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Quantum field theory over $\mathbb{F}_1$

Dori Bejleri, Matilde Marcolli *

Mathematics Department, Mail Code 253-37, Caltech, 1200 E. California Blvd. Pasadena, CA 91125, USA

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ABSTRACT

In this paper we discuss some questions about geometry over the field with one element, motivated by the properties of algebraic varieties that arise in perturbative quantum field theory. We follow the approach to $\mathbb{F}_1$-geometry based on torified-schemes. We first discuss some simple necessary conditions in terms of the Euler characteristic and classes in the Grothendieck ring, then we give a blowup formula for torified varieties and we show that the wonderful compactifications of the graph configuration spaces, that arise in the computation of Feynman integrals in position space, admit an $\mathbb{F}_1$-structure. By a similar argument we show that the moduli spaces of curves $\bar{\mathcal{M}}_{0,n}$ admit an $\mathbb{F}_1$-structure, thus answering a question of Manin. We also discuss conditions on hyperplane arrangements, a possible notion of embedded $\mathbb{F}_1$-structure and its relation to Chern classes, and questions on Chern classes of varieties with regular torifications.

1. Introduction

This paper, as the title readily suggests, is inspired by Oliver Schnetz's interesting paper "Quantum field theory over $\mathbb{F}_q$", [1], with the motivation of investigating under which circumstances one could envision the existence of a quantum field theory over the field with one element. This is meant not so much in the sense of developing the actual physical Lagrangian and Feynman rules in the context of $\mathbb{F}_1$-geometry, but of investigating when certain classes of algebraic varieties that naturally arise in the context of perturbative quantum field theory, and which are already defined over $\mathbb{Z}$, may be carrying an additional $\mathbb{F}_1$-structure.

The relation between quantum field theory and geometry of varieties over finite fields originally arises from the question of "polynomial countability" for a class of hypersurfaces X_Γ defined by the parametric formulation of momentum space Feynman integrals in perturbative quantum field theory. In addition to these much studied graph hypersurfaces X_Γ, there are other algebraic varieties directly connected to the computation of Feynman integrals. In particular, certain complete intersections Λ_Γ, recently studied by Esnault and by Bloch, [2], and some closely related hyperplane arrangements $\mathcal{A}_\Gamma$. There are also the graph configuration spaces $\mathrm{Conf}_\Gamma(X) = X^{\#V(\Gamma)} \smallsetminus \cup_e \Delta_e$ and their wonderful compactifications, which arise in the computation of Feynman integrals in configuration spaces [3,4]. In all of these cases, the motives of these varieties carry some useful information about the corresponding Feynman integral computations.

In a different direction, the idea of the "field with one element" and of the existence of a suitable notion of $\mathbb{F}_1$-geometry, first arose from a comment by J. Tits in a paper from 1956, where he noted that the number of points of $\mathrm{GL}_n(\mathbb{F}_q)$ is a polynomial $N(q) = (q^n - 1)(q^n - q)\dots(q^n - q^{n-1})$ with a zero of degree r at $q = 1$, such that $\lim_{q\to 1} N(q)/(q-1)^r = n!$, and suggested that one might interpret the symmetric group $S_n = \mathrm{GL}_n(\mathbb{F}_1)$ as an algebraic group in "characteristic 1". This suggests, more generally, that for any scheme X defined over $\mathbb{Z}$ for which the counting function

$$N_X(q) = \#X_q(\mathbb{F}_q) \tag{1.1}$$

* Corresponding author.
E-mail addresses: dbejleri@caltech.edu (D. Bejleri), matilde@caltech.edu (M. Marcolli).

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giving the number of points over $\mathbb{F}_q$ of the reduction of X, and for which there is a limit

$$\lim_{q\to 1} N_X(q)/(q-1)^r, \tag{1.2}$$

where r is the order of the zero at $q = 1$, one should interpret this limit as the number of $\mathbb{F}_1$-points, $\#X(\mathbb{F}_1)$ of X. Simple examples include projective space, for which $\#\mathbb{P}^{(n-1)}(\mathbb{F}_1) = n$, and Grassmannians, for which $\#\mathrm{Gr}(k, n)(\mathbb{F}_1) = \binom{n}{k}$, see [5]. Several different forms of geometry over $\mathbb{F}_1$ were developed in recent years (see [6] for a general overview).

There are at present many different versions of geometry over the field with one element (see [6] for a comparative survey). Throughout this paper we adopt the approach to $\mathbb{F}_1$-geometry developed by Javier López Peña and Oliver Lorscheid, based on the existence of affine torifications.

In this paper, we discuss some aspects of the geometry of the varieties associated to Feynman integrals, from the point of view of $\mathbb{F}_1$-geometry. This will lead us to formulate a series of questions, about these classes of varieties as well as, more generally, about $\mathbb{F}_1$-geometry.

In Section 2 we recall the original questions about the motivic properties of the graph hypersurfaces and we show how a simple modification of the standard derivation of the parametric Feynman integral of perturbative QFT leads to the occurrence of other algebraic varieties, including the mixed Tate complete intersection Λ_Γ recently studied by Esnault and by Bloch. We also recall the definition of the varieties $\overline{\mathrm{Conf}}_\Gamma(X)$, the wonderful compactifications of graph configuration spaces, used in [3,4] to study Feynman integrals in configuration spaces.

In Section 3 we describe some simple necessary conditions for the existence of an $\mathbb{F}_1$-structure, based on constraints on the Euler characteristic and on the class in the Grothendieck ring of varieties and we give explicit examples of cases where the graph hypersurfaces satisfy or not these constraints.

In Section 4, we give a blowup formula for torified varieties and we use it to show that, in the case of $X = \mathbb{P}^D$, the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ admit a regular affine torification in the sense of [7], hence they are $\mathbb{F}_1$-varieties. By the same technique, we show that the moduli space of curves $\bar{\mathcal{M}}_{0,n}$ is an $\mathbb{F}_1$-variety and that so are the generalizations $T_{d,n}$, with $T_{1,n} = \bar{\mathcal{M}}_{0,n}$ considered in [8].

In Section 5 we identify hyperplane arrangements as an especially interesting class of varieties about which to investigate when they admit an $\mathbb{F}_1$-structure. This is especially so, in view of their (conjectural) role as "generators" of mixed Tate motives. We identify an explicit necessary condition in terms of the coefficients of the characteristic polynomial of the arrangement and its Möbius function, as a direct consequence of a recent result of Aluffi [9] on the Grothendieck classes of hyperplane arrangements.

In Section 6 we use another result of Aluffi [10] expressing the Chern classes of singular varieties in terms of Euler characteristics of hyperplane sections, to suggest the existence of a notion of embedded $\mathbb{F}_1$-structures.

In Section 7 we formulate the question of the existence of an analog of the Ehlers formula for Chern classes of toric varieties in the case of varieties that admit a regular affine torification. While we show that the argument given in [11] for the Ehlers formula does not directly extend to the regular torified case, we suggest that there may be a reformulation in terms of regular torifications of the results of [12] for Chern classes Schubert varieties.

2. Quantum field theory, algebraic varieties and motives

We recall here some well known facts about the occurrence of algebraic varieties, motives and periods in perturbative quantum field theory, which motivate some of the questions that we analyze in the rest of the paper.

In quantum field theory, Feynman graphs parameterize the perturbative expansion of the Green's functions. To each Feynman graph Γ, there is an associated (typically divergent) integral $U(\Gamma)$ that gives a term in this expansion. Formally (ignoring divergences) each of these integrals can be written in the Feynman parametric form as an integral of an algebraic differential form on the complement of a (singular) projective hypersurface $X_\Gamma \subset \mathbb{P}^{n-1}$, with n the number of edges of the graph, defined by the vanishing of a graph polynomial, the Kirchhoff polynomial Ψ_Γ, see [13,14]. Actually accounting for divergences introduces serious complications, which require performing blowups that separate the locus of integration from the hypersurface and accounting for the ambiguities deriving from monodromies, see [15]. In several physically significant cases, it has been shown [16] that the resulting period obtained from the evaluation of the Feynman amplitude $U(\Gamma)$ is a period of a mixed Tate motive over $\mathbb{Z}$, that is, a $\mathbb{Q}[1/(2\pi i)]$-linear combination of multiple zeta values.

We recall briefly (at the beginning of Section 2.2 below) the standard derivation of the parametric form of Feynman integrals and its relation to the graph hypersurfaces, (see [17], or the overview given in [14]), but first we recall here the main resulting algebro-geometric setting.

2.1. Graph hypersurfaces and the polynomial countability question

Let Γ be a finite graph, occurring as a Feynman diagram for a perturbative scalar massless quantum field theory. We can assume that Γ is connected and one-particle-irreducible (1PI), that is, it cannot be disconnected by removal of a single edge. The Kirchhoff polynomial of Γ is defined as

$$\Psi_\Gamma = \sum_{T\subseteq\Gamma} \prod_{e\notin T} t_e, \tag{2.1}$$

where $T \subset \Gamma$ runs through all the spanning trees of Γ and t_e is an indeterminate corresponding to the edge e. The definition can be extended to non-necessarily connected graphs, by taking a sum over spanning forests made of a union of a spanning tree in each connected component. This a homogeneous polynomial in $n = \#E(\Gamma)$ variables, of degree equal to $b_1(\Gamma)$, hence it defines a projective hypersurface X_Γ in $\mathbb{P}^{n-1}$ given by the set of zeros of $\Psi_\Gamma(t)$. These hypersurfaces are defined as varieties over $\mathbb{Z}$. Thus, it makes sense to consider the reductions $X_{\Gamma,p}$ modulo primes, and the counting of points over finite fields $\mathbb{F}_q$, $q = p^m$, for these varieties.

