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Review

$\mathbb{Q}$-lattices: Quantum statistical mechanics and Galois theory

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Abstract

We review our recent results on the noncommutative geometry of $\mathbb{Q}$-lattices modulo commensurability. We discuss the cases of 1-dimensional and 2-dimensional $\mathbb{Q}$-lattices. In the first case, we show that, by considering commensurability classes of 1-dimensional $\mathbb{Q}$-lattices up to scaling, one recovers the Bost–Connes quantum statistical mechanical system, whose zero temperature KMS states intertwine the symmetries of the system with the Galois action of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. In the 2-dimensional case, commensurability classes of $\mathbb{Q}$-lattices up to scaling give rise to another quantum statistical mechanical system, whose symmetries are the automorphisms of the modular field, and whose (generic) zero temperature KMS states intertwine the action of these symmetries with the Galois action on an embedding in $\mathbb{C}$ of the modular field. Following our joint work with Ramachandran, we then show how the noncommutative spaces associated to commensurability classes of $\mathbb{Q}$-lattices up to scale have a natural geometric interpretation as noncommutative versions of the Shimura varieties $Sh(\mathrm{GL}_1, \{\pm 1\})$ in the Bost-Connes case and $Sh(\mathrm{GL}_2, \mathbb{H}^{\pm})$ in the case of the GL_2 system. We also show how this leads naturally to the construction of a system generalizing the Bost–Connes system that fully recovers the explicit class field theory of imaginary quadratic fields.

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Contents

1. Introduction

This paper summarizes the main aspects of our joint work [4] on quantum statistical mechanics of $\mathbb{Q}$-lattices, with a view towards its relations to class field theory investigated in our joint work with Ramachandran [5]. The noncommutative geometry of the space of $\mathbb{Q}$-lattices modulo the equivalence relation of commensurability provides a setting that unifies several phenomena involving the interaction of noncommutative geometry and number theory. These include, in the one-dimensional case, the Bost–Connes (BC) system [1] with arithmetic spontaneous symmetry breaking and its dual space under the duality given by taking the crossed product with the time evolution. The latter is the noncommutative space underlying the construction of the spectral realization of the zeros of the Riemann zeta function in [3]. The corresponding space in the two-dimensional case contains in its algebra of coordinates the modular Hecke algebras of [6] [7]. The noncommutative compactifications of modular curves of [14] also appear here as a stratum in the compactification of the space of commensurability classes of two-dimensional $\mathbb{Q}$-lattices. Moreover, an interesting and difficult problem is the generalization of the results of [1] to other number fields (For an overview of existing results in this direction we refer the reader to the “further developments” section of [4] and the references quoted therein). The space of commensurability classes of two-dimensional $\mathbb{Q}$-lattices up to scaling, which is the main object of this paper, provides a new approach to the problem, for the case of quadratic fields. In fact, while the BC system is closely related to the Kronecker–Weber construction of the maximal abelian extension of $\mathbb{Q}$, we shall see that the two-dimensional system introduced in [4] is naturally related to the Galois theory of the modular field, which in turn lies at the heart of the explicit class field theory problem for imaginary quadratic fields. A generalization of the results of [1] to imaginary quadratic fields was obtained in [5]. Moreover, the fact that the noncommutative modular curves of [14] appear in the compactification suggests the possible existence of a path towards the case of real quadratic fields, along the lines of Manin's real multiplication program [13]. The fundamental notions in all that follows are those of $\mathbb{Q}$-lattices and commensurability.

Definition 1.1. A $\mathbb{Q}$-lattice in $\mathbb{R}^n$ consists of a pair (Λ, ϕ) of a lattice $\Lambda \subset \mathbb{R}^n$ (a cocompact free abelian subgroup of $\mathbb{R}^n$ of rank *n*) together with a system of labels of its torsion points given by a homomorphism of abelian groups

$$\phi : \mathbb{Q}^n/\mathbb{Z}^n \longrightarrow \mathbb{Q}\Lambda/\Lambda.$$

Two $\mathbb{Q}$-lattices are commensurable,

$$(\Lambda_1, \phi_1) \sim (\Lambda_2, \phi_2),$$

iff $\mathbb{Q}\Lambda_1 = \mathbb{Q}\Lambda_2$ and

$$\phi_1 = \phi_2 \mod \Lambda_1 + \Lambda_2$$

Commensurability defines an equivalence relation among $\mathbb{Q}$-lattices. By definition a $\mathbb{Q}$-lattice is *invertible* when ϕ is an isomorphism (Fig. 1). Two invertible $\mathbb{Q}$-lattices are commensurable if and only if they are equal. While most $\mathbb{Q}$-lattices are not commensurable to an invertible one, the set of invertible $\mathbb{Q}$-lattices gives a cross-section of the equivalence relation on the subset of $\mathbb{Q}$-lattices that have this property. The equivalence relation of commensurability on the space of $\mathbb{Q}$-lattices is subtle enough an operation that the resulting

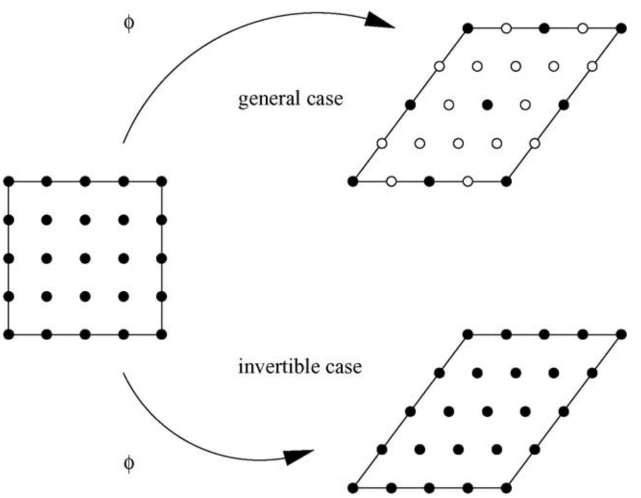

Fig. 1. Generic and invertible two-dimensional $\mathbb{Q}$-lattices.

quotient can only be described efficiently through noncommutative geometry (it is crucial for this that one does not restrict to the invertible ones). In particular, when viewed as a set in the classical sense, the space $\mathcal{L}_n$ of commensurabilty classes of $\mathbb{Q}$-lattices in $\mathbb{R}^n$ has a typical property of noncommutative spaces: it has the cardinality of the continuum but one cannot construct a countable collection of measurable functions that separate points of $\mathcal{L}_n$. If, instead of taking the quotient as a set, one encodes the equivalence relation in a "dynamical" manner, i.e. one builds a convolution algebra from the various identifications, one obtains very interesting algebras, playing the role of coordinate algebras on the spaces $\mathcal{L}_n$. In particular, the topology of the space $\mathcal{L}_n$ is encoded by a C^*-algebra $C^*(\mathcal{L}_n)$. These C^*-algebras and the dynamical systems obtained from the natural time evolution on the C^*-algebras $C^*(\mathcal{L}_1/\mathbb{R}_+^*)$ and $C^*(\mathcal{L}_2/\mathbb{C}^*)$ of $\mathbb{Q}$-lattices up to scaling, are the central objects of this paper.

