{W}}\|_{\nu}$, the operator norm on $H^{\nu}(\Sigma)$ of $\mathcal{C}_{\mathbf{W}}$, satisfies

$$\|\mathcal{C}_{\mathbf{W}}\|_{\nu} < 1.$$

Such a situation is called a *small-norm problem*. We know from Proposition 12 that the operator norm $\|\mathcal{C}_{\mathbf{W}}\|_{\nu}$ is finite, but it will be useful to estimate it in terms of more controllable quantities. Since $\mathcal{C}_{\pm}^{\Sigma}$ are bounded operators from $H^{\nu}_{\mp}(\Sigma)$ to $H^{\nu}(\Sigma)$, there exists a constant K depending only on the contour Σ such that

$$\|\mathcal{C}_{\pm}^{\Sigma}\mathbf{F}\|_{\nu} \le K \|\mathbf{F}\|_{\nu}^{\mp}$$

where on the left-hand side we have the norm on $H^{\nu}(\Sigma)$ while on the right-hand side we have the norm on $H^{\nu}_{\mp}(\Sigma)$ (which is a sum of $H^{\nu}(L)$ norms over loops $L_j^{\mp}$). But then it follows easily that

$$\|\mathcal{C}_{\mathbf{W}}\mathbf{F}\|_{\nu} \le K(\|\mathbf{W}^+\|_{\nu}^+ + \|\mathbf{W}^-\|_{\nu}^-)\|\mathbf{F}\|_{\nu}$$

holds for all $\mathbf{F} \in H^{\nu}(\Sigma)$. We therefore have a small-norm problem if

$$\|\mathbf{W}^+\|_{\nu}^+ + \|\mathbf{W}^-\|_{\nu}^- < \frac{1}{K}.$$

This condition essentially says that the jump matrix $\mathbf{V}$ should be close to the identity matrix in a suitable sense on Σ°.

Because the bounded operators on $H^{\nu}(\Sigma)$ form a Banach algebra, i.e., for bounded operators $\mathcal{A}$, $\mathcal{B}$ acting on $H^{\nu}(\Sigma)$ we have $\|\mathcal{A} \circ \mathcal{B}\|_{\nu} \le \|\mathcal{A}\|_{\nu}\|\mathcal{B}\|_{\nu}$, it follows from the Neumann series formula that

$$\|(\mathcal{I} - \mathcal{C}_{\mathbf{W}})^{-1}\|_{\nu} \le \sum_{n=0}^{\infty} \|\mathcal{C}_{\mathbf{W}}\|_{\nu}^n = \frac{1}{1 - \|\mathcal{C}_{\mathbf{W}}\|_{\nu}}, \quad \|\mathcal{C}_{\mathbf{W}}\|_{\nu} < 1.$$

Therefore, since the norm of the constant function $\mathbb{I} \in H^{\nu}(\Sigma)$ is $\|\mathbb{I}\|_{\nu} = 1$, in the small norm setting we get

$$\|\mathbf{X}\|_{\nu} \le \frac{1}{1 - \|\mathcal{C}_{\mathbf{W}}\|_{\nu}}, \quad \|\mathcal{C}_{\mathbf{W}}\|_{\nu} < 1.$$

If in addition the hypotheses of Proposition 14 hold, the integrals (27) therefore satisfy

$$\|\mathbf{M}_n\| \le \frac{s(\Sigma)}{2\pi(1 - \|\mathcal{C}_{\mathbf{W}}\|_{\nu})} \sup_{w \in \Sigma} \frac{\|\mathbf{W}^+(w) + \mathbf{W}^-(w)\|}{|w - z_0|^{n+1}}, \quad n = 0, 1, 2, \dots,$$

where $s(\Sigma)$ is the total arc length of Σ. By analogy with (22), with the use of a standard factorization the above supremum can typically be estimated in terms of $\mathbf{V} - \mathbb{I}$ and in particular its (rapidly vanishing) asymptotic behavior near $z = z_0$, i.e., in terms of the original data $(\Sigma, \mathbf{V})$ for Riemann-Hilbert Problem 1.

Small norm problems seldom arise on their own in applications. However, in situations where a small parameter $\epsilon \ll 1$ is present in the data $(\Sigma, \mathbf{V})$ of a Riemann-Hilbert problem, one can sometimes introduce a finite sequence of well-motivated substitutions that convert the Riemann-Hilbert problem with data $(\Sigma, \mathbf{V})$ into an equivalent one with data $(\Sigma', \mathbf{V}')$ that is indeed a small-norm problem in the limit $\epsilon \to 0$. This is the main idea behind the *Deift-Zhou steepest descent method* for Riemann-Hilbert problems that we have discussed already in the course.

Fredholm theory. Recall the following definitions, for which an excellent reference is [2, Chapter 27].

Definition 13 (Kernel and cokernel)**.** *Let $\mathcal{A} : B \to B$ be a bounded linear operator on a Banach space B. The kernel of $\mathcal{A}$ is the subspace $\ker(\mathcal{A}) \subset B$ of vectors $f \in B$ such that $\mathcal{A}f = 0$. The range of $\mathcal{A}$ is the subspace $\mathrm{ran}(\mathcal{A}) \subset B$ of vectors $f \in B$ such that $f = \mathcal{A}g$ for some $g \in B$. The cokernel of $\mathcal{A}$ is the vector space (of equivalence classes) $\mathrm{coker}(\mathcal{A}) := B/\mathrm{ran}(\mathcal{A})$.*

Definition 14 (Fredholm operator)**.** *A bounded linear operator $\mathcal{A} : B \to B$ on a Banach space B is a Fredholm operator if $\dim(\ker(\mathcal{A})$ and $\dim(\mathrm{coker}(\mathcal{A}))$ are both finite. The index of a Fredholm operator $\mathcal{A}$ is $\mathrm{ind}(\mathcal{A}) := \dim(\ker(\mathcal{A})) - \dim(\mathrm{coker}(\mathcal{A}))$.*

Definition 15 (Pseudoinverse)**.** *Let $\mathcal{A}$ be a bounded linear operator on a Banach space B. Another bounded linear operator $\mathcal{B}$ acting on B is called a pseudoinverse to $\mathcal{A}$ if*

$$\mathcal{B} \circ \mathcal{A} = \mathcal{I} - \mathcal{K} \quad \text{and} \quad \mathcal{A} \circ \mathcal{B} = \mathcal{I} - \mathcal{K}'$$

where $\mathcal{K}, \mathcal{K}' : B \to B$ are both compact.

The key result of the general theory that we will need is the following.

Theorem 3. *Let $\mathcal{A} : B \to B$ be a bounded linear operator on a Banach space B. Then $\mathcal{A}$ is a Fredholm operator if $\mathcal{A}$ has a pseudoinverse.*

By analogy with the definition of the operator $\mathcal{C}_{\mathbf{W}}$, we may define an operator constructed from the *inverses* of the matrices $\mathbf{B}^{\pm}(z)$: we first set

$$\mathbf{U}^{+}(z) := \mathbf{B}^{+}(z)^{-1} - \mathbb{I} \in H^{\mu}_{+}(\Sigma) \quad \text{and} \quad \mathbf{U}^{-}(z) := \mathbb{I} - \mathbf{B}^{-}(z)^{-1} \in H^{\mu}_{-}(\Sigma),$$

and define $\mathcal{C}_{\mathbf{U}} := \mathcal{C}^{\Sigma}_{+} \circ \mathcal{R}_{\mathbf{U}^-} + \mathcal{C}^{\Sigma}_{-} \circ \mathcal{R}_{\mathbf{U}^+}$. By the same argument as in Proposition 12, $\mathcal{C}_{\mathbf{U}}$ is bounded on $H^{\nu}(\Sigma)$. Note also that since $\mathbf{B}^{\pm}(z)\mathbf{B}^{\pm}(z)^{-1} = \mathbf{B}^{\pm}(z)^{-1}\mathbf{B}^{\pm}(z) = \mathbb{I}$,

$$\mathbf{W}^{\pm}(z)\mathbf{U}^{\pm}(z) = \mathbf{U}^{\pm}(z)\mathbf{W}^{\pm}(z) = \mp(\mathbf{W}^{\pm}(z) + \mathbf{U}^{\pm}(z)). \tag{28}$$

Proposition 15. *Let Σ be a complete admissible contour, and suppose that $\mathbf{V}$ is an admissible jump matrix on Σ with positive exponent $\mu < 1$ having the corresponding admissible factorization $\mathbf{V}(z) = \mathbf{B}^{-}(z)^{-1}\mathbf{B}^{+}(z)$. Then $\mathcal{I} - \mathcal{C}_{\mathbf{U}}$ is a pseudoinverse to $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ on the space $H^{\nu}(\Sigma)$ for each $\nu < \mu$.*