The "polynomial countability" condition for a variety X defined over $\mathbb{Z}$ is the property that the counting function $N_q(X) = \#X_q(\mathbb{F}_q)$ is a polynomial in q. If the motive of the variety X is mixed Tate, then the variety is polynomially countable (for all but finitely many primes) and assuming Tate's conjecture the converse would also hold. In between the motive and the counting function, one has a universal Euler characteristic, which is the class $[X]$ in the Grothendieck ring of varieties $K_0(\mathcal{V}_\mathbb{Z})$. Polynomial countability is implied by this class being a polynomial in the Lefschetz motive $\mathbb{L} = [\mathbb{A}^1]$, the class of the affine line, which is in turn implied by the motive of X being mixed Tate.

It was conjectured by Kontsevich that the graph hypersurfaces X_Γ would always be polynomially countable. This was originally verified for graphs with up to twelve edges in [18], but was later proven false in general by Belkale and Brosnan [19], using a powerful universality results for matroids with which they showed that the classes $[X_\Gamma]$ span a localization of the Grothendieck ring of varieties, hence they are generally not polynomially countable.

Although the result of Belkale and Brosnan [19] showed that the graph hypersurfaces are not in general mixed Tate, one also knows that there are many significant examples (all sufficiently small graphs and several infinite families of graphs) for which the varieties in fact happen to be mixed Tate. In the cases where the mixed Tate property holds, one can then ask a further question of whether these varieties admit an $\mathbb{F}_1$-structure.

In addition to the graph hypersurfaces, there are other algebraic varieties that are naturally associated to the parametric Feynman integrals and that were recently studied by Esnault and by Bloch, which, unlike the graph hypersurfaces, are *always* mixed Tate motives. It is then even more natural to ask for these varieties the question of whether (or when) they admit an $\mathbb{F}_1$-structure.

2.2. Parametric Feynman integrals revisited

We present quickly a variant of the usual derivation of the parametric Feynman integral (as in [17]) and we show that it leads to the construction of a class of algebraic varieties associated to graphs, which map naturally to the graph hypersurfaces. These varieties Λ_Γ were recently considered by Bloch [2] and by Esnault: it is shown in [2] that they are birational covers of the usual graph hypersurfaces whenever the latter are irreducible (that is, for all graphs not obtained by gluing two disjoint graphs at a vertex) and that, unlike the graph hypersurfaces, the Λ_Γ are always mixed Tate.

Recall that the circuit matrix of an oriented graph Γ with a choice of a basis $\{\ell_j\}$ of $H^1(\Gamma, \mathbb{Z})$ is the $\#E(\Gamma) \times b_1(\Gamma)$-matrix defined by

$$\eta_{e,j} = \begin{cases} \pm 1 & e \in \ell_j \text{ with same or opposite orientation} \\ 0 & e \notin \ell_j. \end{cases} \tag{2.2}$$

Theorem 2.1. *For a Feynman graph Γ, let $\mathcal{X}_\Gamma$ be the locus*

$$\mathcal{X}_\Gamma = \{(a, \beta, x) \mid a \in \Sigma_n,\ (a, \beta) \in \Lambda_\Gamma,\ x \in Y^c_{a,m}\}, \tag{2.3}$$

where Σ_n is the n-simplex,

$$Y_{a,m} = \{x \mid x^\tau Q_a(x) + m^2 = 0\} \tag{2.4}$$

and

$$\Lambda_\Gamma = \{(a, \beta) \mid Q_a(\beta) = 0\}, \tag{2.5}$$

with

$$(Q_a(\beta))_j = \sum_{e,r} a_e \eta_{e,r} \eta_{e,j} \beta_r, \tag{2.6}$$

and with $Y^c_{a,m}$ the hypersurface complement. Then the (unrenormalized) parametric Feynman integral $U(\Gamma)$ with trivial external momenta and non-zero mass is given by the integral

$$U(\Gamma) = \iint \frac{\delta\left(1 - \sum_e a_e\right) \delta(Q_a(\beta))}{(m^2 + x^\tau Q_a(x))^n}\, da_1 \cdots da_n\, \omega(x, \beta), \tag{2.7}$$

where $\omega(x, \beta)$ is the volume form on the fiber in $\mathcal{X}_\Gamma$ over a point $a \in \Sigma_n$.

Proof. In the usual computation of the parametric Feynman integral as given in [17], one starts with a Feynman integral, for a Feynman graph Γ, of the form specified by the Feynman rules, namely

$$\int \frac{\delta\left(\sum_{e\in E_{int}} \epsilon_{e,v} k_e + \sum_{e'\in E_{\rm ext}} \epsilon_{e',v} p_{e'}\right)}{q_1(k_1)\cdots q_n(k_n)} d^D k_1 \cdots d^D k_n,$$

where the constraint in the delta function is momentum conservation

$$\sum_{e\in E_{int}} \epsilon_{e,v} k_e + \sum_{e'\in E_{\rm ext}} \epsilon_{e',v} p_{e'} = 0 \tag{2.8}$$

at vertices, with assigned external momenta $p_{e'}$ and the quadrics $q_e(k_e) = k_e^2 + m^2$, and with D the spacetime dimension. The matrix $\epsilon_{e,v}$ is the incidence matrix of the graph Γ.

Then one performs a change of variables,

$$k_e = u_e + \sum_{j=1}^{\ell} \eta_{e,j} x_j, \tag{2.9}$$

where $\eta_{e,j}$ is the circuit matrix of the graph Γ and $\ell = b_1(\Gamma)$, subject to the constraint

$$\sum_e a_e u_e \eta_{e,j} = 0, \quad \forall j = 1, \dots, \ell. \tag{2.10}$$

The u_e are usually taken to be a *fixed choice* of a solution to the resulting equation

$$\sum_{e\in E_{int}} \epsilon_{e,v} u_e + \sum_{e'\in E_{\rm ext}} \epsilon_{e',v} p_{e'}, \tag{2.11}$$

which follows from (2.8) because of the orthogonality relation

$$\sum_e \epsilon_{e,v} \eta_{e,j} = 0, \quad \forall v \in V(\Gamma),\ \forall j = 1, \dots, \ell. \tag{2.12}$$

After applying the Feynman trick and performing this change of variables in the internal momenta, the Feynman integral becomes of the form

$$\iint \frac{\delta\left(1 - \sum_e a_e\right) \delta\left(\sum_{e\in E_{int}} \epsilon_{e,v} u_e + \sum_{e'\in E_{\rm ext}} \epsilon_{e',v} p_{e'}\right)}{\left(\sum_e a_e(u_e^2 + m^2) + x^\tau Q_a(x)\right)^n} da_1 \cdots da_n\, d^D x_1 \cdots d^D x_\ell.$$

There are no mixed terms in the denominator because of the constraint (2.10).

If one looks at this same setting, with the change of variables (2.9) and the constraint (2.10), but setting the external momenta to zero (while keeping a non-zero mass), one finds that the u_e satisfy the momentum conservation

$$\sum_{e\in E_{int}} \epsilon_{e,v} u_e = 0,$$

that is, they are in the kernel of the incidence matrix. By the orthogonality relation (2.12), one can then express them in the form

$$u_e = \sum_{r=1}^{\ell} \eta_{e,r} \beta_r,$$

where the constraint (2.10) then becomes of the form

$$0 = \sum_{e,r} a_e \eta_{e,r} \eta_{e,j} \beta_r = Q_a(\beta). \tag{2.13}$$

While for many choices of a this equation only has the solution $\beta = 0$, which would give the usual choice $u_e = 0$ for zero external momenta, it makes sense here to consider all possible solutions, namely the locus given by the complete intersection variety Λ_Γ defined as in (2.5). This means that one replaces the usual Feynman amplitude with one where one integrates over all these possible solutions.

This means considering an amplitude of the form (2.7), where the n-form $\omega(x,\beta)$ is the form $d^Dk_1\cdots d^Dk_n$, expressed in terms of the dx_j and the $d\beta_j$, after the change of variables as above, and where, in the denominator, one has $\sum_e a_e m^2 = m^2$, because of the constraint that the a_e lie on the simplex, and $\sum_e a_e u_e^2 = 0$, because of having set the external momenta $p_{e'} = 0$. □

2.3. Varieties associated to Feynman integrals in momentum space

The variant of the parametric Feynman integral described in Theorem 2.1 above shows that there are other interesting algebraic varieties, besides the graph hypersurfaces X_Γ, naturally associated to Feynman graphs. We will focus here especially on the first two cases listed below.

2.3.1. The complete intersection Λ_Γ

This is the variety defined by (2.5). It was recently introduced and studied by Esnault and by Bloch, in relation to Hodge structures. It is known, [2], that Λ_Γ is always mixed Tate. Moreover, by writing the Kirchhoff polynomial $\Psi_\Gamma(a) = \det(Q_a)$, it is shown in [2] that the variety $\Lambda_\Gamma \subset X_\Gamma \times \mathbb{P}^{b_1(\Gamma)-1}$, with a projection $\Lambda_\Gamma \to X_\Gamma$ that is a birational map, when X_Γ is irreducible (when Γ is not a union of two disjoint graphs glued together at a vertex).