2. Quantum statistical mechanics

In quantum statistical mechanics, the *algebra of observables* is a C^*-algebra $\mathcal{A}$. Expectation values are assigned to observables through *states*. A state is a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ satisfying normalization and positivity,

$$\varphi(1) = 1, \quad \varphi(a^*a) \geq 0.$$

One can think of a state as a probability measure on the NC space X related to $\mathcal{A}$ by $\mathcal{A}$ = "$C(X)$". The time evolution of a quantum statistical mechanical system is given as a one-parameter family of automorphisms $\sigma_t \in \mathrm{Aut}(\mathcal{A})$ of the C^*-algebra of observables. Given a representation of the C^*-algebra $\mathcal{A}$ as a concrete algebra of operators on a Hilbert space $\mathcal{H}$, one can consider the *Hamiltonian* implementing the time evolution in the representation. This is the operator H satisfying

$$\sigma_t(a) = \mathrm{e}^{\mathrm{i}tH}\, a\, \mathrm{e}^{-\mathrm{i}tH} \text{ for all } a \in \mathcal{A}.$$

One then looks for equilibrium states, depending on a thermodynamical parameter, the inverse temperature $\beta = 1/kT$ (where for simplicity we can put the Boltzmann constant k equal to 1). The analog of integrating against the Gibbs measure on the phase space for a classical Hamiltonian system is given in this quantum mechanical setting by states of the form

$$\varphi(a) = \frac{1}{Z(\beta)} \mathrm{Tr}(a\, \mathrm{e}^{-\beta H}) \tag{2.1}$$

with the partition function given by

$$Z(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta H}). \tag{2.2}$$

The expression (2.1), however, makes sense only under the assumption that the operator $\exp(-\beta H)$ is of trace class. Often, this is the case only in a certain range (low temperature). Thus, one needs a better notion of "equilibrium states", which makes sense more generally and is satisfied in particular by states of the form (2.1). The correct notion is provided by the Kubo–Martin–Schwinger condition (KMS) (cf. [2,8,9]). Given a C^*-dynamical system $(\mathcal{A}, \sigma_t)$ – that is, a C^*-algebra with a one-parameter group of automorphisms—a state φ on $\mathcal{A}$ satisfies the KMS condition at inverse temperature $0 < \beta < \infty$ iff for all $a, b \in \mathcal{A}$ there exists a function $F_{a,b}(z)$ holomorphic on the strip $0 < \Im(z) < \beta$ continuous on the closed strip and bounded, such that for all $t \in \mathbb{R}$

$$F_{a,b}(t) = \varphi(a\sigma_t(b)) \quad \text{and} \quad F_{a,b}(t + \mathrm{i}\beta) = \varphi(\sigma_t(b)a). \tag{2.3}$$

The analogous notion at zero temperature ($\beta = \infty$) is more subtle. In fact, one may use the the same notion of KMS states, that is, the existence for each $a, b \in \mathcal{A}$ of a bounded holomorphic function $F_{a,b}(z)$ on the upper half plane such that $F_{a,b}(t) = \varphi(a\sigma_t(b))$. This definition of KMS_∞ states is often used in the literature. However, it is well known that this condition is considerably weaker than (2.3). For instance, the set KMS_β of KMS states at $\beta < \infty$ is a Choquet simplex (for which we call $\mathcal{E}_\beta$ the set of extremal points). In general, this simplicial structure is lost at $\beta = \infty$, if one adopts this notion of KMS states. In the simple case of the trivial time evolution, for instance, all states satisfy such weaker definition of KMS_∞ while only tracial states satisfy (2.3) at $\beta < \infty$. Thus, a better notion of KMS_∞ condition is obtained by considering states that are weak limits of KMS_β states as $\beta \to \infty$,

$$\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a), \quad \forall a \in \mathcal{A}. \tag{2.4}$$

This restores the property that the set KMS_∞ is a simplex and one can regard the set $\mathcal{E}_\infty$ of its extreme points as an analog of the set of *classical points* on the noncommutative space $\mathcal{A}$. In particular, in the cases of arithmetic interest, one can think of the set $\mathcal{E}_\infty$ as the “classical points” of a noncommutative arithmetic variety. For instance, for the GL(2)-system with C^*-algebra $\mathcal{A} = C^*(\mathcal{L}_2/\mathbb{C}^*)$, the set $\mathcal{E}_\infty$ is the classical Shimura variety

$$\mathcal{E}_\infty \cong \mathrm{GL}_2(\mathbb{Q})\backslash\mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*,$$

while the noncommutative space $\mathcal{L}_2/\mathbb{C}^*$ is a noncommutative arithmetic variety containing $\mathcal{E}_\infty$ as its set of “classical points”. As we shall see below, the arithmetic structure will be specified by an arithmetic subalgebra $\mathcal{A}_\mathbb{Q}$ of $\mathcal{A}$. This will play a key role in the relation between the symmetries of the system and the action of the Galois group on states $\varphi \in \mathcal{E}_\infty$ evaluated on $\mathcal{A}_\mathbb{Q}$ (Fig. 2.).

2.1. Symmetries

An important role in quantum statistical mechanics is played by symmetries. Typically, symmetries of the algebra $\mathcal{A}$ compatible with the time evolution induce symmetries of the equilibrium states $\mathcal{E}_\beta$ at different temperatures. Especially important are the phenomena of *symmetry breaking*. In such cases, there is a global underlying group G of symmetries of the algebra $\mathcal{A}$ but in certain ranges of temperature the choice of an equilibrium state φ breaks the symmetry to a smaller subgroup $G_\varphi = \{g \in G :\ g^*\varphi = \varphi\}$, where g^* denotes the induced action on states. Various systems can exhibit one or more phase transitions, or none at all. A typical situation in physical systems sees a unique KMS state

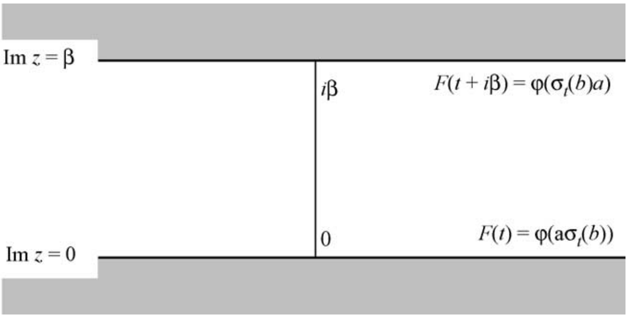

Fig. 2. The KMS condition.

for all values of the parameter above a certain critical temperature ($\beta < \beta_c$). This corresponds to a chaotic phase such as randomly distributed spins in a ferromagnet. When the system cools down and reaches the critical temperature, the unique equilibrium state branches off into a larger set KMS_β and the symmetry is broken by the choice of an extremal state in $\mathcal{E}_\beta$. We will see in detail one such case and a case with multiple phase transitions. A very important point is that we need to consider both symmetries by automorphisms and by endomorphisms. *Automorphisms*: a subgroup $G \subset \mathrm{Aut}(\mathcal{A})$ is compatible with σ_t if for all $g \in G$ and for all $t \in \mathbb{R}$, we have $g\sigma_t = \sigma_t g$. There is then an induced action of G on KMS states and in particular on the set $\mathcal{E}_\beta$. If u is a unitary, acting on $\mathcal{A}$ by

$$\mathrm{Ad}u : a \mapsto uau^*$$

and satisfying $\sigma_t(u) = u$, then we say that $\mathrm{Ad}u$ is an inner automorphism of $(\mathcal{A}, \sigma_t)$. Inner automorphisms act trivially on KMS states. *Endomorphisms*: let $\rho\sigma_t = \sigma_t\rho$ be a $*$-homomorphism. Consider the idempotent $e = \rho(1)$. If $\varphi \in \mathcal{E}_\beta$ is a state such that $\varphi(e) \neq 0$, then there is a well defined pullback $\rho^*\varphi$,

$$\rho^*(\varphi) = \frac{1}{\varphi(e)}\,\varphi \circ \rho. \tag{2.5}$$

Let u be an isometry compatible with the time evolution by

$$\sigma_t(u) = \lambda^{it} u, \quad \lambda > 0. \tag{2.6}$$

One has $u^*u = 1$ and $uu^* = e$. We say that $\mathrm{Ad}u$ defined by $a \mapsto uau^*$ is an inner endomorphism of $(\mathcal{A}, \sigma_t)$. The condition (2.6) ensures that $(\mathrm{Ad}u)^*\varphi$ is well defined according to (2.5) and the KMS condition shows that the induced action of an inner endomorphism on KMS states is trivial. One needs to be especially careful in defining the action of endomorphisms by (2.5). In fact, there are cases where for KMS_∞ states one finds only $\varphi(e) = 0$, yet it is still possible to define an interesting action of endomorphisms by a procedure of “warming up and cooling down”. For this to work one needs sufficiently favorable conditions, namely that the “warming up” map

$$W_\beta(\varphi)(a) = \frac{\mathrm{Tr}(\pi_\varphi(a)\,\mathrm{e}^{-\beta H})}{\mathrm{Tr}(\,\mathrm{e}^{-\beta H})} \tag{2.7}$$

gives a homeomorphism $W_\beta : \mathcal{E}_\infty \to \mathcal{E}_\beta$ for all β sufficiently large. One can then define the action by

$$(\rho^*\varphi)(a) = \lim_{\beta\to\infty} \left(\rho^* W_\beta(\varphi)\right)(a), \tag{2.8}$$

for all $\varphi \in \mathcal{E}_\infty$ and all $a \in \mathcal{A}$.