Proof. Let $\mathcal{K} := \mathcal{I} - (\mathcal{I} - \mathcal{C}_{\mathbf{U}}) \circ (\mathcal{I} - \mathcal{C}_{\mathbf{W}})$. We prove that $\mathcal{K} : H^{\nu}(\Sigma) \to H^{\nu}(\Sigma)$ is compact. We now expand out $\mathcal{K}$, and to keep the formulas as simple as possible, we omit the redundant superscript Σ (because the contour Σ is fixed) from the Cauchy operators $\mathcal{C}^{\Sigma}_{\pm}$, and we simply use the symbol $\mathbf{M}$ in place of the operator $\mathcal{R}_{\mathbf{M}}$ of right multiplication by $\mathbf{M}$ (since $\mathcal{C}_{\mathbf{W}}$ and $\mathcal{C}_{\mathbf{U}}$ involve no left multiplications):

$$\begin{aligned}\mathcal{K} &= \mathcal{C}_{\mathbf{U}} + \mathcal{C}_{\mathbf{W}} - \mathcal{C}_{\mathbf{U}} \circ \mathcal{C}_{\mathbf{W}} \\ &= \left[\mathcal{C}_+ \circ \mathbf{U}^- + \mathcal{C}_- \circ \mathbf{U}^+ + \mathcal{C}_+ \circ \mathbf{W}^- + \mathcal{C}_- \circ \mathbf{W}^+\right] \\ &\quad - \left[\mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_+ \circ \mathbf{W}^- + \mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^+ + \mathcal{C}_- \circ \mathbf{U}^+ \circ \mathcal{C}_+ \circ \mathbf{W}^- + \mathcal{C}_- \circ \mathbf{U}^+ \circ \mathcal{C}_- \circ \mathbf{W}^+\right]\end{aligned}$$

In the first and last terms on the second line, use the Plemelj formula to eliminate the right-most Cauchy operators:

$$\begin{aligned}\mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_+ \circ \mathbf{W}^- &= \mathcal{C}_+ \circ \mathbf{U}^- \circ \mathbf{W}^- + \mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^- \\ &= \mathcal{C}_+ \circ (\mathbf{W}^-\mathbf{U}^-) + \mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^- \\ &= \mathcal{C}_+ \circ (\mathbf{U}^- + \mathbf{W}^-) + \mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^- \\ &= \mathcal{C}_+ \circ \mathbf{U}^- + \mathcal{C}_+ \circ \mathbf{W}^- + \mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^-,\end{aligned}$$

where on the second line $(\mathbf{W}^-\mathbf{U}^-)$ denotes the operator of right-multiplication by the matrix product $\mathbf{W}^-\mathbf{U}^-$ and going to the third line we have used (28). Similarly

$$\mathcal{C}_- \circ \mathbf{U}^+ \circ \mathcal{C}_- \circ \mathbf{W}^+ = \mathcal{C}_- \circ \mathbf{U}^+ + \mathcal{C}_- \circ \mathbf{W}^+ + \mathcal{C}_- \circ \mathbf{U}^+ \circ \mathcal{C}_+ \circ \mathbf{W}^+.$$

The terms linear in $\mathbf{W}^{\pm}$ or $\mathbf{U}^{\pm}$ therefore cancel from $\mathcal{K}$ and we get

$$\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4,$$

where

$$\begin{aligned}\mathcal{K}_1 &:= -\mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^- \\ \mathcal{K}_2 &:= -\mathcal{C}_+ \circ \mathbf{U}^- \circ \mathcal{C}_- \circ \mathbf{W}^+ \\ \mathcal{K}_3 &:= -\mathcal{C}_- \circ \mathbf{U}^+ \circ \mathcal{C}_+ \circ \mathbf{W}^- \\ \mathcal{K}_4 &:= -\mathcal{C}_- \circ \mathbf{U}^+ \circ \mathcal{C}_+ \circ \mathbf{W}^+.\end{aligned}$$

Because $\mathcal{C}_+ \circ \mathcal{C}_- = 0$ on the space $H^\nu_-(\Sigma)$, the range of $\mathbf{U}^- \circ \mathbf{W}^-$ acting on $H^\nu(\Sigma)$, we have

$$\mathcal{K}_1 = \mathcal{C}_+ \circ [\mathcal{C}_-, \mathbf{U}^-] \circ \mathbf{W}^-. \tag{29}$$

Similarly, because $\mathcal{C}_- \circ \mathcal{C}_+ = 0$ on the space $H^\nu_+(\Sigma)$, the range of $\mathbf{U}^+ \circ \mathbf{W}^+$ acting on $H^\nu(\Sigma)$, we have

$$\mathcal{K}_4 = \mathcal{C}_- \circ [\mathcal{C}_+, \mathbf{U}^+] \circ \mathbf{W}^+. \tag{30}$$

By the Plemelj formula, the matrix identity $\mathbf{U}^-(z) = \mathcal{C}_+[\mathbf{U}^-](z) - \mathcal{C}_-[\mathbf{U}^-](z)$ holds, so $\mathcal{K}_2$ can be written in the form

$$\mathcal{K}_2 = -\mathcal{C}_+ \circ \mathbf{U}^-_+ \circ \mathcal{C}_- \circ \mathbf{W}^+ - \mathcal{C}_+ \circ \mathbf{U}^-_- \circ \mathcal{C}_- \circ \mathbf{W}^+, \quad \mathbf{U}^-_\pm(z) := \pm\mathcal{C}_\pm[\mathbf{U}^-](z).$$

By Proposition 10 the second term vanishes because it is $\mathcal{C}_+$ acting on a product of functions in the range of $\mathcal{C}_-$. Then, since $\mathcal{C}_+ \circ \mathcal{C}_- = 0$ on $H^\nu_+(\Sigma)$, the range of $\mathbf{U}^-_+ \circ \mathbf{W}^+$ acting on $H^\nu(\Sigma)$, we have

$$\mathcal{K}_2 = \mathcal{C}_+ \circ [\mathcal{C}_-, \mathbf{U}^-_+] \circ \mathbf{W}^+. \tag{31}$$

Similarly,

$$\mathcal{K}_3 = -\mathcal{C}_- \circ \mathbf{U}^+_+ \circ \mathcal{C}_+ \circ \mathbf{W}^- - \mathcal{C}_- \circ \mathbf{U}^+_- \circ \mathcal{C}_+ \circ \mathbf{W}^-, \quad \mathbf{U}^+_\pm(z) := \pm\mathcal{C}_\pm[\mathbf{U}^+](z),$$

and the first term vanishes by Proposition 10 while a commutator can be introduced in the second as above, with the result that

$$\mathcal{K}_3 = \mathcal{C}_- \circ [\mathcal{C}_+, \mathbf{U}^+_-] \circ \mathbf{W}^-. \tag{32}$$

Consider $\mathcal{K}_1$ given by (29). The operator $[\mathcal{C}_-, \mathbf{U}^-] \circ \mathbf{W}^-$ is the (ordered) product of a bounded map from $H^\nu(\Sigma)$ to $H^\nu_-(\Sigma)$ (Proposition 7), followed by a bounded map from $H^\nu_-(\Sigma)$ to $H^\mu_-(\Sigma)$ (Proposition 9) and is therefore a bounded map from $H^\nu(\Sigma)$ to $H^\mu_-(\Sigma)$. To get back to the larger space $H^\nu_-(\Sigma)$ we may follow this action with the identity operator in the form of the inclusion $\mathcal{I}_{\mu\to\nu}$ which is a compact mapping from $H^\mu_-(\Sigma)$ to $H^\nu_-(\Sigma)$ by Proposition 6. Finally by Proposition 8, the left-most factor in $\mathcal{K}_1$ is a bounded map from $H^\nu_-(\Sigma)$ to $H^\nu(\Sigma)$. Therefore, we see that $\mathcal{K}_1$ is a product of bounded maps with one compact factor. As the compact operators form a two-sided ideal in the algebra of bounded operators [1, 2], all it takes is one compact factor to make the product $\mathcal{K}_1$ a compact map on $H^\nu(\Sigma)$. By virtually the same argument applied to (30), $\mathcal{K}_4 : H^\nu(\Sigma) \to H^\nu(\Sigma)$ is also compact.