2.3.2. The hyperplane arrangement

The construction of the variety Λ_Γ is closely related to a hyperplane arrangement $\mathcal{A}_\Gamma$ associated to the graph Γ, given by the collection of hyperplanes

$$H_e = \{\beta \mid \eta_{e,j}\beta_j = 0\}, \tag{2.14}$$

in the rational vector space $\mathbb{Q}^{b_1(\Gamma)} = H^1(\Gamma, \mathbb{Q})$.

2.3.3. The hypersurfaces $Y_{a,m}$

These are the varieties defined by (2.4). The presence of the non-trivial mass parameter $m \neq 0$ means that one can think of the $Y_{a,m}$ as deformations of the varieties defined by the equation $x^\tau Q_a(x) = 0$.

2.4. Varieties associated to Feynman integrals in configuration space

Another interesting algebro-geometric setting for perturbative quantum field theory arises when one considers Feynman integral calculations in configuration space rather than momentum space. We refer the reader to the two recent papers [3,4] for a detailed discussion of this setting and for the main results. We simply recall here what are the algebraic varieties that arise in this context.

One starts with a smooth projective variety X, which is usually assumed to be $X = \mathbb{P}^D$, with D the spacetime dimension, but that can be taken more general. Given a Feynman graph Γ, inside the product $X^{\#V(\Gamma)}$, with $V(\Gamma)$ the set of vertices of Γ, one considers the diagonals $\Delta_e = \{x_{s(e)} - x_{t(e)} = 0\}$ and one defines the graph configuration space to be

$$\mathrm{Conf}_\Gamma(X) = X^{\#V(\Gamma)} \smallsetminus \cup_e \Delta_e. \tag{2.15}$$

This variety has a wonderful compactification $\overline{\mathrm{Conf}}_\Gamma(X)$, in the sense of [20–22], whose geometric and motivic properties were studied in detail in [3]. It is obtained from $X^{\#V(\Gamma)}$ by an iterated sequence of blowups along diagonals. This gives a completely explicit description of the motive, [21,3]. The varieties $\overline{\mathrm{Conf}}_\Gamma(X)$ include the Fulton–MacPherson compactifications, as the case of the complete graph.

3. Constraints from the Euler characteristic and the Grothendieck class

In this section, we start with a very naive point of view on $\mathbb{F}_1$-geometry, based on the original intuition based on the existence of a good limiting behavior (1.2) for the counting of the number of points over finite field, when one lets $q \to 1$.

By taking this simple point of view, one can derive necessary (but generally far from sufficient) conditions for the existence of an $\mathbb{F}_1$-structure. The simplest such necessary condition can be expressed in terms of constraints on the Euler characteristic.

We begin by recalling some general facts about the Grothendieck ring of varieties and the Euler characteristic. Then we discuss some specific cases of graph hypersurfaces X_Γ (or their hypersurface complements Y_Γ), and of varieties Λ_Γ and $\mathcal{A}_\Gamma$, associated to Feynman graphs.

3.1. The Grothendieck ring of varieties

We will consider here the case where R is either $\mathbb{Z}$ or $\mathbb{Q}$ of $\mathbb{C}$ or $\mathbb{F}_q$. We denote by $\mathcal{V}_R$ the category of quasi-projective varieties defined over R and by $K_0(\mathcal{V}_R)$ the Grothendieck ring of varieties. The latter is constructed by first considering the abelian group of $\mathbb{Z}$-linear combinations of isomorphism classes $[X]$ of varieties $X \in \mathcal{V}_R$ and then modding out by the subgroup generated by all elements of the form $[X] - [Y] - [X \smallsetminus Y]$, where $Y \subseteq X$ is a closed subvariety of X. This is known as the scissor-congruence or inclusion–exclusion relations. Moreover, $K_0(\mathcal{V}_R)$ is made into a ring by defining the product of two classes $[X]$ and $[Y]$ to be the class of the product of X and Y over Spec(R), that is $[X][Y] = [X \times Y]$, and then extending it by linearity. The classes $[X]$ in the Grothendieck ring are also referred to as "virtual motives". The Lefschetz motive is the class $\mathbb{L} = [\mathbb{A}^1]$ and one refers to the subring $\mathbb{Z}[\mathbb{L}] \subset K_0(\mathcal{V}_R)$ as the subring of virtual Tate motives. Thus, we say that a variety X is a virtual Tate motive if its class $[X] = P(\mathbb{L})$ for a polynomial P with $\mathbb{Z}$ coefficients, that is, if $[X] \in \mathbb{Z}[\mathbb{L}]$.

3.2. Additive invariants and the Euler characteristic

The importance of the Grothendieck ring becomes apparent when studying the properties of "additive invariants" of algebraic varieties, the prototype example of which is the Euler characteristic. An additive invariant is a map $\chi : \mathcal{V}_R \to S$, with S a commutative ring, with the following properties:

(1) Isomorphism invariance: $\chi(X) = \chi(Y)$ for all $X \cong Y$.
(2) Inclusion–exclusion: $\chi(X) = \chi(Y) + \chi(X \setminus Y)$, whenever $Y \subseteq X$ is a closed subvariety.
(3) Multiplicative: $\chi(X \times Y) = \chi(X)\chi(Y)$.

Examples of additive invariants are the topological Euler characteristic when $R = \mathbb{Q}$ or $\mathbb{C}$, the Deligne–Hodge polynomial when $R = \mathbb{C}$, or the counting of points over finite fields when $R = \mathbb{Z}$ or $\mathbb{F}_q$.

Note that the standard terminology "additive" refers to the behavior with respect to inclusion–exclusion, but in fact these invariants are also required to be multiplicative with respect to products, so that they are compatible with the ring structure and not just with the group structure in the Grothendieck ring.

The Grothendieck ring is in fact universal with respect to additive invariants. That is, for any additive invariant $\chi : \mathcal{V}_R \to S$, there exists a unique ring homomorphism $\chi_* : K_0(\mathcal{V}_R) \to S$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V}_R & \xrightarrow{\pi} & K_0(\mathcal{V}_R) \\ & {\scriptstyle\chi}\searrow & \vdots\, {\scriptstyle\chi_*} \\ & & S \end{array}$$

where π is the map $X \mapsto [X]$. Thus, one can think of the Grothendieck class $[X]$ as a "universal Euler characteristic" for the variety X.

3.3. The Euler characteristic and $\mathbb{F}_1$-points

The following observation links the Euler characteristic to a necessary condition for the existence of an $\mathbb{F}_1$-structure over a variety defined over $\mathbb{Z}$ (see [23,24]).

Proposition 3.1. *Let $X \in \mathcal{V}_{\mathbb{Z}}$ be a variety over $\mathbb{Z}$ such that $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$ is a virtual Tate motive. Then*

$$\chi(X_{\mathbb{Q}}) = \lim_{q \to 1} N_q(X), \tag{3.1}$$

where $X_{\mathbb{Q}}$ is a model over $\mathbb{Q}$ of X. Thus, a necessary condition for the existence of an $\mathbb{F}_1$-structure over X is that $\chi(X_{\mathbb{Q}}) \geq 0$.

Proof. The assumption that $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$ is a virtual Tate motive means that there exists a polynomial $P(t) \in \mathbb{Z}[t]$ such that $[X] = P(\mathbb{L})$. This implies that, for all but finitely many primes, the counting function satisfies $N_q(X) = P(q)$, since the counting is an additive invariant with $N_q(\mathbb{A}^1) = N_q(\mathbb{L}) = q$. On the other hand, the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{Z}})$ maps to the Grothendieck ring $K_0(\mathcal{V}_{\mathbb{Q}})$ with $\mathbb{L}$ mapping to $\mathbb{L}$, so that we still have $[X] = P(\mathbb{L})$. The Euler characteristic satisfies $\chi(\mathbb{L}) = 1$ and is a ring homomorphism $\chi : K_0(\mathcal{V}_{\mathbb{Q}}) \to \mathbb{Z}$, so that we obtain $\chi(X) = P(1)$, which is the limit of the values $P(q)$ as $q \to 1$. Since the limit of $N_q(X)$ as $q \to 1$ is interpreted as the counting of points over $\mathbb{F}_1$, a necessary condition for the existence of an $\mathbb{F}_1$-structure is that this counting is non-negative. □

Regardless of the particular flavor of $\mathbb{F}_1$-geometry one chooses to work with [6], the existence of an $\mathbb{F}_1$-structure on a variety defined over $\mathbb{Z}$ implies that the motive of the variety X is mixed Tate. In the following we will restrict our attention to the formulation of $\mathbb{F}_1$-structures in terms of torifications given in [7] and we will show how that implies that $[X] \in K_0(\mathcal{V}_{\mathbb{Z}})$ is a virtual Tate motive, with a condition on the coefficients (see Lemma 3.8 below), which implies the necessary condition that $\chi(X_{\mathbb{Q}}) \geq 0$ as above.

3.4. The Euler characteristic of graph hypersurfaces

It is often convenient to switch between the projective hypersurfaces $X_\Gamma \subset \mathbb{P}^{n-1}$ and their affine cones $\hat{X}_\Gamma \subset \mathbb{A}^n$, namely the affine hypersurfaces $\hat{X}_\Gamma$ defined by the vanishing of the same polynomial Ψ_Γ in affine space $\mathbb{A}^n$.

Lemma 3.2. *The class* $[X_\Gamma]$ *is a virtual Tate motive if and only if the class* $[\hat{X}_\Gamma]$ *is a polynomial in* $\mathbb{L}$ *divisible by* $(\mathbb{L}-1)$.