3. The Bost–Connes system

In [1], Bost and Connes constructed a C^*-dynamical system $(\mathcal{A}, \sigma_t)$ with spontaneous symmetry breaking, which encodes the arithmetic of the cyclotomic field $\mathbb{Q}^{\mathrm{cycl}}$, that is, of the maximal abelian extension of $\mathbb{Q}$ by the Kronecker–Weber theorem. The algebra $\mathcal{A}$ of the Bost–Connes system is generated by two types of operators. The first type consists of *phase operators* $e(r)$, parameterized by elements $r \in \mathbb{Q}/\mathbb{Z}$. These can be represented on the Fock space generated by occupation numbers $|n\rangle$ as the operators

$$e(r)|n\rangle = \alpha(\zeta_r^n)|n\rangle. \tag{3.1}$$

Here, we denote by $\zeta_{a/b} = \zeta_b^a$ the *abstract* roots of unity generating $\mathbb{Q}^{\mathrm{cycl}}$ and by $\alpha : \mathbb{Q}^{\mathrm{cycl}} \hookrightarrow \mathbb{C}$ an embedding that identifies $\mathbb{Q}^{\mathrm{cycl}}$ with the subfield of $\mathbb{C}$ generated by the *concrete* roots of unity. The operators (3.1) are familiar in the theory of quantum optics, where they are used to define the quantized optical phase as a state

$$|\theta_{m,N}\rangle = e\left(\frac{m}{N+1}\right) \cdot v_N,$$

where v_N is a superposition of occupation states

$$v_N = \frac{1}{(N+1)^{1/2}} \sum_{n=0}^{N} |n\rangle.$$

In such quantization of the phase, N is chosen as a scale at which the phase is discretized. One needs then to ensure that the results are consistent over changes of scale. The other operators that generate the Bost–Connes algebra can be thought of as implementing the changes of scales in the optical phases in a consistent way. These operators are isometries μ_n parameterized by positive integers $n \in \mathbb{N}^\times = \mathbb{Z}_{>0}$. The changes of scale are described by the action of the μ_n on the $e(r)$ by

$$\mu_n e(r) \mu_n^* = \frac{1}{n} \sum_{ns=r} e(s). \tag{3.2}$$

In addition to this compatibility condition, the operators $e(r)$ and μ_n satisfy other simple relations. These give a presentation of the algebra $\mathcal{A}$ of the form ([1,10]):

- $\mu_n^* \mu_n = 1$, for all $n \in \mathbb{N}^\times$,
- $\mu_k \mu_n = \mu_{kn}$, for all $k, n \in \mathbb{N}^\times$,
- $e(0) = 1$, $e(r)^* = e(-r)$, and $e(r)e(s) = e(r+s)$, for all $r, s \in \mathbb{Q}/\mathbb{Z}$,
- For all $n \in \mathbb{N}^\times$ and all $r \in \mathbb{Q}/\mathbb{Z}$, the relation (3.2) holds.

One considers on the algebra $\mathcal{A}$ the time evolution given by

$$\sigma_t(\mu_n) = n^{it} \mu_n, \quad \sigma_t(e(r)) = e(r). \tag{3.3}$$

3.1. Hecke algebra

The fact that (3.3) defines a natural time evolution is best understood by describing the algebra $\mathcal{A}$ as a Hecke algebra for the pair of groups $(\Gamma_0, \Gamma) = (P_{\mathbb{Z}}, P_{\mathbb{Q}})$, where P is the $ax + b$ group. This is the way the algebra $\mathcal{A}$ was introduced in [1]. Whenever the inclusion $\Gamma_0 \subset \Gamma$ has the property that the left Γ_0 orbits of any $\gamma \in \Gamma/\Gamma_0$ are finite (same for right orbits on the left coset), one can consider the Hecke algebra of the pair (Γ_0, Γ) given by functions on $\Gamma_0 \backslash \Gamma / \Gamma_0$ with the convolution product

$$(f_1 * f_2)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f_1(\gamma \gamma_1^{-1}) f_2(\gamma_1) \tag{3.4}$$

and the involution $f^*(\gamma) := \overline{f(\gamma^{-1})}$. The Hecke algebra defined this way has a regular representation on the Hilbert space $\ell^2(\Gamma_0 \backslash \Gamma)$

$$(\pi(f)\xi)(\gamma) = \sum_{\Gamma_0 \backslash \Gamma} f(\gamma \gamma_1^{-1}) \xi(\gamma_1).$$

The canonical time evolution on the corresponding von Neumann algebra is determined by the ratio of the length of left and right Γ_0 orbits,

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)} \right)^{-it} f(\gamma), \tag{3.5}$$

where, for $\gamma \in \Gamma/\Gamma_0$ we set $L(\gamma) = \#\Gamma_0 \gamma$ and $R(\gamma) = L(\gamma^{-1})$. In the case of the pair (Γ_0, Γ) of parabolic subgroups $(P_{\mathbb{Z}}^+, P_{\mathbb{Q}}^+)$ of $\mathrm{GL}_2^+(\mathbb{Q})$, the Hecke algebra (3.4) gives the Bost–Connes algebra and the time evolution (3.5) is given by (3.3).

3.2. One-dimensional $\mathbb{Q}$-lattices

We now return to the point of view of $\mathbb{Q}$-lattices. As showed in [4], the algebra $\mathcal{A}$ of the Bost–Connes system has a natural interpretation as the noncommutative algebra of coordinates of the space $\mathcal{L}_1/\mathbb{R}_+^*$ of one-dimensional $\mathbb{Q}$-lattices (up to scaling) modulo commensurability. In fact, a one-dimensional $\mathbb{Q}$-lattice can always be written in the form

$$(\Lambda, \phi) = (\lambda\, \mathbb{Z}, \lambda\, \rho) \tag{3.6}$$

for some $\lambda > 0$ and some

$$\rho \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \varprojlim \mathbb{Z}/n\mathbb{Z} = \hat{\mathbb{Z}}. \tag{3.7}$$

By considering lattices up to scaling, we eliminate the factor $\lambda > 0$ so that one-dimensional $\mathbb{Q}$-lattices up to scale are completely specified by the choice of the element $\rho \in \hat{\mathbb{Z}}$. Thus, the algebra of coordinates of the space of one-dimensional $\mathbb{Q}$-lattices up to scale is the commutative C^*-algebra

$$C(\hat{\mathbb{Z}}) \simeq C^*(\mathbb{Q}/\mathbb{Z}). \tag{3.8}$$

The identification in (3.8) results from the fact that $\hat{\mathbb{Z}}$ is the Pontrjagin dual of $\mathbb{Q}/\mathbb{Z}$. The equivalence relation of commensurability is implemented by the action of the semigroup $\mathbb{N}^\times$ on $\mathbb{Q}$-lattices. The corresponding action on the algebra (3.8) is by

$$\alpha_n(f)(\rho) = \begin{cases} f(n^{-1}\rho), & \rho \in n\hat{\mathbb{Z}} \\ 0, & \text{otherwise}. \end{cases} \tag{3.9}$$

Thus, the quotient of the space of one-dimensional $\mathbb{Q}$-lattices up to scale by the commensurability relation and its algebra of coordinates of is given by the semigroup crossed product

$$C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}^\times. \tag{3.10}$$

This is another description of the Bost–Connes algebra, as (3.10) has the right set of generators and relations, with (3.2) implementing the semigroup action (3.9).