Next consider $\mathcal{K}_2$ given by (31). Note that by Proposition 8, $\mathbf{U}^-_+ \in H^\mu(\Sigma)$ as the action of $\mathcal{C}_+$ on a function in $H^\mu_-(\Sigma)$. Thus, the operator of right-multiplication by $\mathbf{W}^+$ is bounded from $H^\nu(\Sigma)$ to $H^\nu_+(\Sigma)$ (Proposition 7), $[\mathcal{C}_-, \mathbf{U}^-_+]$ is bounded from $H^\nu_+(\Sigma)$ to $H^\mu(\Sigma)$ (Proposition 9) or equivalently is compact from $H^\nu_+(\Sigma)$ to $H^\nu(\Sigma)$ (Proposition 6), and $\mathcal{C}_-$ is bounded on $H^\nu(\Sigma)$ (Proposition 8). Thus $\mathcal{K}_2$ is compact on $H^\nu(\Sigma)$. By virtually the same arguments applied to the formula (32), $\mathcal{K}_3 : H^\nu(\Sigma) \to H^\nu(\Sigma)$ is also compact.

As a sum $\mathcal{K} = \mathcal{K}_1 + \mathcal{K}_2 + \mathcal{K}_3 + \mathcal{K}_4$ of compact operators on $H^\nu(\Sigma)$, $\mathcal{K}$ is itself compact on $H^\nu(\Sigma)$. Writing $\mathcal{K}' := \mathcal{I} - (\mathcal{I} - \mathcal{C}_{\mathbf{W}}) \circ (\mathcal{I} - \mathcal{C}_{\mathbf{U}})$, a simple lexicographical swap $\mathbf{W} \leftrightarrow \mathbf{U}$ in the above arguments shows that $\mathcal{K}'$ is also compact on $H^\nu(\Sigma)$. □

Corollary 3. *Under the hypotheses of Proposition 15, $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ is a Fredholm operator on $H^\nu(\Sigma)$.*

Proof. Combine Proposition 15 with Theorem 3. □

We present the following important result without (complete) proof. See [6] for details.

Proposition 16 (Zhou's index theorem)**.** *Under the hypotheses of Proposition 15,* $\operatorname{ind}(\mathcal{I} - \mathcal{C}_{\mathbf{W}}) = 0$.

In fact, Zhou proves a more general result. He considers the case of invertible jump matrices $\mathbf{V} : \Sigma^\circ \to \mathbb{C}^{N\times N}$ that need not have unit determinant, and establishes the following remarkable formula for the Fredholm index of $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$:

$$\mathrm{ind}(\mathcal{I} - \mathcal{C}_{\mathbf{W}}) = N\omega(\det(\mathbf{V})),$$

where $\omega(f)$ denotes the *winding number* of a function $f : \Sigma^\circ \to \mathbb{C}$, i.e.,

$$\omega(f) := \frac{1}{2\pi}\int_\Sigma \mathrm{d}(\arg(f(z))).$$

The integral above is the total increment of the phase angle of f as z traverses the arcs of Σ according to their orientation. Obviously, $\omega(fg) = \omega(f) + \omega(g)$, and so by the factorization $\mathbf{V}(z) = \mathbf{B}^-(z)^{-1}\mathbf{B}^+(z)$ with $\mathbf{B}^\pm \in H^\nu_\pm(\Sigma)$ we have $\omega(\det(\mathbf{V})) = \omega(\det(\mathbf{B}^+)) - \omega(\det(\mathbf{B}^-))$. Since Σ may be viewed as either the collection of loops $\{L_j^+\}$ or $\{L_j^-\}$, both $\omega(\det(\mathbf{B}^+))$ and $\omega(\det(\mathbf{B}^-))$ are integers, and therefore $\omega(\det(\mathbf{V})) \in \mathbb{Z}$. Of course in the special case that $\det(\mathbf{V}(z)) = 1$, $\omega(\det(\mathbf{V})) = 0$, so Zhou's formula reduces to the statement of Proposition 16.

We can give a simple proof of Zhou's index theorem under suitable additional hypotheses on $\mathbf{W}^\pm$. For this we first need another general result of Fredholm theory (see, e.g., [2]).

Theorem 4 (Homotopy invariance of the Fredholm index)**.** *Let $\mathcal{A}(t)$ be a one-parameter family of Fredholm operators on a Banach space B, $0 \le t \le 1$, such that $\mathcal{A}(t)$ is a continuous function of t with respect to operator norm. Then* $\mathrm{ind}(\mathcal{A}(t))$ *is independent of t; in particular* $\mathrm{ind}(\mathcal{A}(1)) = \mathrm{ind}(\mathcal{A}(0))$.

To use this theorem, we should try to connect $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ to a simple operator for which we know the index by a suitable homotopy. The idea is the following: the identities

$$\mathbb{I} \pm \mathbf{U}^\pm(z) = \mathbf{B}^\pm(z)^{-1} = (\mathbb{I} \pm \mathbf{W}^\pm(z))^{-1}$$

show that the matrices $\mathbf{U}^\pm$ are determined once $\mathbf{W}^\pm$ are known. We may therefore introduce an artificial parameter $t \in [0, 1]$ by setting $\mathbf{W}^\pm(z; t) := t\mathbf{W}^\pm(z)$ and attempt to determine the corresponding matrix functions $\mathbf{U}^\pm(z; t)$ by inversion of $\mathbb{I} \pm \mathbf{W}^\pm(z; t) = \mathbb{I} \pm t\mathbf{W}^\pm(z)$. Assuming invertibility of $\mathbb{I} \pm \mathbf{W}^\pm(z; t)$ for $0 \le t \le 1$, we may therefore define corresponding bounded operators $\mathcal{C}_{\mathbf{W}(t)}$ and $\mathcal{C}_{\mathbf{U}(t)}$ acting on $H^\nu(\Sigma)$. Clearly $\mathcal{C}_{\mathbf{W}(0)} = \mathcal{C}_{\mathbf{U}(0)} = 0$. One condition guaranteeing the required invertibility is simply that $\mathbf{W}^\pm(z)$ be nilpotent matrices.

Proposition 17. *Let Σ be a complete admissible contour and $\mathbf{V} : \Sigma^\circ \to \mathbb{C}^{N\times N}$ an admissible jump matrix with exponent $\mu < 1$ and admissible factorization $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$. Suppose that $\mathbf{W}^\pm \in H^\nu_\pm(\Sigma)$ are (pointwise) nilpotent matrices. Then $\mathcal{I} - \mathcal{C}_{\mathbf{W}(t)}$ is continuous on $[0, 1]$ with respect to operator norm on $H^\nu(\Sigma)$, and if $\nu < \mu$, $\mathcal{I} - \mathcal{C}_{\mathbf{U}(t)}$ is a pseudoinverse to $\mathcal{I} - \mathcal{C}_{\mathbf{W}(t)}$ on $H^\nu(\Sigma)$.*

Proof. If $\mathbf{W}^\pm(z)$ are nilpotent for each z, all of the eigenvalues of $\mathbf{W}^\pm(z)$ vanish, so all of the eigenvalues of $\mathbb{I} \pm \mathbf{W}^\pm(z; t) = \mathbb{I} \pm t\mathbf{W}^\pm(z)$ are equal to 1 for all t. It follows that the matrices $\mathbf{U}^\pm(z; t)$ exist and lie in $H^\nu_\pm(\Sigma)$ as functions of z for each t. Hence both $\mathcal{C}_{\mathbf{W}(t)}$ and $\mathcal{C}_{\mathbf{U}(t)}$ are bounded operators on the same space $H^\nu(\Sigma)$ for each t. Since $\mathcal{I} - \mathcal{C}_{\mathbf{W}(t)} = \mathcal{I} - t\mathcal{C}_{\mathbf{W}}$,

$$\|(\mathcal{I} - \mathcal{C}_{\mathbf{W}(t_2)}) - (\mathcal{I} - \mathcal{C}_{\mathbf{W}(t_1)})\|_\nu = |t_2 - t_1| \cdot \|\mathcal{C}_{\mathbf{W}}\|_\nu$$

where the norms are the operator norm on $H^\nu(\Sigma)$, which proves the continuity of $\mathcal{I} - \mathcal{C}_{\mathbf{W}(t)}$ with respect to t. The fact that $\mathcal{I} - \mathcal{C}_{\mathbf{U}(t)}$ is a pseudoinverse to $\mathcal{I} - \mathcal{C}_{\mathbf{W}(t)}$ follows exactly the proof of Proposition 15, as that only relied on the algebraic relation (28) between $\mathbf{W}^\pm$ and $\mathbf{U}^\pm$ which is formally the same for all t. □

Corollary 4 (Zhou's index theorem — special case of nilpotent $\mathbf{W}^\pm$)**.** *Let Σ be a complete admissible contour and $\mathbf{V} : \Sigma^\circ \to \mathbb{C}^{N\times N}$ an admissible jump matrix with exponent $\mu < 1$ and admissible factorization $(\Sigma, \mathbf{B}^+, \mathbf{B}^-)$. If $\mathbf{W}^\pm$ are nilpotent, then for each $\nu < \mu$, $\mathcal{I} - \mathcal{C}_{\mathbf{W}} : H^\nu(\Sigma) \to H^\nu(\Sigma)$ is a Fredholm operator with Fredholm index* $\mathrm{ind}(\mathcal{I} - \mathcal{C}_{\mathbf{W}}) = 0$.