Proof. It is easy to check (see [25]) that the classes in the Grothendieck ring are related by the simple relation,

$$[X_\Gamma] = \frac{[\hat{X}_\Gamma]-1}{\mathbb{L}-1}. \tag{3.2}$$

The statement then follows immediately. □

We say that an edge e in the graph Γ is a bridge if the removal of e disconnects Γ (in particular Γ is not a 1PI graph) and we say that e is a looping edge if the two endpoints of e coincide with the same vertex of Γ.

It was shown in [26] (see also [18] for a formulation in terms of counting functions) that the classes of the affine graph hypersurface complements satisfy a deletion–contraction relation of the following form:

$$\begin{aligned}
&[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = \mathbb{L}\cdot[\mathbb{A}^{n-1}\smallsetminus \hat{X}_{\Gamma\smallsetminus e}] = \mathbb{L}\cdot[\mathbb{A}^{n-1}\smallsetminus \hat{X}_{\Gamma/e}], \quad e \text{ bridge}\\
&[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = (\mathbb{L}-1)[\mathbb{A}^{n-1}\smallsetminus \hat{X}_{\Gamma\smallsetminus e}] = (\mathbb{L}-1)[\mathbb{A}^{n-1}\smallsetminus \hat{X}_{\Gamma/e}] \quad e \text{ looping edge}\\
&[\mathbb{A}^n \smallsetminus \hat{X}_\Gamma] = \mathbb{L}\cdot[\mathbb{A}^{n-1}\smallsetminus (\hat{X}_{\Gamma\smallsetminus e}\cap \hat{X}_{\Gamma/e})] - [\mathbb{A}^{n-1}\smallsetminus \hat{X}_{\Gamma\smallsetminus e}], \quad \text{otherwise.}
\end{aligned} \tag{3.3}$$

Proposition 3.3. *If* Γ *is not a forest and has at least one bridge or one looping edge, then* $\chi(X_\Gamma) = n$.

Proof. If e is a bridge in Γ, and Γ is not a forest, then (as observed in [25,27]) the hypersurface complement $\mathbb{P}^{n-2}\smallsetminus X_{\Gamma\smallsetminus e}$ is a $\mathbb{G}_m$-bundle over the product $(\mathbb{P}^{n_1-1}\smallsetminus X_{\Gamma_1}) \times (\mathbb{P}^{n_2-1}\smallsetminus X_{\Gamma_2})$, where $\Gamma\smallsetminus e = \Gamma_1 \amalg \Gamma_2$ with $n_i = \#E(\Gamma_i)$ for $i = 1, 2$. Thus, $[Y_{\Gamma\smallsetminus e}] = (\mathbb{L}-1)[Y_{\Gamma_1}][Y_{\Gamma_2}]$, since in the Grothendieck ring the class of the $\mathbb{G}_m$-bundle is equal to the class of the product of base and fiber. Moreover, from the deletion–contraction relation (3.3) for the bridge case and the formula (3.2) relating the classes of the affine and projective hypersurfaces we obtain

$$[X_\Gamma] = \mathbb{L}[X_{\Gamma\smallsetminus e}] + 1,$$

from which we obtain

$$[Y_\Gamma] = \sum_{k=0}^{n-1}\mathbb{L}^k - [X_\Gamma] = \sum_{k=0}^{n-1}\mathbb{L}^k - 1 - \mathbb{L}[X_{\Gamma\smallsetminus e}] = \mathbb{L}\left(\sum_{k=0}^{n-2}\mathbb{L}^k - [X_{\Gamma\smallsetminus e}]\right) = \mathbb{L}[Y_{\Gamma\smallsetminus e}].$$

Thus, we have $[Y_\Gamma] = \mathbb{L}(\mathbb{L}-1)[Y_{\Gamma_1}][Y_{\Gamma_2}]$, which implies, from the fact that the Euler characteristic is a ring homomorphism and that $\chi(\mathbb{L}) = 1$, that $\chi(Y_\Gamma) = 0$, hence $\chi(X_\Gamma) = \chi(\mathbb{P}^{n-1}) = n$.

If e is a looping edge in the graph, then we can rewrite the deletion–contraction formula (3.3) in the equivalent form

$$[\hat{X}_\Gamma] = \mathbb{L}^{n-1} + (\mathbb{L}-1)[\hat{X}_{\Gamma/e}], \tag{3.4}$$

just by writing out the above as $\mathbb{L}^n - [\hat{X}_\Gamma] = \mathbb{L}^n - \mathbb{L}^{n-1} - \mathbb{L}[\hat{X}_{\Gamma/e}] + [\hat{X}_{\Gamma/e}]$. Using then the relation (3.2) between the classes of the affine and projective hypersurfaces, we can rewrite this as

$$[X_\Gamma] = (\mathbb{L}-1)[X_{\Gamma/e}] + 1 + \frac{\mathbb{L}^{n-1}-1}{\mathbb{L}-1}. \tag{3.5}$$

We then use the fact that the Euler characteristic is a ring homomorphism and that $(\mathbb{L}^{n-1}-1)/(\mathbb{L}-1) = 1+\mathbb{L}+\cdots+\mathbb{L}^{n-2}$, so that we obtain $\chi(X_\Gamma) = (\chi(\mathbb{L})-1)\chi(X_{\Gamma/e}) + 1 + \sum_{k=0}^{n-2}\chi(\mathbb{L}^k)$. Since $\chi(\mathbb{L}) = 1$, we obtain $\chi(X_\Gamma) = 1 + (n-1) = n$. □

This explains the frequent occurrence of the value $\chi(Y_\Gamma) = 0$ among sufficiently small graphs, as many have either bridges or looping edges. It also shows that all these graphs satisfy the necessary condition of Proposition 3.1 on the positivity of the Euler characteristic. For graphs that have no bridges and no looping edges, as observed in [26], the deletion–contraction relation only provides a constraint on the Euler characteristic of the form

$$\chi(X_\Gamma) = n + \chi(X_{\Gamma/e}\cap X_{\Gamma\smallsetminus e}) - \chi(X_{\Gamma\smallsetminus e}), \tag{3.6}$$

if the graph has $b_2(\Gamma) \geq 2$, and $\chi(X_\Gamma) = n-1$ when $b_1(\Gamma) = 1$. The latter is the hyperplane case, which is an example giving an occurrence of the value $+1$ for $\chi(Y_\Gamma)$.

For example, the Euler characteristic was computed explicitly in [25] in the case of the banana graphs (graphs with two vertices and n parallel edges between them), where one has, for $n \geq 3$,

$$\chi(X_{\Gamma_n}) = n + (-1)^n,$$

and, by (3.6), one finds $\chi(X_{\Gamma_n/e}\cap X_{\Gamma_n\smallsetminus e}) = n + (-1)^n - n + n - 1 + (-1)^{n-1} = n-1$.

Remark 3.4. The banana graphs Γ_n with n even and $n \geq 3$ give examples where $\chi(Y_{\Gamma_n}) = -1$. Thus, in this case, while it is known from [25] that $Y_{\Gamma_n} = \mathbb{P}^{n-1} \smallsetminus X_{\Gamma_n}$ is a virtual Tate motive, it violates the Euler characteristic condition of Proposition 3.1 for the existence of an $\mathbb{F}_1$-structure. The hypersurface X_{Γ_n} instead is a virtual Tate motive that satisfies the condition of Proposition 3.1.

In all the cases listed above (see Proposition 3.3) where $\chi(X_\Gamma) = n$, the projective hypersurface complement $Y_\Gamma = \mathbb{P}^{n-1} \smallsetminus X_\Gamma$ satisfies $\chi(Y_\Gamma) = 0$. In such cases, assuming that the varieties are virtual Tate motives, the "number of points over $\mathbb{F}_1$" is given by the limit

$$N_1(Y_\Gamma) = \lim_{q\to 1} \frac{P_{Y_\Gamma}(q)}{(q-1)^{r_\Gamma}}, \tag{3.7}$$

where $[Y_\Gamma] = P_{Y_\Gamma}(\mathbb{L}) \in K_0(\mathcal{V}_\mathbb{Z})$, for $P_{Y_\Gamma}(t) \in \mathbb{Z}[t]$ and r_Γ is the order of vanishing of $P_{Y_\Gamma}(q)$ when $q \to 1$.

Corollary 3.5. *If Γ is not a forest and it has either a bridge or a looping edge, then the order of vanishing r_Γ in (3.7) is equal to $r_\Gamma = 1 + s_\Gamma$ where s_Γ is as follows.*

(1) *If e is a bridge, then $(\mathbb{L}-1)^{s_\Gamma}$ is the maximal power of $\mathbb{L}-1$ that divides the product $[Y_{\Gamma_1}][Y_{\Gamma_2}]$, with $\Gamma \smallsetminus e = \Gamma_1 \amalg \Gamma_2$.*
(2) *If e is a looping edge, then $(\mathbb{L}-1)^{s_\Gamma}$ is the maximal power of $\mathbb{L}-1$ that divides $[Y_{\Gamma/e}]$.*

Proof. (1) If e is a bridge, then we use, as in Proposition 3.3 the fact that $[Y_\Gamma] = \mathbb{L}[Y_{\Gamma\smallsetminus e}] = \mathbb{L}(\mathbb{L}-1)[Y_{\Gamma_1}][Y_{\Gamma_2}]$, which gives the result for r_Γ. (2) Similarly, if e is a looping edge, then we have the expression (3.5) for the class $[X_\Gamma]$ as in Proposition 3.3, which gives $[X_\Gamma] = (\mathbb{L}-1)[X_{\Gamma/e}] + 1 + \sum_{k=0}^{n-2} \mathbb{L}^k$. We then get

$$[Y_\Gamma] = \sum_{k=0}^{n-1} \mathbb{L}^k - [X_\Gamma] = \mathbb{L}^{n-1} - 1 - (\mathbb{L}-1)[X_{\Gamma/e}]$$

$$= (\mathbb{L}-1)\left(\sum_{k=0}^{n-2} \mathbb{L}^k - [X_{\Gamma/e}]\right) = (\mathbb{L}-1)[Y_{\Gamma/e}].$$

So we again obtain the value of r_Γ as stated. □

The very simple instances considered here above all lead to values of $\chi(Y_\Gamma)$ that are 0, +1 or −1. In fact, this is conjectured to be the always the case.