3.3. *Structure of KMS states*

The Bost–Connes algebra has irreducible representations on the Hilbert space $\mathcal{H} = \ell^2(\mathbb{N}^\times)$. These are parameterized by elements $\alpha \in \hat{\mathbb{Z}}^* = \mathrm{GL}_1(\hat{\mathbb{Z}})$. Any such element defines an embedding $\alpha : \mathbb{Q}^{\mathrm{cycl}} \hookrightarrow \mathbb{C}$ and the corresponding representation is of the form

$$\begin{aligned} \pi_\alpha(e(r))\,\epsilon_k &= \alpha(\zeta_r^k)\,\epsilon_k \\ \pi_\alpha(\mu_n)\,\epsilon_k &= \epsilon_{nk} \end{aligned} \tag{3.11}$$

The Hamiltonian implementing the time evolution (3.9) on $\mathcal{H}$ is

$$H\,\epsilon_k = \log k\,\epsilon_k \tag{3.12}$$

Thus, the partition function of the Bost–Connes system is the Riemann zeta function

$$Z(\beta) = \mathrm{Tr}(\mathrm{e}^{-\beta H}) = \sum_{k=1}^{\infty} k^{-\beta} = \zeta(\beta). \tag{3.13}$$

Bost and Connes showed in Ref. [1] that KMS states have the following structure, with a phase transition at $\beta = 1$.

- In the range $\beta \leq 1$, there is a unique KMS_β state. Its restriction to $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ is of the form

$$\varphi_\beta\left(e\left(\frac{a}{b}\right)\right) = b^{-\beta} \prod_{p \text{ prime},\ p|b} \frac{1-p^{\beta-1}}{1-p^{-1}}.$$

- For $1 < \beta \leq \infty$, the set of extremal KMS states $\mathcal{E}_\beta$ can be identified with $\hat{\mathbb{Z}}^*$. It has a free and transitive action of this group induced by an action on $\mathcal{A}$ by automorphisms. The extremal KMS_β state corresponding to $\alpha \in \hat{\mathbb{Z}}^*$ is of the form

$$\varphi_{\beta,\alpha}(x) = \frac{1}{\zeta(\beta)} \operatorname{Tr}(\pi_\alpha(x)\, \mathrm{e}^{-\beta H}). \tag{3.14}$$

- At $\beta = \infty$, the Galois group $\mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$ acts on the values of states $\varphi \in \mathcal{E}_\infty$ on an arithmetic subalgebra $\mathcal{A}_\mathbb{Q} \subset \mathcal{A}$. These have the property that $\varphi(\mathcal{A}_\mathbb{Q}) \subset \mathbb{Q}^{\mathrm{cycl}}$ and that the isomorphism (class field theory isomorphism) $\theta : \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q}) \xrightarrow{\cong} \hat{\mathbb{Z}}^*$ intertwines the Galois action on values with the action of $\hat{\mathbb{Z}}^*$ by symmetries, namely

$$\gamma\, \varphi(x) = \varphi(\theta(\gamma)\, x), \tag{3.15}$$

for all $\varphi \in \mathcal{E}_\infty$, for all $\gamma \in \mathrm{Gal}(\mathbb{Q}^{\mathrm{cycl}}/\mathbb{Q})$ and for all $x \in \mathcal{A}_\mathbb{Q}$.

Here, the arithmetic subalgebra can be taken as the algebra over $\mathbb{Q}$ generated by the $e(r)$ and μ_n, μ_n^*, or equivalently as the Hecke algebra of compactly supported $\mathbb{Q}$-valued functions on $\Gamma_0\backslash\Gamma$ with the convolution product (3.4). As we shall see, a different description of the arithmetic subalgebra is given in [4] in terms of homogeneous weight zero functions of $\mathbb{Q}$-lattices. The choice of an "arithmetic subalgebra" corresponds to endowing the noncommutative space $\mathcal{A}$ with an arithmetic structure. The subalgebra corresponds to the rational functions and the values of KMS_∞ states at elements of this subalgebra should be thought of as "values of rational functions at classical points" (cf. [5]). What is remarkable about the ground states of this system is that, when evaluated on the rational observables of the system, they only affect values that are algebraic numbers. Moreover, these span the maximal abelian extension of $\mathbb{Q}$ and the class field theory isomorphism intertwines the two actions of the idèle class group, as symmetry group of the system, and of the Galois group, as permutations of the expectation values of the rational observables. In general, the fact that the Galois action on the values of states would preserve positivity (i.e. would give values of other states) is a very unusual property. We refer to such states as "fabulous states".

3.4. Noncommutative geometry and class field theory

The main result of Bost–Connes [1] on the structure of KMS states for the system described above suggests the possibility of a connection between noncommutative geometry and class field theory. If $\mathbb{K}$ is a number field with $[\mathbb{K} : \mathbb{Q}] = n$, and $\bar{\mathbb{K}}$ is an algebraic closure of $\mathbb{K}$, then one has the Galois group $\mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})$. This group of symmetries is a very beautiful object, and quite mysterious even in the case of $\mathbb{K} = \mathbb{Q}$. On the other hand, one can consider a smaller field than $\bar{\mathbb{K}}$, namely the *maximal abelian extension* $\mathbb{K}^{ab}$ of $\mathbb{K}$. This has the property that

$$\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}) = \mathrm{Gal}(\bar{\mathbb{K}}/\mathbb{K})^{ab}.$$

The Kronecker–Weber theorem shows that for $\mathbb{K} = \mathbb{Q}$

$$\mathbb{Q}^{ab} = \mathbb{Q}^{\mathrm{cycl}} \quad \text{and} \quad \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*.$$

Finding an analogous result for more general number fields is the content of Hilbert's 12th problem, the problem of explicit class field theory. For a number field $\mathbb{K}$ one knows that there is an identification (the class field theory isomorphism)

$$\theta : C_{\mathbb{K}}/D_{\mathbb{K}} \xrightarrow{\simeq} \mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K}), \tag{3.16}$$

where $C_{\mathbb{K}} = \mathbb{A}^*_{\mathbb{K}}/\mathbb{K}^*$ is the group of idèle classes and $D_{\mathbb{K}}$ the connected component of the identity in $C_{\mathbb{K}}$. In the explicit class field theory problem one wants to obtain an explicit set of generators for $\mathbb{K}^{ab}$ and an explicit description of the action of $\mathrm{Gal}(\mathbb{K}^{ab}/\mathbb{K})$. Remarkably, a complete solution to Hilbert's 12th problem exists only for $\mathbb{Q}$ and for the imaginary quadratic fields $\mathbb{Q}(\sqrt{-d})$ for $d > 1$ a positive integer. The first challenge is posed by the case of real quadratic fields $\mathbb{Q}(\sqrt{d})$. It is natural to ask whether noncommutative geometry can provide some new insight on the Hilbert 12th problem, at least for the case of real quadratic fields. A series of beautiful reflections on this theme is given in Manin's real multiplication project [13]. The Bost–Connes system has also an adèlic description [1], where the algebra $\mathcal{A}$ is Morita equivalent to the crossed product

$$C_0(\mathbb{A}_f) \rtimes \mathbb{Q}^*_+ \tag{3.17}$$

(cf. [11]) with $\mathbb{A}_f = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the finite adèles of $\mathbb{Q}$. The set of extremal KMS states below critical temperature can also be described as the adèlic quotient

$$\mathcal{E}_\infty \simeq \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A})/\mathbb{R}^*_+, \tag{3.18}$$

with $\mathbb{A} = \mathbb{A}_f \times \mathbb{R}$ the full adèles of $\mathbb{Q}$. Given a number field $\mathbb{K}$ with $[\mathbb{K} : \mathbb{Q}] = n$, there is an embedding $\mathbb{K}^* \hookrightarrow \mathrm{GL}_n(\mathbb{Q})$ of its multiplicative group in $\mathrm{GL}_n(\mathbb{Q})$. Such embedding induces an embedding of $\mathrm{GL}_1(\mathbb{A}_{\mathbb{K},f})$ where $\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K}$ are the finite adèles of $\mathbb{K}$ into $\mathrm{GL}_n(\mathbb{A}_f)$. This suggests a possible strategy to develop an approach to explicit class field theory via the construction of "fabulous states" for quantum statistical mechanical systems associated to other number fields, by studying GL_n analogs of the Bost–Connes system. This was done (especially in the case of GL_2) in [4]. In the case of GL_2, one sees that the geometry of modular curves and the algebra of modular forms appear naturally. These are the main ingredients also in the solution of the explicit class field theory problem for imaginary quadratic fields (cf. [16]).