Proof. The identity operator $\mathcal{I} = \mathcal{I} - \mathcal{C}_{\mathbf{W}(0)}$ obviously has $\ker(\mathcal{I}) = \mathrm{coker}(\mathcal{I}) = \{\mathbf{0}\}$, so $\mathrm{ind}(\mathcal{I} - \mathcal{C}_{\mathbf{W}(0)}) = 0$. By Proposition 17 and Theorem 4, we then have $\mathrm{ind}(\mathcal{I} - \mathcal{C}_{\mathbf{W}}) = \mathrm{ind}(\mathcal{I} - \mathcal{C}_{\mathbf{W}(1)}) = 0$. □

Fredholm alternative. Since according to Proposition 16, $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ has Fredholm index zero on $H^\nu(\Sigma)$, the *Fredholm alternative* applies, i.e., $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ is invertible on $H^\nu(\Sigma)$ — and hence Riemann-Hilbert Problem 1 has a solution — provided $\ker(\mathcal{I} - \mathcal{C}_{\mathbf{W}}) = \{\mathbf{0}\}$. This has practical implications, as we will see.

Application to nonlinear Schrödinger equations. We now study the Riemann-Hilbert problems for the focusing and defocusing nonlinear Schrödinger equations under the assumption that the reflection coefficient R is a Schwartz-class function of $\lambda \in \mathbb{R}$. We summarize these here for future reference.

Riemann-Hilbert Problem 2 (Riemann-Hilbert problem for defocusing NLS $\mathrm{i}\psi_t + \frac{1}{2}\psi_{xx} - |\psi|^2\psi = 0$)**.** *Let $R : \mathbb{R} \to \mathbb{C}$ be the reflection coefficient for initial data $\psi(x,0) = \psi_0(x)$, $\psi_0 \in L^1(\mathbb{R})$, and assume further that $R \in \mathscr{S}(\mathbb{R})$ (recall that necessarily $|R(\lambda)|^2 < 1$ for all $\lambda \in \mathbb{R}$). Find a 2×2 matrix $\mathbf{M}^{\mathrm{D}}(\lambda; x, t)$ with the following properties:*

- ***Analyticity:*** *$\mathbf{M}^{\mathrm{D}}(\lambda; x, t)$ is an analytic function of λ for $\lambda \in \mathbb{C} \setminus \mathbb{R}$.*
- ***Jump Condition:*** *The matrix $\mathbf{M}^{\mathrm{D}}(\lambda; x, t)$ takes continuous boundary values $\mathbf{M}^{\mathrm{D}}_{\pm}(\lambda; x, t)$ on the real axis from $\mathbb{C}_{\pm}$, and they are related by the condition*

$$\mathbf{M}^{\mathrm{D}}_{+}(\lambda; x, t) = \mathbf{M}^{\mathrm{D}}_{-}(\lambda; x, t)\mathbf{V}^{\mathrm{D}}(\lambda; x, t), \quad \lambda \in \mathbb{R},$$

where

$$\mathbf{V}^{\mathrm{D}}(\lambda; x, t) := \begin{bmatrix} 1 - |R(\lambda)|^2 & -\mathrm{e}^{-2\mathrm{i}(\lambda x + \lambda^2 t)} R(\lambda)^* \\ \mathrm{e}^{2\mathrm{i}(\lambda x + \lambda^2 t)} R(\lambda) & 1 \end{bmatrix}.$$

- ***Normalization:*** *As $\lambda \to \infty$, $\mathbf{M}^{\mathrm{D}}(\lambda; x, t) \to \mathbb{I}$.*

Riemann-Hilbert Problem 3 (Riemann-Hilbert problem for focusing NLS $\mathrm{i}\psi_t + \frac{1}{2}\psi_{xx} + |\psi|^2\psi = 0$)**.** *Let $R : \mathbb{R} \to \mathbb{C}$ be the reflection coefficient, $\{\lambda_n\}_{n=1}^N$ the eigenvalues in the upper half-plane, and $\{c_n\}_{n=1}^N$ the corresponding residue constants for initial data $\psi(x,0) = \psi_0(x)$, $\psi_0 \in P_0 \subset L^1(\mathbb{R})$. Assume that $R \in \mathscr{S}(\mathbb{R})$. Seek a 2×2 matrix $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ satisfying the following properties:*

- ***Analyticity:*** *$\mathbf{M}^{\mathbf{F}}(\lambda; x, t)$ is an analytic function of λ for $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \{\lambda_1, \dots, \lambda_N, \lambda_1^*, \dots, \lambda_N^*\})$.*
- ***Residues:*** *At $\lambda = \lambda_n$ and $\lambda = \lambda_n^*$, $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ has simple poles and the residues satisfy the conditions*

(33)
$$\operatorname*{Res}_{\lambda=\lambda_n} \mathbf{M}^{\mathrm{F}}(\lambda; x, t) = \lim_{\lambda \to \lambda_n} \mathbf{M}^{\mathrm{F}}(\lambda; x, t) \begin{bmatrix} 0 & 0 \\ c_n(x,t) & 0 \end{bmatrix},$$

and

(34)
$$\operatorname*{Res}_{\lambda=\lambda_n^*} \mathbf{M}^{\mathrm{F}}(\lambda; x, t) = \lim_{\lambda \to \lambda_n^*} \mathbf{M}^{\mathrm{F}}(\lambda; x, t) \begin{bmatrix} 0 & -c_n(x,t)^* \\ 0 & 0 \end{bmatrix},$$

where $c_n(x,t) := c_n \mathrm{e}^{2\mathrm{i}(\lambda_n x + \lambda_n^2 t)}$.

- ***Jump Condition:*** *The matrix $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ takes continuous boundary values $\mathbf{M}^{\mathrm{F}}_{\pm}(\lambda; x, t)$ on the real axis from $\mathbb{C}_{\pm}$, and they are related by the condition*

$$\mathbf{M}^{\mathrm{F}}_{+}(\lambda; x, t) = \mathbf{M}^{\mathrm{F}}_{-}(\lambda; x, t)\mathbf{V}^{\mathrm{F}}(\lambda; x, t),$$

where

(35)
$$\mathbf{V}^{\mathrm{F}}(\lambda; x, t) := \begin{bmatrix} 1 + |R(\lambda)|^2 & \mathrm{e}^{-2\mathrm{i}(\lambda x + \lambda^2 t)} R(\lambda)^* \\ \mathrm{e}^{2\mathrm{i}(\lambda x + \lambda^2 t)} R(\lambda) & 1 \end{bmatrix}.$$

- ***Normalization:*** *As $\lambda \to \infty$, $\mathbf{M}^{\mathrm{F}}(\lambda; x, t) \to \mathbb{I}$.*

The solution of the initial-value problem for the nonlinear Schrödinger equation in each case is given by the same formula:

(36)
$$\psi(x,t) = 2\mathrm{i} \lim_{\lambda \to \infty} \lambda M_{12}^{\mathrm{D,F}}(\lambda; x, t).$$

Swapping poles for jumps across circles. The first issue we have to deal with in applying the general theory is that Riemann-Hilbert Problem 3 deals with an unknown having not only jumps across contours but also poles. However, it is easy to get around this problem with the following simple device. For each pole $\lambda_n \in \mathbb{C}_+$, let D_n be a small disk centered at λ_n of sufficiently small radius to be lie in the open upper half-plane and to be disjoint from all other disks. A particular matrix with a simple pole at $\lambda = \lambda_n$ satisfying the condition (33) is

$$\mathbf{P}_n(\lambda; x, t) := \begin{bmatrix} 1 & 0 \\ c_n(x,t)(\lambda - \lambda_n)^{-1} & 1 \end{bmatrix}. \tag{37}$$