Conjecture 3.6 (*Aluffi*)**.** *The graph hypersurface complements Y_Γ always have Euler characteristic $\chi(Y_\Gamma)$ which is either* 0 *or* 1 *or* −1.

This conjecture was so far confirmed by computer calculations on a significant number of sufficiently small graphs [28], and for some other classes of graphs, using techniques from [29]. If this conjecture holds, then the Euler characteristics of the graph hypersurfaces X_Γ themselves would always be non-negative, so this simple necessary condition would always be satisfied.

3.5. Constraints from the Grothendieck class

We now look at the existence of $\mathbb{F}_1$-structures on $\mathbb{Z}$-varieties from a more sophisticated point of view, which is the one developed in the work of Javier López Peña and Oliver Lorscheid [7], based on the existence of affine torifications. We show that this imposes a very simple necessary condition on the form of the class in the Grothendieck ring, which we then discuss in some cases of varieties of Feynman graphs.

Definition 3.7 (*López Peña and Lorscheid [7]*)**.** A torified scheme is a scheme X together with a disjoint union of tori $T = \amalg_j \mathbb{G}_m^{d_j}$ and a morphism of schemes $e_X : T \to X$, such that the restriction $e_X|_{\mathbb{G}_m^{d_j}}$ to each torus is an embedding and the morphism induces bijections $e_X(k) : T(k) \to X(k)$, for every field k.

Lemma 3.8. *Let X be a smooth quasi-projective variety defined over $\mathbb{Z}$, which admits a torification $e_X : T \to X$ as above. Then the class $[X]$ in the Grothendieck ring can be written as a polynomial $[X] = \sum_{k\geq 0} a_k \mathbb{T}^k$, in the class $\mathbb{T} = [\mathbb{G}_m] = \mathbb{L}-1$, with non-negative integer coefficients $a_k \geq 0$.*

Proof. The torification condition implies that the counting function (1.1) satisfies $N_X(q) = \sum_j N_{\mathbb{G}_m^{d_j}}(q)$. The Tate conjecture predicts that the numbers $N_X(q)$ determine the motive of X, hence its class in the Grothendieck ring, which should therefore be of the form $[X] = \sum_j [\mathbb{G}_m^{d_j}] = \sum_j \mathbb{T}^{d_j}$. □

Thus, the form of the class in the Grothendieck ring gives a second necessary condition for the existence of an $\mathbb{F}_1$-structure, more refined than the condition on the Euler characteristic. Notice that, if the class of X in the Grothendieck ring has an expression of the form $[X] = \sum_k a_k \mathbb{T}^k$, then the Euler characteristic $\chi(X) = a_0$, since all the positive dimensional tori have trivial Euler characteristic.

3.6. The case of graph hypersurfaces

We show in some examples the condition of Lemma 3.8 for graph hypersurfaces. As the first example, we consider the banana graphs, for which the class $[X_\Gamma]$ in the Grothendieck ring was computed explicitly in [25].

Lemma 3.9. *Let Γ_n be the banana graph with two vertices and n parallel edges between them. The class $[X_{\Gamma_n}]$ satisfies the condition of Lemma 3.8, while the class of the complement $[Y_{\Gamma_n}] = [\mathbb{P}^{n-1} \smallsetminus X_{\Gamma_n}]$ does not.*

Proof. The class in the Grothendieck ring is a function of $\mathbb{T}$ of the form (Theorem 3.10 of [25]):

$$[X_{\Gamma_n}] = \frac{(1+\mathbb{T})^n - 1}{\mathbb{T}} - \frac{\mathbb{T}^n - (-1)^n}{\mathbb{T}+1} - n\mathbb{T}^{n-2}. \tag{3.8}$$

While it appears at first that this expression may contain negative coefficients of some powers of $\mathbb{T}$, one can check that in fact it does satisfy the necessary condition of Lemma 3.8. For example, for $n = 15$, we find

$$\begin{aligned}[X_{\Gamma_{15}}] = \ & 14 + 106\,\mathbb{T} + 454\,\mathbb{T}^2 + 1366\,\mathbb{T}^3 + 3002\,\mathbb{T}^4 + 5006\,\mathbb{T}^5 + 6434\,\mathbb{T}^6 + 6436\,\mathbb{T}^7 + 5004\,\mathbb{T}^8 + 3004\,\mathbb{T}^9 \\ & + 1364\,\mathbb{T}^{10} + 456\,\mathbb{T}^{11} + 104\,\mathbb{T}^{12} + \mathbb{T}^{13}.\end{aligned}$$

In fact, as already observed in [25], we can write the first term in (3.8) as

$$[\mathbb{P}^{n-1}] = \frac{\mathbb{L}^n - 1}{\mathbb{L}-1} = \frac{(1+\mathbb{T})^n - 1}{\mathbb{T}} = \sum_{k=1}^{n} \binom{n}{k} \mathbb{T}^{k-1}, \tag{3.9}$$

while the second term is of the form

$$-\left(\frac{\mathbb{T}^n - (-1)^n}{\mathbb{T}+1} + n\mathbb{T}^{n-2}\right) = -(\mathbb{T}^{n-1} + (n-1)\mathbb{T}^{n-2} + \mathbb{T}^{n-3} - \mathbb{T}^{n-4} + \cdots + \pm 1). \tag{3.10}$$

Notice then that the coefficients of (3.9) are always greater than or equal than the coefficients of the corresponding powers of $\mathbb{T}$ being subtracted in (3.10), since the coefficient of $\mathbb{T}^r$ in (3.9) greater than or equal to one, in all cases, and to $n-1$ in the case of $\mathbb{T}^{n-2}$. The class of the hypersurface complement, on the other hand, is given by (Corollary 3.13 of [25])

$$[Y_{\Gamma_n}] = \mathbb{T}^{n-1} + (n-1)\mathbb{T}^{n-2} + \mathbb{T}^{n-3} - \mathbb{T}^{n-4} + \cdots + \pm 1,$$

which has some negative coefficients. □

The deletion–contraction relation for classes in the Grothendieck ring of graph hypersurface complements, proved in [26] and recalled here above in (3.3), has as consequence certain recursion relations for the Grothendieck classes for graphs that are obtained from one another by iterating certain simple operations, such as edge splitting or edge doubling. These recursions were derived in [26] and can be used to obtain examples of graphs for which the class $[\hat{Y}_\Gamma]$ of the affine graph hypersurface complement satisfies the condition of Lemma 3.8. For example, Proposition 5.11 of [26] shows that the *lemon wedge graph Λ_m* with m sections gives

$$[\hat{Y}_{\Lambda_m}] = (\mathbb{T}+1)^{m+1} \sum_{j=0}^{m} \binom{m-j}{j} \mathbb{T}^{m-j}. \tag{3.11}$$

Question 3.10. *Do graph hypersurfaces X_Γ, $\hat{X}_\Gamma$ (or hypersurface complements Y_Γ, $\hat{Y}_\Gamma$) that satisfy the condition of Lemma 3.8 admit a torification? Do they admit an affine torification? Are there natural conditions on the graphs, or interesting families of graphs for which this is the case?*

In relation to this question, one can see for example that there are other interesting graph hypersurfaces that satisfy the necessary condition of Lemma 3.8. For example, Brown and Schnetz computed explicitly the classes in the Grothendieck ring for wheels with n spokes graphs and for zig-zag graphs, [30]. We see, for example, that when we rewrite as functions of $\mathbb{T} = \mathbb{L} - 1$ the classes $w_n = [W_n]$ of Proposition 49 of [30], for the wheel graphs W_n, we obtain

$$w_3 = 1 + 6\,\mathbb{T} + 12\,\mathbb{T}^2 + 11\,\mathbb{T}^3 + 5\,\mathbb{T}^4 + \mathbb{T}^5$$

$$w_4 = 1 + 8\,\mathbb{T} + 26\,\mathbb{T}^2 + 45\,\mathbb{T}^3 + 44\,\mathbb{T}^4 + 24\,\mathbb{T}^5 + 7\,\mathbb{T}^6 + \mathbb{T}^7$$

$$w_5 = 1 + 10\,\mathbb{T} + 45\,\mathbb{T}^2 + 115\,\mathbb{T}^3 + 180\,\mathbb{T}^4 + 178\,\mathbb{T}^5 + 111\,\mathbb{T}^6 + 42\,\mathbb{T}^7 + 9\,\mathbb{T}^8 + \mathbb{T}^9$$

$$w_6 = 1 + 12\,\mathbb{T} + 66\,\mathbb{T}^2 + 218\,\mathbb{T}^3 + 474\,\mathbb{T}^4 + 703\,\mathbb{T}^5 + 716\,\mathbb{T}^6 + 495\,\mathbb{T}^7 + 226\,\mathbb{T}^8 + 65\,\mathbb{T}^9 + 11\,\mathbb{T}^{10} + \mathbb{T}^{11}$$