4. The GL_2-system

In this section, we will describe the main features of the GL_2 analog of the Bost–Connes system, according to the results of [4]. In the following, to avoid confusion, we use the notation $\mathcal{A}_1$ and $\mathcal{A}_{1,\mathbb{Q}}$ for the C^*-algebra of the Bost–Connes system and its arithmetic subalgebra and $\mathcal{A}_2$ and $\mathcal{A}_{2,\mathbb{Q}}$ for the analogs in the GL_2 case. Any two-dimensional $\mathbb{Q}$-lattice can be written in the form

$$(\Lambda, \phi) = (\lambda(\mathbb{Z} + \mathbb{Z}\tau), \lambda\rho),$$

for some $\lambda \in \mathbb{C}^*$, some $\tau \in \mathbb{H}$, and some $\rho \in M_2(\hat{\mathbb{Z}}) = \mathrm{Hom}(\mathbb{Q}^2/\mathbb{Z}^2, \mathbb{Q}^2/\mathbb{Z}^2)$. Thus, the space of two-dimensional $\mathbb{Q}$-lattices up to the scale factor $\lambda \in \mathbb{C}^*$ and up to isomorphisms, is given by

$$M_2(\hat{\mathbb{Z}}) \times \mathbb{H} \quad \mathrm{mod} \quad \Gamma = \mathrm{SL}(2, \mathbb{Z}). \tag{4.1}$$

The commensurability relation giving the space $\mathcal{L}_2/\mathbb{C}^*$ is implemented by the partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$. More precisely, we proceed as follows. We choose a basis $\{e_1 = 1, e_2 = -\mathrm{i}\}$ of $\mathbb{C}$ as a vector space over $\mathbb{R}$, with respect to which we define the action of $\mathrm{GL}_2^+(\mathbb{R})$ on $\mathbb{C}$. If we set $\Lambda_0 = \mathbb{Z}e_1 + \mathbb{Z}e_2 = \mathbb{Z} + \mathrm{i}\mathbb{Z}$, an element $\rho \in M_2(\hat{\mathbb{Z}})$ defines a homomorphism

$$\rho : \mathbb{Q}^2/\mathbb{Z}^2 \to \mathbb{Q}\Lambda_0/\Lambda_0, \quad \rho(a) = \rho_1(a)e_1 + \rho_2(e)e_2.$$

Consider the quotient of the space

$$\tilde{\mathcal{U}} := \{(g, \rho, \alpha) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathrm{GL}_2^+(\mathbb{R}) \, : \, g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{4.2}$$

by the action of $\Gamma \times \Gamma$ given by

$$(\gamma_1, \gamma_2)(g, \rho, \alpha) = (\gamma_1 g \gamma_2^{-1}, \gamma_2\rho, \gamma_2\alpha). \tag{4.3}$$

The groupoid $\mathcal{R}_2$ of the equivalence relation of commensurability on two-dimensional $\mathbb{Q}$-lattices (not considered up to scaling for the moment) is a locally compact groupoid, which can be parameterized by the quotient of (4.2) by $\Gamma \times \Gamma$ via the map $r : \tilde{\mathcal{U}} \to \mathcal{R}_2$,

$$r(g, \rho, \alpha) = \left((\alpha^{-1}g^{-1}\Lambda_0, \alpha^{-1}\rho), (\alpha^{-1}\Lambda_0, \alpha^{-1}\rho)\right). \tag{4.4}$$

We then consider the quotient by scaling. Upon identifying $\mathbb{C}^* \subset \mathrm{GL}_2^+(\mathbb{R})$ by

$$a + \mathrm{i}b \in \mathbb{C}^* \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}),$$

the quotient $\mathrm{GL}_2^+(\mathbb{R})/\mathbb{C}^*$ can be identified with the hyperbolic plane $\mathbb{H}$ in the usual way

$$\alpha = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}) \mapsto \frac{a\mathrm{i} + b}{c\mathrm{i} + d} \in \mathbb{H}.$$

If $(\Lambda_k, \phi_k)\, k = 1, 2$ are a pair of commensurable two-dimensional $\mathbb{Q}$-lattices, then for any $\lambda \in \mathbb{C}^*$, the $\mathbb{Q}$-lattices $(\lambda\Lambda_k, \lambda\phi_k)$ are also commensurable, with

$$r(g, \rho, \alpha\lambda^{-1}) = \lambda r(g, \rho, \alpha).$$

However, the action of $\mathbb{C}^*$ on $\mathbb{Q}$-lattices is not free due to the presence of lattices (such as Λ_0) with nontrivial automorphisms. Thus, the quotient $Z = \mathcal{R}_2/\mathbb{C}^*$ is no longer a groupoid. Still, one can define a convolution algebra for Z by restricting the convolution product of $\mathcal{R}_2$ to homogeneous functions of weight zero, where a function f has weight k if it satisfies

$$f(g, \rho, \alpha\lambda) = \lambda^k f(g, \rho, \alpha), \quad \forall \lambda \in \mathbb{C}^*.$$

The space Z is the quotient of the space

$$\mathcal{U} := \{(g, \rho, z) \in \mathrm{GL}_2^+(\mathbb{Q}) \times M_2(\hat{\mathbb{Z}}) \times \mathbb{H} | g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{4.5}$$

by the action of $\Gamma \times \Gamma$. Here, the space $M_2(\hat{\mathbb{Z}}) \times \mathbb{H}$ has a partially defined action of $\mathrm{GL}_2^+(\mathbb{Q})$ given by

$$g(\rho, z) = (g\rho, g(z)),$$

where $g(z)$ denotes the action as fractional linear transformation. Thus, the quotient $\mathcal{L}_2/\mathbb{C}^*$ of the space of two-dimensional $\mathbb{Q}$-lattices up to scale by the relation of commensurability is a noncommutative space whose algebra of coordinates is a Hecke algebra obtained as follows. Consider the space $C_c(Z)$ of continuous compactly supported functions on Z. These can be seen, equivalently, as functions on $\mathcal{U}$ as in (4.5) invariant under the $\Gamma \times \Gamma$ action $(g, \rho, z) \mapsto (\gamma_1 g \gamma_2^{-1}, \gamma_2 z)$. One endows $C_c(Z)$ with the convolution product

$$(f_1 * f_2)(g, \rho, z) = \sum_{s \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q}) : s\rho \in M_2(\hat{\mathbb{Z}})} f_1(gs^{-1}, s\rho, s(z)) f_2(s, \rho, z) \tag{4.6}$$

and the involution $f^*(g, \rho, z) = \overline{f(g^{-1}, g\rho, g(z))}$. The time evolution is given by

$$\sigma_t(f)(g, \rho, z) = \det(g)^{it} f(g, \rho, z). \tag{4.7}$$

For $\rho \in M_2(\hat{\mathbb{Z}})$, let

$$G_\rho := \{g \in \mathrm{GL}_2^+(\mathbb{Q}) : g\rho \in M_2(\hat{\mathbb{Z}})\} \tag{4.8}$$

and consider the Hilbert space $\mathcal{H}_\rho = \ell^2(\Gamma \backslash G_\rho)$. A two-dimensional $\mathbb{Q}$-lattice $L = (\Lambda, \phi) = (\rho, z)$ determines a representation of the Hecke algebra by bounded operators on $\mathcal{H}_\rho$, setting

$$(\pi_L(f)\xi)(g) = \sum_{s \in \Gamma \backslash G_\rho} f(gs^{-1}, s\rho, s(z))\, \xi(s). \tag{4.9}$$