Observe now that the matrix $\mathbf{N}(\lambda; x, t) := \mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda; x, t)^{-1}$ analytic for $\lambda \in D_n \setminus \{\lambda_n\}$ has only a removable singularity at λ_n. Indeed, (33) implies that the Laurent expansion of $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ about $\lambda = \lambda_n$ takes the form

$$\mathbf{M}^{\mathrm{F}}(\lambda; x, t) = \frac{(\mathbf{a}_{-1}, \mathbf{0})}{\lambda - \lambda_n} + (\mathbf{a}_0, \mathbf{b}_0) + O(\lambda - \lambda_n), \quad \text{where} \quad \mathbf{a}_{-1} = c_n(x,t)\mathbf{b}_0,$$

for some vector coefficients $\mathbf{a}_0(x,t)$ and $\mathbf{b}_0(x,t)$. Therefore, if $\mathbf{e}_1$ and $\mathbf{e}_2$ denote the standard unit vectors in $\mathbb{C}^2$,

$$\mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda; x, t)^{-1} = \left(\frac{(c_n(x,t)\mathbf{b}_0, \mathbf{0})}{\lambda - \lambda_n} + (\mathbf{a}_0, \mathbf{b}_0) + O(\lambda - \lambda_n)\right)\left(\frac{(-c_n(x,t)\mathbf{e}_2, \mathbf{0})}{\lambda - \lambda_n} + (\mathbf{e}_1, \mathbf{e}_2)\right)$$

has a limit as $\lambda \to \lambda_n$, because

$$(c_n(x,t)\mathbf{b}_0, \mathbf{0}) \cdot (-c_n(x,t)\mathbf{e}_2, \mathbf{0}) = \mathbf{0} \quad \text{and} \quad (c_n(x,t)\mathbf{b}_0, \mathbf{0}) \cdot (\mathbf{e}_1, \mathbf{e}_2) + (\mathbf{a}_0, \mathbf{b}_0) \cdot (-c_n(x,t)\mathbf{e}_2, \mathbf{0}) = \mathbf{0}.$$

Similarly, a matrix corresponding to (37) with a simple pole at $\lambda = \lambda_n^*$ satisfying the condition (34) is

$$(\mathrm{i}\sigma_2)\mathbf{P}_n(\lambda^*; x, t)^*(\mathrm{i}\sigma_2)^{-1} = \begin{bmatrix} 1 & -c_n(x,t)^*(\lambda - \lambda_n^*)^{-1} \\ 0 & 1 \end{bmatrix} = \mathbf{P}_n(\lambda^*; x, t)^{-\dagger},$$

where the superscript $-\dagger$ denotes taking both the conjugate transpose and the inverse. It follows by completely analogous reasoning that the matrix $\mathbf{N}(\lambda; x, t) := \mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda^*; x, t)^{\dagger}$ has a removable singularity at $\lambda = \lambda_n^*$.

We may therefore define a new matrix unknown $\mathbf{N}(\lambda; x, t)$ in terms of $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ satisfying Riemann-Hilbert Problem 3 by the "piecewise" formula

$$\mathbf{N}(\lambda; x, t) := \begin{cases} \mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda; x, t)^{-1}, & \lambda \in D_n, \quad n = 1, \dots, N, \\ \mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda^*; x, t)^{\dagger}, & \lambda \in D_n^*, \quad n = 1, \dots, N, \\ \mathbf{M}^{\mathrm{F}}(\lambda; x, t), & \lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \overline{D}_1 \cup \dots \cup \overline{D}_N \cup \overline{D}_1^* \cup \dots \cup \overline{D}_N^*). \end{cases}$$

Now, $\mathbf{N}(\lambda; x, t)$ has no poles, but in addition to a jump across the real axis, it has jumps across the circular disk boundaries. Indeed, if we take ∂D_n to have negative (clockwise) orientation, then

$$\mathbf{N}_+(\lambda; x, t) = \mathbf{M}^{\mathrm{F}}(\lambda; x, t) = \mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda; x, t)^{-1}\mathbf{P}_n(\lambda; x, t) = \mathbf{N}_-(\lambda; x, t)\mathbf{P}_n(\lambda; x, t)$$

holds for $\lambda \in \partial D_n$, $n = 1, \dots, N$, and if we take ∂D_n^* to have positive orientation, then

$$\mathbf{N}_+(\lambda; x, t) = \mathbf{M}^{\mathrm{F}}(\lambda; x, t)\mathbf{P}_n(\lambda^*; x, t)^{\dagger} = \mathbf{N}_-(\lambda; x, t)\mathbf{P}_n(\lambda^*; x, t)^{\dagger}$$

holds for $\lambda \in \partial D_n^*$, $n = 1, \dots, N$. Since $\mathbf{N}(\lambda; x, t) = \mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ outside of the disks D_n and their conjugates, we now see that $\mathbf{N}(\lambda; x, t)$ satisfies the conditions of an equivalent Riemann-Hilbert problem closely related to Riemann-Hilbert Problem 3 but with the residue conditions replaced by jump conditions across small circles centered at the points $\{\lambda_n, \lambda_n^*\}_{n=1}^N$. In formulating this problem, we will simply relabel $\mathbf{N}(\lambda; x, t)$ as $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$:

Riemann-Hilbert Problem 4 (Pole-free Riemann-Hilbert Problem for focusing NLS)**.** *Let $R : \mathbb{R} \to \mathbb{C}$ be the reflection coefficient, $\{\lambda_n\}_{n=1}^N$ the eigenvalues in the upper half-plane, and $\{c_n\}_{n=1}^N$ the corresponding residue constants for initial data $\psi(x, 0) = \psi_0(x)$, $\psi_0 \in P_0 \subset L^1(\mathbb{R})$. Assume that $R \in \mathscr{S}(\mathbb{R})$. Seek a 2×2 matrix $\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ satisfying the following properties:*

- ***Analyticity:*** *$\mathbf{M}^{\mathrm{F}}(\lambda; x, t)$ is an analytic function of λ for $\lambda \in \mathbb{C} \setminus (\mathbb{R} \cup \{\partial D_1, \dots, \partial D_N, \partial D_1^*, \dots, \partial D_N^*\})$.*

- ***Jump Conditions:*** *The matrix* $\mathbf{M}^{\rm F}(\lambda;x,t)$ *takes continuous boundary values* $\mathbf{M}^{\rm F}_\pm(\lambda;x,t)$ *on the real axis from* $\mathbb{C}_\pm$*, as well as from the left and right on* $\partial D_1,\dots,\partial D_N$ *oriented negatively and* $\partial D_1^*,\dots,\partial D_N^*$ *oriented positively. The boundary values are related by*
$$\mathbf{M}^{\rm F}_+(\lambda;x,t)=\mathbf{M}^{\rm F}_-(\lambda;x,t)\mathbf{V}^{\rm F}(\lambda;x,t),\quad \lambda\in\mathbb{R},$$
where $\mathbf{V}^{\rm F}(\lambda;x,t)$ *is given by* (35)*, by*
$$\mathbf{M}^{\rm F}_+(\lambda;x,t)=\mathbf{M}^{\rm F}_-(\lambda;x,t)\mathbf{P}_n(\lambda;x,t),\quad \lambda\in\partial D_n,\quad n=1,\dots,N,$$
and
$$\mathbf{M}^{\rm F}_+(\lambda;x,t)=\mathbf{M}^{\rm F}_-(\lambda;x,t)\mathbf{P}_n(\lambda^*;x,t)^\dagger,\quad \lambda\in\partial D_n^*,\quad n=1,\dots,N,$$
where $\mathbf{P}_n(\lambda;x,t)$ *is defined by* (37) *in which* $c_n(x,t):=c_n\mathrm{e}^{2\mathrm{i}(\lambda_n x+\lambda_n^2 t)}$.
- ***Normalization:*** *As* $\lambda\to\infty$, $\mathbf{M}^{\rm F}(\lambda;x,t)\to\mathbb{I}$.

Since no change was made in $\mathbf{M}^{\rm F}(\lambda;x,t)$ for $|\lambda|$ sufficiently large, again the solution of the initial value problem is given by (36).

Compactification of the contours. The next obstruction to applying the general theory is that both Riemann-Hilbert Problems 2 and 4 involve an unbounded jump contour $\mathbb{R}$, oriented left-to-right. To apply the general theory as it has been formulated, it is necessary to first map this contour to a bounded contour by a fractional linear mapping, e.g.,
$$z=z(\lambda):=\frac{\lambda-\mathrm{i}T}{\lambda+\mathrm{i}T}\quad\text{with inverse}\quad \lambda=\lambda(z):=-\mathrm{i}T\frac{z+1}{z-1},$$
which maps the upper (lower) half λ-plane to the interior (exterior) of the unit circle in the z-plane. The point $\lambda=\infty$ is mapped to $z=1$, and $z=\infty$ is the image of $\lambda=-\mathrm{i}T$. Here, $T>0$ is chosen so large that all disks D_n, D_n^* lie within the circle centered at the origin of radius T. This ensures that no point of the jump contour for Riemann-Hilbert Problem 4 is mapped to $z=\infty$.