$$\begin{aligned}w_7 = \ & 1 + 14\,\mathbb{T} + 91\,\mathbb{T}^2 + 364\,\mathbb{T}^3 + 994\,\mathbb{T}^4 + 1939\,\mathbb{T}^5 + 2751\,\mathbb{T}^6 + 2846\,\mathbb{T}^7 + 2126\,\mathbb{T}^8 + 1121\,\mathbb{T}^9 \\ & + 402\,\mathbb{T}^{10} + 93\,\mathbb{T}^{11} + 13\,\mathbb{T}^{12} + \mathbb{T}^{13}\end{aligned}$$

which satisfy the positivity condition of Lemma 3.8. Similarly, the condition is satisfied in the case of the zig-zag graphs, with the classes z_n given by

$$z_3 = 1 + 6\,\mathbb{T} + 12\,\mathbb{T}^2 + 11\,\mathbb{T}^3 + 5\,\mathbb{T}^4 + \mathbb{T}^5$$
$$z_4 = 1 + 8\,\mathbb{T} + 26\,\mathbb{T}^2 + 45\,\mathbb{T}^3 + 44\,\mathbb{T}^4 + 24\,\mathbb{T}^5 + 7\,\mathbb{T}^6 + \mathbb{T}^7$$
$$z_5 = 1 + 10\,\mathbb{T} + 45\,\mathbb{T}^2 + 116\,\mathbb{T}^3 + 182\,\mathbb{T}^4 + 178\,\mathbb{T}^5 + 109\,\mathbb{T}^6 + 41\,\mathbb{T}^7 + 9\,\mathbb{T}^8 + \mathbb{T}^9$$
$$z_6 = 1 + 12\,\mathbb{T} + 66\,\mathbb{T}^2 + 221\,\mathbb{T}^3 + 484\,\mathbb{T}^4 + 714\,\mathbb{T}^5 + 716\,\mathbb{T}^6 + 484\,\mathbb{T}^7 + 216\,\mathbb{T}^8 + 62\,\mathbb{T}^9 + 11\,\mathbb{T}^{10} + \mathbb{T}^{11}$$
$$z_7 = 1 + 14\,\mathbb{T} + 91\,\mathbb{T}^2 + 368\,\mathbb{T}^3 + 1014\,\mathbb{T}^4 + 1986\,\mathbb{T}^5 + 2807\,\mathbb{T}^6 + 2860\,\mathbb{T}^7 + 2080\,\mathbb{T}^8 + 1062\,\mathbb{T}^9 + 372\,\mathbb{T}^{10} + 87\,\mathbb{T}^{11} + 13\,\mathbb{T}^{12} + \mathbb{T}^{13}.$$

One can similarly see, from (21) of [30], that the classes b_n for the series–parallel graphs also satisfy the condition of Lemma 3.8, by expanding the powers of $\mathbb{L}$ on the right-hand-side of the equation in powers of $\mathbb{T}$ and binomial coefficients.

Recent work of Müller-Stach and Westrich [31] also provides new methods that can be used to address Question 3.10 for some classes of graphs. Namely, in [31] they identify a combinatorial condition on the graphs that ensure the existence of a linear faithful torus action on the graph hypersurface. The condition is satisfies, for instance, by a class of "polygonal graphs" (graphs obtained as successive gluing of polygons along non-repeated connected subsets of edges), which is a $\star$-graph, namely the set of edges used for gluing has trivial b_1 (hence $b_1(\Gamma)$ equals the number of polygons). This class of graphs include some of the examples discussed above, such as the wheels with n-spokes. The existence of a faithful torus action gives rise to a torification of the variety given by the torus orbits. Thus, we can state the following result, which includes the case of the wheels with n-spokes.

Proposition 3.11. *Let Γ be a polygonal $\star$-graph, as above. Then the graph hypersurface X_Γ admits a torification.*

Proof. Proposition 2.4 of [31] shows that the combinatorial condition that Γ is a polygonal $\star$-graph implies that the non-zero upper triangular matrix entries in the matrix $(M_\Gamma(t))_{ij} = \sum_{k=1}^n t_k \eta_{ki}\eta_{kj}$ are linearly independent polynomials in $t = (t_1, \dots, t_n)$. In turn, as shown in Theorem 1.8 of [31], this condition implies the existence of a faithful linear torus action on the graph hypersurface $X_\Gamma = \{\det M_\Gamma = 0\}$ by a torus of rank $r \geq b_1(\Gamma) - 1 + n - \ell(\Gamma)$ where $\ell(\Gamma)$ is the number of non-zero entries in the upper triangle of M_Γ. The orbits of this torus action then determine a torification of X_Γ. □

3.7. The complete intersection Λ_Γ

We now discuss briefly the varieties Λ_Γ associated to Feynman graphs. One knows [2] that, unlike the graph hypersurfaces, these are mixed Tate for all graphs Γ. One can then ask whether the necessary conditions for the existence of $\mathbb{F}_1$-structures coming from the Euler characteristic and the Grothendieck class are satisfied in this case.

It is shown in [2] that the varieties Λ_Γ have a stratification defined by sets $S_\Gamma^{(m)} \subset \mathbb{P}^{b_1(\Gamma)-1}$ of the form $S_\Gamma^{(m)} = T_\Gamma^{(m)} \smallsetminus T_\Gamma^{(m+1)}$, with each $T_\Gamma^{(m)}$ a union of linear subspaces, which is defined as the locus of $\{\beta \in \mathbb{P}^{b_1(\Gamma)-1} \mid \epsilon(\beta) \geq m\}$, where $\epsilon(\beta)$ is the codimension of the span of the $\eta_{e,\cdot}$ with $\sum_i \eta_{e,i}\beta_i \neq 0$ in $\mathbb{Q}^{b_1(\Gamma)}$. The restriction $\Lambda_\Gamma|_{S_\Gamma^{(m)}}$ is a projective bundle over $S_\Gamma^{(m)}$ with fiber $\mathbb{P}^{n-b_1(\Gamma)+m-1}$. Thus, the class in the Grothendieck ring decomposes as

$$[\Lambda_\Gamma] = \sum_m [S_\Gamma^{(m)}]\,[\mathbb{P}^{n-b_1(\Gamma)+m-1}].$$

Thus, to show that Λ_Γ satisfies the two necessary conditions, it suffices to know that $[S_\Gamma^{(m)}] = P_{\Gamma,m}(\mathbb{T})$, for a polynomial $P_{\Gamma,m}$ with non-negative coefficients, which also implies that $\chi(S_\Gamma^{(m)}) \geq 0$ for each stratum.

Question 3.12. *Do the complete intersections Λ_Γ admit an affine torification?*

There are explicit cases discussed in [2], such as the wheel with three spokes graph, where the variety Λ_Γ is a projective bundle over a projective space. This suggests that, at least for certain classes of graphs, the question above may be checked directly.

4. Graph configuration spaces and $\mathbb{F}_1$-geometry

We now consider the varieties associated to Feynman integrals in configuration spaces, as in [3,4]. We see that, unlike the case of the varieties of Feynman graphs in momentum space, we can say a lot more about $\mathbb{F}_1$-structures in the configuration space setting.

4.1. The Euler characteristic and the Grothendieck class

We recall the following notation from [3], see also [21,22]. We assume that Γ is a finite graph without looping edges (edges whose endpoints coincide).

Recall from [22] that a $\mathcal{G}_\Gamma$-*nest* is a collection $\{\gamma_1, \ldots, \gamma_\ell\}$ of biconnected induced subgraphs (induced meaning that the subgraph has all the same edges between the given subset of vertices as the original graph), with the property that any two subgraphs γ and γ' in the set either have $\gamma \cap \gamma' = \emptyset$, or $\gamma \cap \gamma' = \{v\}$, a single vertex, or $\gamma \subseteq \gamma'$ or $\gamma' \subseteq \gamma$. As in [22] we use the notation $M_{\mathcal{N}} := \{(\mu_\gamma)_{\Delta_\gamma \in \mathcal{G}_\Gamma} : 1 \le \mu_\gamma \le r_\gamma - 1,\ \mu_\gamma \in \mathbb{Z}\}$ with $r_\gamma = r_{\gamma,\mathcal{N}} := \dim(\cap_{\gamma' \in \mathcal{N}:\gamma' \subset \gamma} \Delta_{\gamma'}) - \dim \Delta_\gamma$ and $\|\mu\| := \sum_{\Delta_\gamma \in \mathcal{G}_\Gamma} \mu_\gamma$. We also denote by $\Gamma/\delta_{\mathcal{N}}(\Gamma)$ the quotient $\Gamma//(\gamma_1 \cup \cdots \cup \gamma_r)$, where $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$ is the $\mathcal{G}$-nest and the quotient double bar $\Gamma//\gamma$ means the graph obtained from Γ by shrinking each connected component of the subgraph γ to a (different) vertex.

The class in the Grothendieck ring of the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ was computed in [3]: it is explicitly of the form

$$[\overline{\mathrm{Conf}}_\Gamma(X)] = [X]^{\#V(\Gamma)} + \sum_{\mathcal{N} \in \mathcal{G}_\Gamma-\text{nests}} [X]^{\#V(\Gamma/\delta_{\mathcal{N}}(\Gamma))} \sum_{\mu \in M_{\mathcal{N}}} \mathbb{L}^{\|\mu\|}. \tag{4.1}$$

The necessary conditions for the existence of torifications, in terms of Euler characteristic and Grothendieck class, can be checked easily as follows.