In particular, when the $\mathbb{Q}$-lattice $L = (\Lambda, \phi)$ is invertible, one obtains

$$\mathcal{H}_\rho \cong \ell^2(\Gamma \backslash M_2^+(\mathbb{Z})).$$

In this case, the Hamiltonian implementing the time evolution (4.7) is given by the operator

$$H\,\epsilon_m = \log\det(m)\,\epsilon_m. \tag{4.10}$$

Thus, in the special case of invertible $\mathbb{Q}$-lattices, (4.9) yields a *positive energy* representation. In general, for $\mathbb{Q}$-lattices which are not commensurable to an invertible one, the corresponding Hamiltonian H is not bounded below. The Hecke algebra (4.6) admits a C^*-algebra completion $\mathcal{A}_2$, where the norm is the sup over all representations π_L. The partition function for this GL_2-system is given by

$$Z(\beta) = \sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})} \det(m)^{-\beta} = \sum_{k=1}^{\infty} \sigma(k)\,k^{-\beta} = \zeta(\beta)\zeta(\beta-1), \tag{4.11}$$

where $\sigma(k) = \sum_{d|k} d$. This already hints to the fact that the system might have more than one phase transition. In fact, the form of the partition function suggests the possibility that two distinct phase transitions might happen at $\beta = 1$ and 2.

5. KMS states and symmetries

The structure of KMS states for the GL_2-system is analysed in [4]. The main result is the following.

Theorem 5.1. *The KMS_β states of the GL_2-system have the following properties*:

(1) *In the range $\beta \le 1$, there are no KMS states.*
(2) *In the range $\beta > 2$, the set of extremal KMS states is given by the classical Shimura variety*

$$\mathcal{E}_\beta \cong \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*. \tag{5.1}$$

This shows that the extremal KMS states at sufficiently low temperature are parameterized by the invertible $\mathbb{Q}$-lattices. The explicit expression for these extremal KMS_β states is obtained as

$$\varphi_{\beta,L}(f) = \frac{1}{Z(\beta)} \sum_{m\in\Gamma\backslash M_2^+(\mathbb{Z})} f(1, m\rho, m(z))\,\det(m)^{-\beta} \tag{5.2}$$

where $L = (\rho, z)$ is an invertible $\mathbb{Q}$-lattice. The difficult part of the proof is to show that indeed all extremal KMS_β states are of this form. When $\beta \to 1$ from above, the different pure phases merge, so it is reasonable to expect that in the intermediate range $1 < \beta < 2$ there will be a unique KMS_β state. Thus, the system exhibits two distinct phase transitions at $\beta = 2$ and 1. The main step in the proof of Theorem 5.1 is the construction of a subalgebra generated by projections $\pi_p(k, l)$, where p is a prime number and k, l are integers with $k \le l$, with the following properties

- If φ is a KMS_β state for the GL_2-system, then it satisfies

$$\begin{aligned}&\varphi(\pi_p(k,l)) = p^{-(k+l)\beta} p^{l-k}(1+p^{-1})(1-p^{-\beta})(1-p^{1-\beta}), && k<l\\ &\varphi(\pi_p(l,l)) = p^{-2l\beta}(1-p^{-\beta})(1-p^{1-\beta}), && k=l.\end{aligned}$$

- If p_j are distinct prime numbers, then

$$\varphi\left(\prod_j \pi_{p_j}(k_j,l_j)\right) = \prod_j \varphi(\pi_{p_j}(k_j,l_j)).$$

In particular, these properties show that there cannot be any KMS state in the range $0<\beta<1$.

5.1. Symmetries

In the range $2<\beta\le\infty$, there is a very interesting action of symmetries on the KMS states of the GL_2-system. The symmetry group of $\mathcal{A}_2$ (including both automorphisms and endomorphisms) can be identified with the group

$$\mathrm{GL}_2(\mathbb{A}_f) = \mathrm{GL}_2^+(\mathbb{Q})\mathrm{GL}_2(\hat{\mathbb{Z}}). \tag{5.3}$$

Here, the group $\mathrm{GL}_2(\hat{\mathbb{Z}})$ acts by automorphisms,

$$\theta_\gamma(f)(g,\rho,z) = f(g,\rho\gamma,z). \tag{5.4}$$

Geometrically, this is the group of deck transformations of coverings of modular curves. In fact, when we consider the (compact) modular curve $X(n)$ over the cyclotomic field $\mathbb{Q}(\zeta_n)$, these form a tower over the base $X(1)=\mathbb{P}^1$ over $\mathbb{Q}$, and the group $\mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\pm 1$ is the group of automorphisms of the projection $X(n)\to X(1)$ (cf. [15,5]) so that one obtains the automorphism group

$$\mathrm{GL}_2(\hat{\mathbb{Z}})/\pm 1 = \varprojlim_n \mathrm{GL}_2(\mathbb{Z}/n\mathbb{Z})/\{\pm 1\}. \tag{5.5}$$

On the other hand, the group $\mathrm{GL}_2^+(\mathbb{Q})$ in (5.3) acts by endomorphisms,

$$\theta_m(f)(g,\rho,z) = \begin{cases} f(g,\rho\tilde{m}^{-1},z), & \rho\in m\,M_2(\hat{\mathbb{Z}})\\ 0, & \text{otherwise}\end{cases} \tag{5.6}$$

where $\tilde{m}=\det(m)m^{-1}$. The subgroup $\mathbb{Q}^*\hookrightarrow \mathrm{GL}_2(\mathbb{A}_f)$ acts by inner, hence the group of symmetries of the set of extremal states $\mathcal{E}_\beta$ is of the form

$$S = \mathbb{Q}^*\backslash \mathrm{GL}_2(\mathbb{A}_f). \tag{5.7}$$

In the case of $\mathcal{E}_\infty$ states (defined as weak limits), the action of $\mathrm{GL}_2^+(\mathbb{Q})$ is more subtle to define. In fact, (5.6) does not directly induce a nontrivial action on $\mathcal{E}_\infty$. However, there is a nontrivial action induced by the action on $\mathcal{E}_\beta$ states for sufficiently large β. The action on the KMS_∞ states is obtained by a "warming up and cooling down procedure", as in (2.7) and (2.8).

5.2. Lattice functions

In the case of the BC system, the arithmetic subalgebra $\mathcal{A}_{1,\mathbb{Q}}$ can be regarded as the algebra generated by the μ_n, μ_n^* and by homogeneous functions of weight zero on one-dimensional $\mathbb{Q}$-lattices obtained as a normalization of the functions

$$\epsilon_{k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-k} \tag{5.8}$$

by covolume, namely by the functions $e_{k,a} := c^k \epsilon_{k,a}$, where $c(\Lambda)$ is proportional to the covolume $|\Lambda|$ and satisfies

$$(2\pi\sqrt{-1})\, c(\mathbb{Z}) = 1.$$

In fact, it suffices to consider the $e_{1,a}$.

It is natural therefore to expect that the analogous $\mathcal{A}_{2,\mathbb{Q}}$ of the GL_2-system will involve Eisenstein series

$$E_{2k,a}(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-2k} \tag{5.9}$$

and

$$X_a(\Lambda, \phi) = \sum_{y \in \Lambda + \phi(a)} y^{-2} - {\sum_{y \in \Lambda}}' y^{-2} \tag{5.10}$$

normalized to weight zero, in a similar fashion. This points to the fact that modular functions should appear naturally as the rational subalgebra of the GL_2-system. This can also be noticed from the fact that the group of symmetries S described in (5.7) is in fact the Galois group of the field of modular functions, by a deep arithmetic result of Shimura [16]. As we shall see below, in fact $\mathcal{A}_{2,\mathbb{Q}}$ will turn out to be a subalgebra of unbounded multipliers of $\mathcal{A}_2$. Modular functions will appear naturally from a simple set of conditions specifying the arithmetic nature of these multipliers.