Let $\Sigma_{\rm D}$ ($\Sigma_{\rm F}$) be the image in the z-plane of the jump contour for Riemann-Hilbert Problem 2 (Riemann-Hilbert Problem 4). See Figure 3 for the defocusing case and Figure 4 for the focusing case. It is easy to see that $\Sigma_{\rm D}$ is a complete admissible contour without any self-intersection points, i.e., $X=\emptyset$. However,

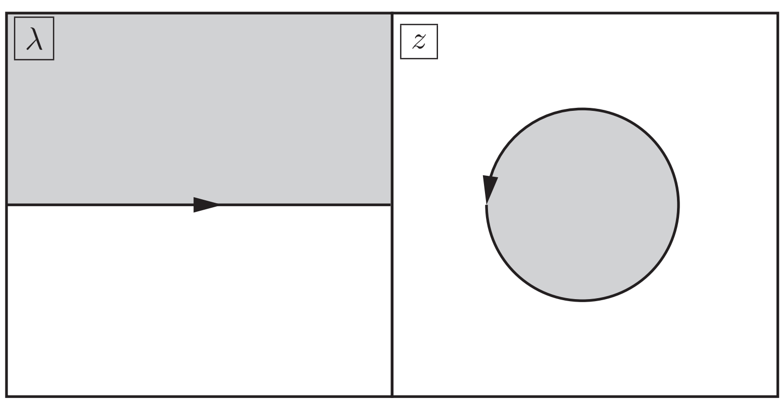

Figure 3. Left: the original contour for Riemann-Hilbert Problem 2 in the λ-plane. Right: the image contour $\Sigma_{\rm D}$ in the z-plane. The domain Ω^+ is shaded.

the components of the domains $\Omega^\pm$ are not all simply-connected in the case of $\Sigma_{\rm F}$, so an augmentation of the jump contour with additional arcs carrying the identity as the jump matrix is needed to arrive at a complete admissible contour in the focusing case. The additional arcs are indicated with dashed lines in Figure 5. With this modification of $\Sigma_{\rm F}$, the corresponding jump matrices on $\Sigma_{\rm D}$ and $\Sigma_{\rm F}$ obtained by composing the jump matrices in the λ-plane with $\lambda=\lambda(z)$ are easily seen to be admissible in both cases with Hölder exponent $\mu=1$. Since neither $\Sigma_{\rm D}$ nor $\Sigma_{\rm F}$ has any exceptional points, it is easy to see that the

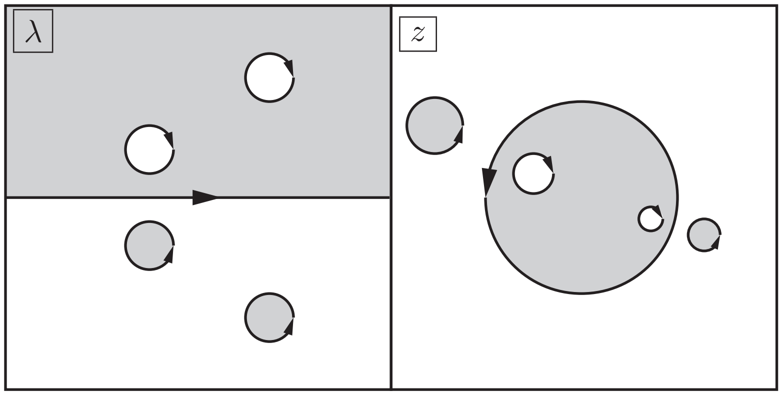

Figure 4. Left: the original contour for Riemann-Hilbert Problem 4 in the λ-plane. Right: the image contour Σ_{F} in the z-plane. The domain Ω^+ is shaded.

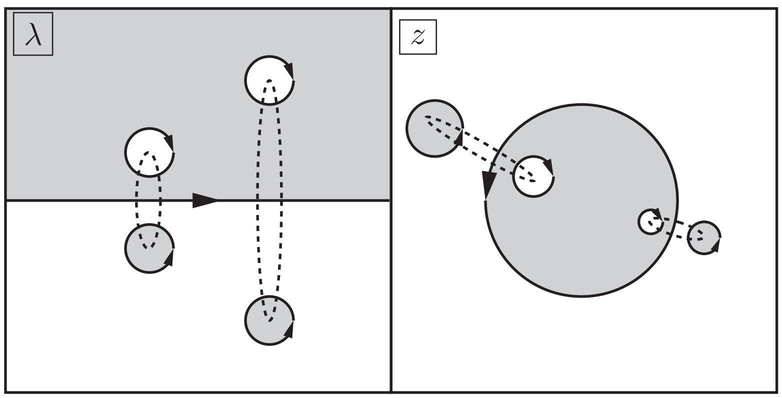

Figure 5. The additional arcs added to complete the contour in the focusing case.

trivial factorization $\mathbf{B}^+(z) = \mathbf{V}(z)$ and $\mathbf{B}^-(z) = \mathbb{I}$ is admissible. It therefore follows that in both focusing and defocusing cases, $\mathcal{I} - \mathcal{C}_{\mathbf{W}}$ is Fredholm with index zero on $H^\nu(\Sigma_{\mathrm{D}})$ or $H^\nu(\Sigma_{\mathrm{F}})$ provided $\nu < 1$.

Thus the analyticity and jump conditions of the Riemann-Hilbert problems for both focusing and defocusing NLS have been recast in the form of Riemann-Hilbert Problem 1. Note however, that every solution $\mathbf{M}(z)$ of that problem is normalized to the identity $\mathbb{I}$ at the point $z = \infty$. Therefore after substituting for $z = z(\lambda)$ to return to the λ-plane we obtain a matrix function that tends to the identity matrix as $\lambda \to -iT$, whereas it is instead required to tend to the identity as $\lambda \to \infty$. Next observe that because $\mathbf{V}^{\mathrm{D}}(\lambda; x, t) - \mathbb{I}$ and $\mathbf{V}^{\mathrm{F}}(\lambda; x, t) - \mathbb{I}$ vanish to all orders as $\lambda \to \infty$ (because $R \in \mathscr{S}(\mathbb{R})$) the corresponding jump matrix $\mathbf{V}(z)$ for Riemann-Hilbert Problem 1 is in each case such that $\mathbf{V}(z) - \mathbb{I}$ vanishes at the corresponding point $z = 1$ to all orders in $|z - 1|$. Therefore by Proposition 14 $\mathbf{M}(z)$ has an asymptotic power series expansion about $z = 1$, and in particular $\mathbf{M}(1)$ makes sense and $\det(\mathbf{M}(1)) = 1$. Consider the matrix $\mathbf{M}(1)^{-1}\mathbf{M}(z)$. It is easy to check that this matrix satisfies all of the conditions of Riemann-Hilbert Problem 1 but with the normalization condition instead replaced by $\mathbf{M}(1)^{-1}\mathbf{M}(z) \to \mathbb{I}$ as $z \to 1$ (in particular multiplication of the jump condition on the left by any constant matrix, or even entire matrix, leaves the condition invariant). Bringing this matrix function back to the λ-plane by substituting $z = z(\lambda)$ then gives a solution of either Riemann-Hilbert Problem 2 or 4.

Application of the Fredholm alternative. By the Fredholm alternative, it therefore remains to show that the only solution $\mathbf{X}_0 \in H^\nu(\Sigma)$, $\Sigma = \Sigma_{\mathrm{D}}$ or $\Sigma = \Sigma_{\mathrm{F}}$, of the equation $(\mathcal{I} - \mathcal{C}_{\mathbf{W}})\mathbf{X}_0 = \mathbf{0}$ is $\mathbf{X}_0(z) \equiv \mathbf{0}$. Let $\mathbf{X}_0 \in H^\nu(\Sigma)$ be any solution of this equation, and define a matrix $\mathbf{M}_0(z)$ by setting

$$\mathbf{M}_0(z) := \mathcal{C}^\Sigma[\mathbf{X}_0(\mathbf{W}^+ + \mathbf{W}^-)](z), \quad z \in \mathbb{C} \setminus \Sigma \tag{38}$$

(compare with (25)). First we observe the following.