Lemma 4.1. *Suppose that the smooth projective variety X is defined over $\mathbb{Z}$ and admits an affine torification, which gives it the structure of an $\mathbb{F}_1$-variety in the sense of [7]. Then the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ satisfy the necessary conditions for the existence of an $\mathbb{F}_1$-structure, $\chi(\overline{\mathrm{Conf}}_\Gamma(X)) \ge 0$ and $[\overline{\mathrm{Conf}}_\Gamma(X)] = \sum_{k\ge 0} a_k \mathbb{T}^k$ with $a_k \ge 0$.*

Proof. Both conditions follow immediately from the formula (4.1). In fact, we see that

$$\chi(\overline{\mathrm{Conf}}_\Gamma(X)) = \chi(X)^{\#V(\Gamma)} + \sum_{\mathcal{N} \in \mathcal{G}_\Gamma-\text{nests}} \chi(X)^{\#V(\Gamma/\delta_{\mathcal{N}}(\Gamma))} \# M_{\mathcal{N}}, \tag{4.2}$$

using $\chi(\mathbb{L}) = 1$ and the fact that χ is a ring homomorphism from the Grothendieck ring of varieties to $\mathbb{Z}$. The assumption on X implies that $\chi(X) \ge 0$, hence (4.2) also gives a non-negative number. For the second condition, the assumption on X implies that $[X] = \sum_{j\ge 0} b_j \mathbb{T}^j$, with all the coefficients $b_j \ge 0$. Moreover, all powers $\mathbb{L}^n$ can be written as a polynomial in $\mathbb{T}$ with non-negative coefficients, using the simple identity

$$\sum_{k=0}^{n} (\mathbb{L} - 1)^k \binom{n}{k} = \mathbb{L}^n.$$

It then follows that the second condition is also satisfied. □

4.2. Affine torifications

According to the approach to $\mathbb{F}_1$-geometry developed in [7], the existence of a torification is not in itself sufficient for the existence of an $\mathbb{F}_1$-structure on a variety defined over $\mathbb{Z}$. However, if the torification is in addition *affine* then one obtains from it an $\mathbb{F}_1$-*gadget* and an $\mathbb{F}_1$-variety, [7]. The $\mathbb{F}_1$-structure can in turn depend on the choice of the torification.

Definition 4.2 (*López Peña and Lorscheid [7]*)**.** The *affine* condition on a torification $e_X : T \to X$ consists of the requirement that there exists an affine covering $\{\mathcal{U}_\alpha\}$ of X that is compatible with the torification, in the sense that, for each affine open set $\mathcal{U}_\alpha$ in the covering, there is a subcollection of tori T_α in the torification T of X, such that the restriction $e_X|_{T_\alpha} : T_\alpha \to \mathcal{U}_\alpha$ is also a torification.

4.3. Affine torifications and toric bundles

As a preliminary step towards discussing blowups of $\mathbb{F}_1$-varieties, we show that torifications extend to equivariant projective bundles over toric varieties.

A vector bundle $\mathcal{E}$ over a smooth projective toric variety X is said to be equivariant (or toric) if the torus action on X lifts to an action on $\mathcal{E}$ that is linear on the fibers. The bundle $\mathcal{E}$ is said to have an equivariant structure. A result of [32] shows that $\mathcal{E}$ is equivariant if an only if the bundles $\gamma^*\mathcal{E}$ are all isomorphic to $\mathcal{E}$, for all elements γ in the torus T. For further characterizations and results on toric bundles see [33,34]. Vector bundles that are sums of line bundles admit an equivariant structure, and so do tangent and cotangent bundle.

Lemma 4.3. *Let X be a smooth projective toric variety and let $\mathcal{E}$ be an equivariant vector bundle on X. Then the projective bundle $\mathbb{P}(\mathcal{E})$ admits an affine torification.*

Proof. The equivariant structure on $\mathcal{E}$ induces a torus action on the projectivized bundle $\mathbb{P}(\mathcal{E})$, so that the projection map $\pi : \mathbb{P}(\mathcal{E}) \to X$ is equivariant. Thus, although $\mathbb{P}(\mathcal{E})$ is in general not itself a toric variety, the decomposition of $\mathbb{P}(\mathcal{E})$ into orbits of the torus action gives a torification of $\mathbb{P}(\mathcal{E})$. To see that one in fact has an affine torification, notice that the variety X itself has an affine open covering $\{\mathcal{U}_\sigma\}$ compatible with the torification, by Proposition 1.13 of [7], where the open sets $\mathcal{U}_\sigma$ are the ones associated to the cones σ. The vector bundle $\mathcal{E}$ is determined, as an equivariant bundle, by its restrictions $\mathcal{E}|_{\mathcal{U}_\sigma}$ and by the equivariant gluings over the intersections $\mathcal{U}_{\sigma_i} \cap \mathcal{U}_{\sigma_j}$. Moreover, as observed in [33], the restrictions $\mathcal{E}|_{\mathcal{U}_\sigma}$ decompose as sums of line bundles

$$\mathcal{E}|_{\mathcal{U}_\sigma} \simeq \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r.$$

In the case of a sum of line bundles, one knows that the projectivization $\mathbb{P}(\mathcal{E}|_{\mathcal{U}_\sigma})$ is in fact a toric variety, and this implies that the torification is affine. The equivariant gluing guarantees the compatibility of the torifications over the intersections $\mathcal{U}_{\sigma_i} \cap \mathcal{U}_{\sigma_j}$ so that one obtains an affine torification for $\mathbb{P}(\mathcal{E})$. □

4.4. $\mathbb{F}_1$-structures and blowups

We now consider conditions under which one can perform blowups of $\mathbb{F}_1$-varieties.

Lemma 4.4. *Let X be a torified smooth projective variety with an affine torification $e_X : T_X \to X$. Let $Y \subset X$ be a smooth subvariety which is toric, with an affine torification $e_X|_{T_Y} : T_Y \to Y$ that consists of a subcollection of tori T_Y in the torification T_X and which are orbits of the torus action on Y. Assume moreover that the normal bundle $\mathcal{N}_X(Y)$ has an equivariant structure. Then the blowup $\mathrm{Bl}_Y(X)$ of X along Y has an affine torification.*

Proof. The exceptional divisor E of the blowup can be identified with the projectivization $\mathbb{P}(\mathcal{N}_X(Y))$. Thus, by Lemma 4.3 this admits an affine torification. Moreover, the compatibility between the affine torification of the toric variety Y and the affine torification of the (possibly non-toric) ambient variety X imply that the affine torification obtained in this way on E is compatible with the affine torification of X, so that one obtains an affine torification of $\mathrm{Bl}_Y(X)$. □

4.5. Wonderful compactifications and $\mathbb{F}_1$-structures

We now consider the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$, in the case where $X = \mathbb{P}^D$, used in [3,4] to compute Feynman integrals in configuration space.

We recall briefly the blowup construction of $\overline{\mathrm{Conf}}_\Gamma(X)$, see [3,21,22]. Let $\mathcal{G}_{k,\Gamma} \subseteq \mathcal{G}_\Gamma$ be the subcollection of *biconnected* induced subgraphs with $\#V(\gamma) = k$. Let $Y_0 = X^{\#V(\Gamma)}$ and let Y_k be the blowup of Y_{k-1} along the (iterated) dominant transform of $\Delta_\gamma \in \mathcal{G}_{n-k+1,\Gamma}$. If Γ is itself biconnected, then Y_1 is the blowup of Y_0 along the deepest diagonal Δ_Γ, otherwise $Y_1 = Y_0$. (Similarly, we have $Y_k = Y_{k-1}$ whenever there are no biconnected induced subgraphs with exactly $n-k+1$ vertices.) The resulting sequence of blowups

$$Y_{n-1} \to \cdots \to Y_2 \to Y_1 \to X^{\#V(\Gamma)}, \tag{4.3}$$

where $n = \#V(\Gamma)$, does not depend on the order in which the blowups are performed, for $\gamma \in \mathcal{G}_{n-k+1,\Gamma}$, for a fixed k. The variety Y_{n-1} obtained in this way is the wonderful compactification $\overline{\mathrm{Conf}}_\Gamma(X)$.

Lemma 4.5. *Let X be a toric variety and let $\Delta_\gamma = \cap_{e\in E(\gamma)} \Delta_e$ be the diagonal of a biconnected induced subgraph $\gamma \subseteq \Gamma$ in $X^{\#V(\Gamma)}$. Then Δ_γ is toric and there is an affine torification of $X^{\#V(\Gamma)}$ that induces a compatible affine torification on Δ_γ.*

Proof. The product of toric varieties is toric, hence $X^{\#V(\Gamma)}$ is a toric variety. Since the graph γ is connected, we have (Lemma 1 of [3]) $\Delta_\gamma \simeq X^{\#V(\Gamma/\gamma)}$, so this is also a toric variety, and we can decompose $X^{\#V(\Gamma)} \cong X^{\#V(\Gamma/\gamma)} \times X^{\#V(\gamma)}$, with compatible affine torifications. □

We then have the following result.