5.3. The modular field

We recall briefly some basic facts and results about the modular field. Let F denote the field of modular functions over $\mathbb{Q}^{ab}$, namely the union of the fields F_N of modular functions of level N rational over the cyclotomic field $\mathbb{Q}(\zeta_n)$, that is, such that the q-expansion in powers of $q^{1/N} = \exp(2\pi \mathrm{i}\tau/N)$ has all coefficients in $\mathbb{Q}(\mathrm{e}^{2\pi \mathrm{i}/N})$. The action of the Galois group $\hat{\mathbb{Z}}^* \simeq \mathrm{Gal}(\mathbb{Q}^{ab}/\mathbb{Q})$ on the coefficients determines a homomorphism

$$\mathrm{cycl} : \hat{\mathbb{Z}}^* \to \mathrm{Aut}(F). \tag{5.11}$$

The modular field has an explicit set of generators given by the Fricke functions ([16,12]). If $\wp$ is the Weierstrass $\wp$-function, which gives the parameterization

$$w \mapsto (1, \wp(w;\tau,1), \wp'(w;\tau,1))$$

of the elliptic curve

$$y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$$

by the quotient $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, then the Fricke functions are homogeneous functions of one-dimensional lattices of weight zero, parameterized by $v \in \mathbb{Q}^2/\mathbb{Z}^2$, of the form

$$f_v(z) = -2^7 3^5 \frac{g_2(z)g_3(z)}{\Delta(z)} \wp(\lambda_z(v); z, 1), \tag{5.12}$$

where $\Delta(z) = g_2^3 - 27g_3^2$ is the discriminant and $\lambda_z(v) := v_1 z + v_2$. The following important result of Shimura completely determines the Galois group of the modular field:

$$\mathrm{Aut}(F) \cong \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f).$$

If $\tau \in \mathbb{H}$ is a generic point, then the evaluation map $f \mapsto f(\tau)$ is an embedding $F \hookrightarrow \mathbb{C}$. We denote by F_τ the image in $\mathbb{C}$. This yields an identification

$$\theta_\tau : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f). \tag{5.13}$$

5.4. Arithmetic subalgebra

We determine a natural arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ of unbounded multipliers of $\mathcal{A}_2$. Unbounded multipliers on $\mathcal{A}_2$ are endowed with the same convolution product (4.6). Elements of $\mathcal{A}_{2,\mathbb{Q}}$ are continuous functions on Z (cf. (4.5)), with finite support in the variable $g \in \Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})$. For convenience, we adopt the notation

$$f_{(g,\rho)}(z) = f(g, \rho, z)$$

so that $f_{(g,\rho)} \in C(\mathbb{H})$. Let $p_N : M_2(\hat{\mathbb{Z}}) \to M_2(\mathbb{Z}/N\mathbb{Z})$ be the canonical projection. We say that *f* is of level *N* if

$$f_{(g,\rho)} = f_{(g,p_N(\rho))}, \quad \forall (g, \rho).$$

Then *f* is completely determined by the functions

$$f_{(g,m)} \in C(\mathbb{H}), \quad \text{for } m \in M_2(\mathbb{Z}/N\mathbb{Z}).$$

Notice that the invariance

$$f(g\gamma, \rho, z) = f(g, \gamma\rho, \gamma(z)),$$

for all $\gamma \in \Gamma$ and for all $(g, \rho, z) \in \mathcal{U}$, implies that we have

$$f_{(g,m)|\gamma} = f_{(g,m)}, \quad \forall \gamma \in \Gamma(N) \cap g^{-1} \Gamma g. \tag{5.14}$$

so that f is invariant under a congruence subgroup. Thus, we define the arithmetic $\mathcal{A}_{2,\mathbb{Q}}$ as follows.

Definition 5.1. A continuous function on Z is in the arithmetic subalgebra $\mathcal{A}_{2,\mathbb{Q}}$ if it satisfies the following properties:

(1) The support of f in $\Gamma \backslash \mathrm{GL}_2^+(\mathbb{Q})$ is finite.
(2) The function f is of finite level with

$$f_{(g,m)} \in F, \quad \forall (g, m).$$

(3) The function f satisfies the *cyclotomic condition*:

$$f_{(g,\alpha(u)m)} = \mathrm{cycl}(u)\, f_{(g,m)},$$

for all $g \in \mathrm{GL}_2^+(\mathbb{Q})$ diagonal and all $u \in \hat{\mathbb{Z}}^*$, with

$$\alpha(u) = \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$$

and cycl as in (5.11).

If we took all but the last condition, this would allow the algebra $\mathcal{A}_{2,\mathbb{Q}}$ to contain the cyclotomic field $\mathbb{Q}^{ab} \subset \mathbb{C}$, but this would prevent the existence of “fabulous states", because the “fabulous” property would not be compatible with $\mathbb{C}$-linearity. The cyclotomic condition forces the spectrum of the corresponding elements of $\mathcal{A}_{2,\mathbb{Q}}$ to contain all Galois conjugates of any such root, so that these elements cannot be scalar. This is achieved via a simple an natural consistency condition on the roots of unity that appear in the coefficients of the q-series. The algebra $\mathcal{A}_{2,\mathbb{Q}}$ defined by the properties above is a subalgebra of unbounded multipliers of $\mathcal{A}_2$, which is globally invariant under the group of symmetries S.

5.5. Galois action on $\mathcal{E}_\infty$

Consider a state $\varphi = \varphi_{\infty,L} \in \mathcal{E}_\infty$, where the invertible $\mathbb{Q}$-lattice $L = (\rho, \tau)$ is generic, in the sense that $\tau \in \mathbb{H}$ is generic so that one has the identification (5.13).

Theorem 5.2. *For $\varphi_{\infty,L} \in \mathcal{E}_\infty$ with $L = (\rho, \tau)$ generic, the values of the state on elements of the arithmetic subalgebra lie in the image in $\mathbb{C}$ of the modular field,*

$$\varphi(\mathcal{A}_{2,\mathbb{Q}}) \subset F_\tau, \tag{5.15}$$

and the isomorphism

$$\theta_\varphi : \mathrm{Gal}(F_\tau/\mathbb{Q}) \xrightarrow{\simeq} \mathbb{Q}^* \backslash \mathrm{GL}_2(\mathbb{A}_f), \tag{5.16}$$

given by

$$\theta_\varphi(\gamma) = \rho^{-1}\,\theta_\tau(\gamma)\,\rho, \tag{5.17}$$

for θ_τ as in (5.13), intertwines the Galois action on the values of the state with the action of symmetries,

$$\gamma\,\varphi(f) = \varphi(\theta_\varphi(\gamma) f), \quad \forall f \in \mathcal{A}_{2,\mathbb{Q}}, \quad \forall \gamma \in \mathrm{Gal}(F_\tau/\mathbb{Q}). \tag{5.18}$$

6. Noncommutative Shimura varieties

This point of view is stressed in our joint work with Ramachandran [5]. With the notation $Sh(G, X) := G(\mathbb{Q})\backslash G(\mathbb{A}_f) \times X$, the Shimura variety associated to the tower of modular curves is described by the adèlic quotient

$$\begin{aligned} Sh(\mathrm{GL}_2, \mathbb{H}^\pm) &= \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H}^\pm \\ &= \mathrm{GL}_2^+(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_f) \times \mathbb{H} = \mathrm{GL}_2^+(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A})/\mathbb{C}^*. \end{aligned} \tag{6.1}$$