Proposition 18. *Suppose that $\mathbf{X}_0 \in H^\nu(\Sigma)$ solves $(\mathcal{I} - \mathcal{C}_{\mathbf{W}})\mathbf{X}_0 = \mathbf{0} \in H^\nu(\Sigma)$, i.e., $\mathbf{X}_0 \in \ker(\mathcal{I} - \mathcal{C}_{\mathbf{W}})$. Then $\mathbf{M}_0 : \mathbb{C} \setminus \Sigma \to \mathbb{C}^{N\times N}$ is the zero function if and only if $\mathbf{X}_0$ is the zero element of $H^\nu(\Sigma)$.*

Proof. Clearly if $\mathbf{X}_0(z) \equiv \mathbf{0}$ then also $\mathbf{M}_0(z)$ is the zero function. Suppose now that $\mathbf{M}_0(z)$ is the zero function. Then in particular, its boundary value taken on Σ from Ω^- vanishes, i.e.,

$$\mathcal{C}^\Sigma_-[\mathbf{X}_0(\mathbf{W}^+ + \mathbf{W}^-)](z) = 0, \quad z \in \Sigma^\circ. \tag{39}$$

Since $\mathbf{X}_0(z) = \mathcal{C}_{\mathbf{W}}[\mathbf{X}_0](z) = \mathcal{C}^\Sigma_+[\mathbf{X}_0\mathbf{W}^-](z) + \mathcal{C}^\Sigma_-[\mathbf{X}_0\mathbf{W}^+](z)$, by the Plemelj formula, we have

$$\begin{aligned}\mathbf{X}_0(z) &= \mathbf{X}_0(z)\mathbf{W}^-(z) + \mathcal{C}^\Sigma_-[\mathbf{X}_0\mathbf{W}^-](z) + \mathcal{C}^\Sigma_-[\mathbf{X}_0\mathbf{W}^+](z) \\ &= \mathbf{X}_0(z)\mathbf{W}^-(z) + \mathcal{C}^\Sigma_-[\mathbf{X}_0(\mathbf{W}^- + \mathbf{W}^+)](z),\end{aligned}$$

or, recalling $\mathbf{B}^-(z) = \mathbb{I} - \mathbf{W}^-(z)$,

$$\begin{aligned}\mathbf{X}_0(z)\mathbf{B}^-(z) &= \mathcal{C}^\Sigma_-[\mathbf{X}_0(\mathbf{W}^+ + \mathbf{W}^-)](z) \\ &= \mathbf{0}, \quad z \in \Sigma^\circ,\end{aligned}$$

according to (39). But $\mathbf{B}^-(z)$ is an invertible matrix for each $z \in \Sigma^\circ$, so $\mathbf{X}_0(z) = \mathbf{0}$. □

Observe that the function $\mathbf{M}_0(z)$ given by (38) is analytic where it is defined and Hölder continuous with exponent $\nu < 1$ up to the boundary Σ. Also, by analogous steps as in the proof of Theorem 2, its boundary values satisfy $\mathbf{M}_{0+}(z) = \mathbf{M}_{0-}(z)\mathbf{V}(z)$ at every point of Σ, where $\mathbf{V}$ is the corresponding jump matrix. Therefore, $\mathbf{M}_0(z)$ satisfies all of the conditions of Riemann-Hilbert Problem 1 with the exception of the normalization condition, which gets replaced by $\mathbf{M}_0(z) \to \mathbf{0}$ as $z \to \infty$ according to (38). Using Proposition 18, we deduce that in the Fredholm index zero situation, Riemann-Hilbert Problem 1 has a (unique) solution provided that the only *vanishing solution* of the same problem, i.e., replacing the normalization to $\mathbb{I}$ at $z = \infty$ with normalization to $\mathbf{0}$, is $\mathbf{M}_0(z) \equiv \mathbf{0}$.

Furthermore, since as an analytic function of z for $|z|$ sufficiently large, any nonzero vanishing solution $\mathbf{M}_0(z)$ decays as $z \to \infty$ like z^{-p} for some positive integer p, the product

$$\tilde{\mathbf{M}}_0(z) := (z-1)^p\mathbf{M}_0(z)$$

will have a nonzero limit as $z \to \infty$ but will vanish to (at least) order p as $z \to 1$, which corresponds to $\lambda \to \infty$. Since $\tilde{\mathbf{M}}_0(z)$ is identically zero if and only if $\mathbf{M}_0(z)$ is, for unique solvability of Riemann-Hilbert Problems 2 and 4 it is sufficient to rule out nonzero solutions of these two problems in which the conditions $\mathbf{M}^{\mathrm{D,F}}(\lambda; x, t) \to \mathbb{I}$ as $\lambda \to \infty$ are replaced with $\mathbf{M}^{\mathrm{D,F}}(\lambda; x, t) = O(\lambda^{-1})$ as $\lambda \to \infty$ respectively.

Zhou's vanishing lemma. The non-existence of nontrivial vanishing solutions for Riemann-Hilbert Problems 2 and 4 follows from the following result, due to Zhou [6, Theorem 9.3].

Proposition 19 (Vanishing lemma)**.** *Let Σ be a complete contour in the λ-plane that is Schwarz-symmetric (invariant under reflection through the real axis, including orientation). Let $\mathbf{V}$ be an admissible jump matrix on Σ that satisfies:*

$$\mathbf{V}(\lambda) + \mathbf{V}(\lambda)^\dagger \textit{ is positive definite for } \lambda \in \mathbb{R}, \tag{40}$$

and

$$\mathbf{V}(\lambda^*) = \mathbf{V}(\lambda)^\dagger, \quad \lambda \in \Sigma \setminus \mathbb{R}. \tag{41}$$

Then the only matrix function $\mathbf{M}_0(\lambda)$ analytic for $\lambda \in \mathbb{C} \setminus \Sigma$ and continuous up to the boundary with $\mathbf{M}_{0+}(\lambda) = \mathbf{M}_{0-}(\lambda)\mathbf{V}(\lambda)$ for $\lambda \in \Sigma^\circ$, and that satisfies $\mathbf{M}_0(\lambda) = O(\lambda^{-(1+\epsilon)/2})$ as $\lambda \to \infty$ for any $\epsilon > 0$, is $\mathbf{M}_0(\lambda) \equiv \mathbf{0}$.

Note that every complete Schwarz-symmetric contour necessarily contains the real axis $\mathbb{R}$, but as usual the jump matrix may be artificially taken to be the identity on $\mathbb{R}$ if necessary.

Proof. Consider the matrix function $\mathbf{A}(\lambda) := \mathbf{M}_0(\lambda)\mathbf{M}_0(\lambda^*)^\dagger$. Clearly, $\mathbf{A}(\lambda)$ is analytic for $\lambda \in \mathbb{C}\setminus\Sigma$ because $\Sigma = \Sigma^*$ (Schwarz symmetry), and $\mathbf{A}(\lambda)$ is also continuous up to Σ. Let Σ_j be an arc of Σ in the open upper half-plane. We may calculate the jump of $\mathbf{A}(\lambda)$ across Σ_j as follows: first observe that

$$\mathbf{A}_+(\lambda) = \mathbf{M}_{0+}(\lambda)\mathbf{M}_{0-}(\lambda^*)^\dagger,$$

because if $\lambda \to \Sigma_j$ from the left ("+" side), then $\lambda^* \to \Sigma_j^*$ from the right ("−" side), where the "±" subscripts indicate boundary values on Σ_j (for λ) and Σ_j^* (for λ^*), for which the orientation is induced by Schwarz reflection symmetry. Applying the jump conditions across Σ_j and Σ_j^* respectively thus gives

$$\begin{aligned}\mathbf{A}_+(\lambda) &= \mathbf{M}_{0-}(\lambda)\mathbf{V}(\lambda)[\mathbf{M}_{0+}(\lambda^*)\mathbf{V}(\lambda^*)^{-1}]^\dagger \\ &= \mathbf{M}_{0-}(\lambda)\mathbf{V}(\lambda)\mathbf{V}(\lambda^*)^{-\dagger}\mathbf{M}_{0+}(\lambda^*)^\dagger.\end{aligned}$$

Next, using the identity (41) gives

$$\begin{aligned}\mathbf{A}_+(\lambda) &= \mathbf{M}_{0-}(\lambda)\mathbf{M}_{0+}(\lambda^*)^\dagger \\ &= \mathbf{A}_-(\lambda),\end{aligned}$$

by similar reasoning as in the first step. Therefore, $\mathbf{A}(\lambda)$ is continuous in the upper half-plane (as well as in the lower half-plane by the identity $\mathbf{A}(\lambda) = \mathbf{A}(\lambda^*)^\dagger$). It then follows by Morera's Theorem and the Generalized Cauchy Integral Theorem that $\mathbf{A}(z)$ is analytic for $\lambda \in \mathbb{C}\setminus\mathbb{R}$, and continuous to the real axis from either half-plane. Also, $\mathbf{A}(\lambda) = O(\lambda^{-(1+\epsilon)})$ as $\lambda \to \infty$, so $\mathbf{A}(\lambda)$ is integrable at $\lambda = \infty$. It therefore follows again from the Generalized Cauchy Integral Theorem and Jordan's Lemma (closing the contour by a large semicircle in $\mathbb{C}_\pm$) that

$$\int_{-\infty}^{+\infty} \mathbf{A}_\pm(\lambda)\,\mathrm{d}\lambda = \mathbf{0}.$$

But, for $\lambda \in \mathbb{R}$, we have

$$\mathbf{A}_+(\lambda) = \mathbf{M}_{0+}(\lambda)\mathbf{M}_{0-}(\lambda)^\dagger = \mathbf{M}_{0-}(\lambda)\mathbf{V}(\lambda)\mathbf{M}_{0-}(\lambda)^\dagger$$

and

$$\mathbf{A}_-(\lambda) = \mathbf{M}_{0-}(\lambda)\mathbf{M}_{0+}(\lambda)^\dagger = \mathbf{M}_{0-}(\lambda)[\mathbf{M}_{0+}(\lambda)\mathbf{V}(\lambda)]^\dagger = \mathbf{M}_{0-}(\lambda)\mathbf{V}(\lambda)^\dagger\mathbf{M}_{0-}(\lambda)^\dagger.$$

Therefore

$$\int_{-\infty}^{+\infty} \mathbf{M}_{0-}(\lambda)[\mathbf{V}(\lambda)+\mathbf{V}(\lambda)^\dagger]\mathbf{M}_{0-}(\lambda)^\dagger\,\mathrm{d}\lambda = \int_{-\infty}^{+\infty}\mathbf{A}_+(\lambda)\,\mathrm{d}\lambda + \int_{-\infty}^{+\infty}\mathbf{A}_-(\lambda)\,\mathrm{d}\lambda = \mathbf{0}+\mathbf{0} = \mathbf{0}. \tag{42}$$

Now let $\mathbf{u}_n(\lambda)^\dagger$, $n = 1,\ldots,N$, denote the rows of $\mathbf{M}_{0-}(\lambda)$, and denote the quadratic form of $\mathbf{V}(\lambda)+\mathbf{V}(\lambda)^\dagger$ by

$$Q(\mathbf{u};\lambda) := \mathbf{u}^\dagger[\mathbf{V}(\lambda)+\mathbf{V}(\lambda)^\dagger]\mathbf{u}, \quad \lambda \in \mathbb{R}.$$

By the hypothesis (40), $Q(\mathbf{u};\lambda) \geq 0$ for all $\lambda \in \mathbb{R}$ and $Q(\mathbf{u};\lambda) = 0$ if and only if $\mathbf{u} = \mathbf{0}$. Taking the trace of (42) gives

$$\sum_{n=1}^N \int_{-\infty}^{+\infty} Q(\mathbf{u}_n(\lambda);\lambda)\,\mathrm{d}\lambda = 0,$$

but as a sum and integral of nonnegative terms, this implies that $Q(\mathbf{u}_n(\lambda);\lambda) = 0$ for all $\lambda \in \mathbb{R}$ and all $n = 1,\ldots,N$. Therefore (40) implies that $\mathbf{u}_n(\lambda) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$ and all $n = 1,\ldots,N$, i.e., $\mathbf{M}_{0-}(\lambda) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$. By $\mathbf{M}_{0+}(\lambda) = \mathbf{M}_{0-}(\lambda)\mathbf{V}(\lambda)$ we therefore also get $\mathbf{M}_{0+}(\lambda) = \mathbf{0}$ for all $\lambda \in \mathbb{R}$.

Since $\mathbf{M}_{0+}(\lambda) = \mathbf{M}_{0-}(\lambda)$ for all $\lambda \in \mathbb{R} \subset \Sigma$, the Morera/Generalized Cauchy argument applies to show that $\mathbf{M}_0(\lambda)$ is analytic in a neighborhood of every point of Σ° on the real axis. Since also $\mathbf{M}_0(\lambda) = \mathbf{0}$ for $\lambda \in \mathbb{R}$, it follows by analytic continuation that $\mathbf{M}_0(\lambda) = \mathbf{0}$ holds as an identity all the way up to the first complex arcs of Σ. But then applying the jump condition for $\mathbf{M}_0(\lambda)$ on these arcs shows that again both boundary values agree and vanish, so the argument continues to the next arcs of Σ, and so on until the complex plane is exhausted. Thus $\mathbf{M}_0(\lambda) \equiv \mathbf{0}$ on the whole complex plane as desired. □

Global solvability: defocusing case. To confirm the hypotheses of the vanishing lemma in this case, observe that it is sufficient to show that $\mathbf{V}^{\mathrm{D}}(\lambda;x,t)+\mathbf{V}^{\mathrm{D}}(\lambda;x,t)^{\dagger}$ is positive definite for all $\lambda\in\mathbb{R}$. But

$$\mathbf{V}^{\mathrm{D}}(\lambda;x,t)+\mathbf{V}^{\mathrm{D}}(\lambda;x,t)^{\dagger}=\begin{bmatrix}2(1-|R(\lambda)|^2) & 0\\ 0 & 2\end{bmatrix}$$

and this is clearly positive definite because $|R(\lambda)|^2<1$ holds for $\lambda\in\mathbb{R}$. This finally proves:

Theorem 5 (Global solvability of the inverse-scattering problem for defocusing NLS)**.** *Riemann-Hilbert Problem 2 has a unique solution for all $(x,t)\in\mathbb{R}^2$, hence determining the corresponding solution of the initial-value problem for the defocusing NLS equation via the formula* (36)*.*

Global solvability: focusing case. To confirm the hypotheses of the vanishing lemma in this case, first note that the contour consisting of the real axis oriented left-to-right together with the positively-oriented circles ∂D_n, $n=1,\ldots,N$ and the negatively-oriented circles ∂D_n^*, $n=1,\ldots,N$, has the necessary Schwarz symmetry. Also, clearly $\mathbf{V}(\lambda^*)=\mathbf{V}(\lambda)^{\dagger}$ holds for all complex λ in the jump contour, i.e., on all of the circles. Therefore it remains again to analyze the jump matrix on the real axis. In this case, we have

$$\mathbf{V}^{\mathrm{F}}(\lambda;x,t)+\mathbf{V}^{\mathrm{F}}(\lambda;x,t)^{\dagger}=2\begin{bmatrix}1+|R(\lambda)|^2 & \mathrm{e}^{-2\mathrm{i}(\lambda x+\lambda^2 t)}R(\lambda)^*\\ \mathrm{e}^{2\mathrm{i}(\lambda x+\lambda^2 t)}R(\lambda) & 1\end{bmatrix}=\mathbf{G}(\lambda;x,t)^{\dagger}\mathbf{G}(\lambda;x,t),$$

where

$$\mathbf{G}(\lambda;x,t):=\sqrt{2}\begin{bmatrix}1 & 0\\ \mathrm{e}^{2\mathrm{i}(\lambda x+\lambda^2 t)}R(\lambda) & 1\end{bmatrix}$$

is clearly an invertible matrix ($\det(\mathbf{G}(\lambda;x,t))=2$). However, every matrix of the form $\mathbf{G}^{\dagger}\mathbf{G}$ with $\mathbf{G}$ invertible is positive definite, so all hypotheses of the vanishing lemma have been confirmed. Noting that Riemann-Hilbert Problems 3 and 4 are completely equivalent, we have finally proved the following.

Theorem 6 (Global solvability of the inverse-scattering problem for focusing NLS)**.** *Riemann-Hilbert Problem 3 has a unique solution for all $(x,t)\in\mathbb{R}^2$, hence determining the corresponding solution (for suitable generic initial data) of the initial-value problem for the focusing NLS equation via the formula* (36)*.*

Observe that the result holds true even in the special case that $R(\lambda)$ vanishes identically, which gives an indirect proof that the determinant of the linear algebra system for the N-soliton solution of the focusing NLS equation is nonzero for all $(x,t)\in\mathbb{R}^2$.

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