The inverse limit $\varprojlim \Gamma\backslash\mathbb{H}$ over congruence subgroups $\Gamma \subset \mathrm{SL}(2, \mathbb{Z})$ gives a connected component, while by taking congruence subgroups in $\mathrm{SL}(2, \mathbb{Q})$ one obtains the adèlic version $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$. The set of components of $Sh(\mathrm{GL}_2, \mathbb{H}^\pm)$ is given by

$$\pi_0(Sh(\mathrm{GL}_2, \mathbb{H}^\pm)) = Sh(\mathrm{GL}_1, \{\pm 1\}), \tag{6.2}$$

where

$$Sh(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathrm{GL}_1(\mathbb{A}_f) \times \{\pm 1\} = \mathbb{Q}_+^* \backslash \mathbb{A}_f^* \tag{6.3}$$

is the Shimura variety associated to the cyclotomic tower (cf. [5,15]). As we shall see below, (6.3) can be thought of as the “set of classical points” of the noncommutative space of the BC system, where the algebra $\mathcal{A}_1$ is Morita equivalent to $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*$. The result of [1] shows in particular that, at zero temperature, the BC system settles onto its “commutative points” (extremal KMS_∞ states) which form the classical Shimura variety (6.3). Similarly, the results of Theorems 5.1 and 5.2 show the analogous behavior in the GL_2-system. At zero temperature, the system settles onto its “commutative points” given by the Shimura variety (6.1). This leads us naturally to think of the algebras of the BC system and of the GL_2-system as *noncommutative Shimura varieties*. The first is associated to the adèlic quotient

$$Sh^{(nc)}(\mathrm{GL}_1, \{\pm 1\}) := \mathrm{GL}_1(\mathbb{Q})\backslash(\mathbb{A}_f \times \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash \mathbb{A}^{\cdot}/\mathbb{R}_+^* \tag{6.4}$$

with $\mathbb{A}^{\cdot} := \mathbb{A}_f \times \mathbb{R}^*$. This has a *compactification*, obtained by replacing $\mathbb{A}^{\cdot}$ by $\mathbb{A}$, as in [3],

$$\overline{Sh^{(nc)}}(\mathrm{GL}_1, \{\pm 1\}) = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^*. \tag{6.5}$$

The compactification consists of adding the trivial lattice (with a possibly nontrivial $\mathbb{Q}$-structure). The dual space (namely the principal $\mathbb{R}_+^*$-bundle obtained by taking the crossed product by time evolution σ_t) is the space of adèle classes

$$\mathcal{L}_1 = \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A} \to \mathrm{GL}_1(\mathbb{Q})\backslash\mathbb{A}/\mathbb{R}_+^* \tag{6.6}$$

that gives the spectral realization of zeros of the Riemann ζ function in [3]. This dual space corresponds to considering commensurability classes of one-dimensional $\mathbb{Q}$-lattices (not up to scaling). In the case of the GL_2-system, similarly, we have a noncommutative Shimura variety

$$Sh^{(nc)}(\mathrm{GL}_2, \mathbb{H}^{\pm}) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f) \times \mathbb{H}^{\pm}). \tag{6.7}$$

This also admits a compactification, now given by adding the boundary $\mathbb{P}^1(\mathbb{R})$ to $\mathbb{H}^{\pm}$, as in the noncommutative compactification of modular curves of [14],

$$\overline{Sh^{(nc)}}(\mathrm{GL}_2, \mathbb{H}^{\pm}) := \mathrm{GL}_2(\mathbb{Q})\backslash(M_2(\mathbb{A}_f) \times \mathbb{P}^1(\mathbb{C})) = \mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A})/\mathbb{C}^*, \tag{6.8}$$

where $\mathbb{P}^1(\mathbb{C}) = \mathbb{H}^{\pm} \cup \mathbb{P}^1(\mathbb{R})$. This corresponds to adding to the space of commensurability classes of two-dimensional $\mathbb{Q}$-lattices the pseudolattices (in the sense of [13]), here considered together with a $\mathbb{Q}$-structure. In this case we can also consider the dual system. This is a $\mathbb{C}^*$-bundle

$$\mathcal{L}_2 = \mathrm{GL}_2(\mathbb{Q})\backslash M_2(\mathbb{A}). \tag{6.9}$$

On this dual space modular forms appear naturally instead of modular functions and the algebra of coordinates contains the modular Hecke algebra of Connes–Moscovici ([6,7]) as arithmetic subalgebra. The identification (6.2) then gives the compatibility between the GL_1 and the GL_2-system. At the level of the classical commutative spaces, this is given by the map

$$\det \times \mathrm{sign} : Sh(\mathrm{GL}_2, \mathbb{H}^{\pm}) \to Sh(\mathrm{GL}_1, \{\pm 1\}), \tag{6.10}$$

which corresponds in fact to passing to the set π_0 of connected components.

7. Class field theory

In our joint work with Ramachandran [5], we use the GL_2-system to extend the relation between noncommutative geometry and class field theory illustrated in the BC system for the case of $\mathbb{Q}$ to the next important case, that of imaginary quadratic fields. Thus, we assume that $\mathbb{K} = \mathbb{Q}(\sqrt{-d})$. A point $\tau \in \mathbb{H}$ is a CM (complex multiplication) point for $\mathbb{K}$ if we have $\mathbb{K} = \mathbb{Q}(\tau)$. This is a *nongeneric* case, in the sense of the properties of the modular field under the evaluation map. In fact, in this case, the evaluation $F \to F_\tau \subset \mathbb{C}$ does *not* give an embedding. The image in $\mathbb{C}$ of the modular field can be characterized ([16]) and is the maximal abelian extension of $\mathbb{K}$,

$$F_\tau \simeq \mathbb{K}^{ab}. \tag{7.1}$$

The values $\{f(\tau),\ f \in F\}$ give a set of *generators* of $\mathbb{K}^{ab}$ and the Galois action is described explicitly in the following way ([16])

$$\begin{tikzcd}
1 \arrow[r] & \mathrm{K}^* \arrow[r] & \mathrm{GL}_1(\mathrm{A}_{\mathrm{K},f}) \arrow[r, "\simeq"] \arrow[d] & \mathrm{Gal}(\mathrm{K}^{ab}/\mathrm{K}) \arrow[r] & 1 \\
1 \arrow[r] & \mathrm{Q}^* \arrow[r] & \mathrm{GL}_2(\mathrm{A}_f) \arrow[r, "\simeq"] & \mathrm{Aut}\,(F) \arrow[r] & 1,
\end{tikzcd}$$

where $\mathbb{A}_{\mathbb{K},f} = \mathbb{A}_f \otimes \mathbb{K}$. This gives a complete solution to the problem of explicit class field theory for imaginary quadratic fields. As in the BC system one sees the explicit class field theory of $\mathbb{Q}^{ab}$ appear in the symmetries of $\mathcal{E}_\infty$ states, the class field theory for imaginary quadratic fields appears naturally in relation to the GL_2-system. In fact, one can consider a special class of two-dimensional $\mathbb{Q}$-lattices, given by those that also have the similarly defined structure of a one-dimensional $\mathbb{K}$-lattice. The commensurability relation (compatible with the $\mathbb{K}$-structure) gives a system $\mathcal{A}_\mathbb{K}$ which is closely related to both the original BC system and the GL_2-system and has properties in common with both. The arithmetic structure of the GL_2-system induces a corresponding arithmetic structure $\mathcal{A}_{\mathbb{K},\mathbb{Q}}$ on the $\mathcal{A}_\mathbb{K}$ system, which also inherits a natural time evolution. The Galois theory of KMS_∞ states of the GL_2-system has a parallel result for the $\mathcal{A}_\mathbb{K}$ system, which mirrors the relation between the explicit class field theory of imaginary quadratic fields and the Galois theory of the modular field described above (see [5] for details). The next fundamental question in the direction of generalizations of the BC system to other number fields is how to approach the more complicated case of real quadratic fields, $\mathbb{Q}(\sqrt{d})$, for which there is not yet a complete solution to the explicit class field theory problem. Manin's real multiplication program [13] suggests that the right geometric setup may still be found within the GL_2-system, by looking at the boundary strata of $\overline{Sh^{(nc)}}(\mathrm{GL}_2, \mathbb{H}^\pm)$.

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