Theorem 4.6. *Let $X = \mathbb{P}^D$. Then, for all Feynman graphs Γ the configuration spaces $\overline{\mathrm{Conf}}_\Gamma(X)$ admit an affine torification, hence they admit a structure of $\mathbb{F}_1$-variety.*

Proof. Recall that (Definition 2.7 of [22]) for a blowup $\pi : \mathrm{Bl}_Z(Y) \to Y$ of a smooth subvariety $Z \subset Y$ of a smooth projective variety Y, the dominant transform of an irreducible subvariety V of Y is the proper transform of V, if V is not contained in Z, and the scheme-theoretic inverse image $\pi^{-1}(V)$ if $V \subset Z$. Let Y_k be the k-th step of the iterated blowup construction (4.3) of $\overline{\mathrm{Conf}}_\Gamma(X)$ and let $\Delta_\gamma^{(k)}$ be the dominant transform of the diagonal Δ_γ of a subgraph $\gamma \in \mathcal{G}_{n-k+1,\Gamma}$. We show that, at each step, there is an affine torification on Y_k that induces a compatible affine torification on the $\Delta_\gamma^{(k)} \subset Y_k$. To see this, we need to check that, at each stage of the blowup construction, one can repeatedly apply Lemmas 4.3 and 4.4. Thus, we need to ensure that, at each step, the projectivized normal bundles of the blowup loci have an equivariant structure. An explicit description of these projectivized normal bundles was given in Section 2.5 of [3], in terms of *screen configurations*, as in Section 1 of [35]. Namely, one has the following setting. Let $\mathbb{T}_x = T_x(X)$, the tangent space of X at x. The screen configuration space of a subgraph γ at x is $\mathbb{P}(\mathbb{T}_x^{V(\gamma)}/\mathbb{T}_x)$. Notice that the projectivized bundle $\mathbb{P}(\mathbb{T}^{V(\gamma)}/\mathbb{T})$ admits an equivariant structure, as it is built out of copies of the tangent bundle $\mathbb{T} = T(X)$, which is an equivariant vector bundle. The blow up loci in the iterative construction of $\overline{\mathrm{Conf}}_\Gamma(X)$ are the dominant transforms of the diagonals Δ_γ in $\mathcal{G}_{n-k+1,\Gamma}$ and at each stage the exceptional divisor is identified with the projectivized normal bundle

$$\mathbb{P}(\mathcal{N}(\Delta_\gamma \subset \cap_{\gamma'\in\mathcal{N}:\gamma'\subsetneq\gamma}\Delta_{\gamma'})),$$

where $\mathcal{N}$ is a $\mathcal{G}$-nest containing γ (see Proposition 3 of [3]), or with $\mathbb{P}(\mathcal{N}(\Delta_\gamma \subset X^{V(\Gamma)}))$if $\{\gamma' \in \mathcal{N} : \gamma' \subsetneq \gamma\} = \emptyset$. One also has an identification (Theorem 1 of [3])

$$\mathbb{P}(\mathcal{N}(\Delta_\gamma \subset \cap_{\gamma'\in\mathcal{N}:\gamma'\subsetneq\gamma}\Delta_{\gamma'})) \simeq \mathbb{P}(T(X^{V(\Gamma//(\gamma_1\cup\cdots\cup\gamma_r))})/T(X^{V(\Gamma/\gamma)})),$$

where $\mathcal{N} = \{\gamma_1, \ldots, \gamma_r\}$ and the latter can in turn be identified with

$$T(X^{V(\Gamma//(\gamma_1\cup\cdots\cup\gamma_r))})/T(X^{V(\Gamma/\gamma)} \simeq \mathbb{T}^{V(\gamma//(\gamma_1\cup\cdots\cup\gamma_r))}/\mathbb{T}),$$

hence by the previous observation it carries an equivariant structure. One concludes using Lemmas 4.3 and 4.4 that at each stage of the construction, the blowups then have an affine torification. One can then prove the statement inductively. At the first step $Y_0 = X^{\#V(\Gamma)}$ and one has a compatible affine torification on the deepest diagonal Δ_Γ by Lemma 4.5. (We can assume Γ itself is biconnected for simplicity.) Then the general argument here above shows that the dominant transforms in $Y_1 = \mathrm{Bl}_{\Delta_\Gamma}(X^{\#V(\Gamma)})$ of the diagonals Δ_γ, with $\gamma \in \mathcal{G}_{n-1,\Gamma}$ has an affine torification compatible with the affine torification obtained on the blowup Y_1. Similarly, if we assume that the result holds for all the Y_r with $r \leq k-1$, we can apply the general argument described above to obtain compatible regular affine torifications on the blowup Y_k and on the dominant transforms of the Δ_γ, with $\gamma \in \mathcal{G}_{n-k+1,\Gamma}$. We can then conclude that the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ are $\mathbb{F}_1$-varieties in the sense of [7]. □

The result can be extended similarly to cases where X is a smooth projective toric variety, though the case of $X = \mathbb{P}^D$ is the one that is physically significant for quantum field theory in configuration spaces.

4.6. $\mathbb{F}_1$-geometry and the moduli space of curves

The question of whether the moduli spaces $\bar{\mathcal{M}}_{0,n}$ of stable curves of genus zero with n marked points can be defined over $\mathbb{F}_1$ was initially posed by Manin in Section 4.2 of [36].

We have discussed in the previous section the existence of affine torifications on the wonderful compactifications $\overline{\mathrm{Conf}}_\Gamma(X)$ of the graph configuration spaces. It is well known that, when Γ is the complete graph Γ_n, this is just the Fulton–MacPherson space $X[n]$. Moreover, it is known that the moduli space $\bar{\mathcal{M}}_{0,n+1}$ can be realized as a subscheme of $X[n]$ for X any smooth curve, [8]. Thus, it is natural to ask the question of whether the method discussed in the previous section can be adapted to show that the moduli spaces $\bar{\mathcal{M}}_{0,n}$ are defined over $\mathbb{F}_1$.

There are various constructions of $\bar{\mathcal{M}}_{0,n}$. The one that appears most directly useful from our point of view is the one described in [8], as part of a larger family of varieties $T_{d,n}$, which are compactifications of the configuration spaces of n distinct points in $\mathbb{A}^d$ up to translation and homothety, with the boundary strata parameterizing n-pointed stable rooted trees of d-dimensional projective spaces, so that $T_{1,n} = \bar{\mathcal{M}}_{0,n}$.

Theorem 4.7. *The moduli spaces $\bar{\mathcal{M}}_{0,n}$ and the varieties $T_{d,n}$ of* [8]*, for $d \geq 2$ and $n \geq 2$, are defined over $\mathbb{F}_1$ in the sense of* [7]*, that is, they have an affine torification.*

Proof. We follow the steps in the iterated blowup construction of [8] and we show that we can apply to them the same results of Section 4 above. The varieties $T_{d,n}$ defined in [8], with $T_{1,n} = \bar{\mathcal{M}}_{0,n}$ are obtained as an iterated sequence of blowups (Theorem 3.3.1 of [8])

$$T_{d,n+1} = F^n_{d,n} \to \cdots \to F^1_{d,n} \to F^0_{d,n} = T_{d,n}, \tag{4.4}$$

where $F^1_{d,n} \to F^0_{d,n}$ is a projective bundle morphism, and the other morphisms $F^{i+1}_{d,n} \to F^i_{d,n}$ are blowups along a union of subvarieties isomorphic to products $T_{d,n-i} \times T_{d,i+1}$. Thus, $T_{d,n+1}$ is obtained as a sequence of blowups from a projective bundle over $T_{d,n}$. By Proposition 3.4.1 of [8] we also know that $T_{1,3} \simeq \mathbb{P}^1$, while for $d > 1$ we have $T_{d,2} \simeq \mathbb{P}^{d-1}$. Since $\bar{\mathcal{M}}_{0,n}$ is of dimension $n-3$, we are interested in the cases with $n > 3$, and we can take $\bar{\mathcal{M}}_{0,4} = T_{1,3} \simeq \mathbb{P}^1$ as the starting point for the construction. Similarly, we use $T_{d,2} \simeq \mathbb{P}^{d-1}$ as the starting point for the case of $d > 1$. To obtain the result, we again need to check that at each stage of the iterated blowup construction the normal bundles have an equivariant structure. To this purpose we can use the fact that the construction of the spaces $F^i_{d,n}$ can be done inside strata of the compactification of the Fulton–MacPherson configuration spaces [35]. In fact, it is shown in Section 3 of [8] that, for a smooth variety X of dimension d (which we can take to be $\mathbb{P}^d$ for convenience), the spaces $T_{d,n}$ are realized as the fibers of the projection map

$$X[n] = \overline{\mathrm{Conf}}_{\Gamma_n}(X) \to \Delta_{\Gamma_n} \simeq X,$$

over the boundary stratum D_{Γ_n} of $X[n]$ that corresponds to the diagonal Δ_{Γ_n}. Therefore, as explained in Section 3 of [8], the normal bundles can again be described in terms of screen configurations, so we can apply the same argument of Theorem 4.6 to show that the normal bundles have an equivariant structure so that we can apply Lemmas 4.3 and 4.4, to obtain affine torifications on the $T_{d,n}$ and in particular on $\bar{\mathcal{M}}_{0,n}$. □

5. Hyperplane arrangements and $\mathbb{F}_1$-geometry

We discuss in this section a question about $\mathbb{F}_1$-geometry, which is aimed at identifying other natural and sufficiently interesting classes of varieties that can, under suitable conditions, carry $\mathbb{F}_1$-structures. We identify the hyperplane arrangements as a sufficiently broad class of varieties (in the specific sense described in Section 5.1 below), for which the question of the existence of an $\mathbb{F}_1$-structure (in the sense of an affine torification as in [7]) is related to well known invariants.

5.1. Hyperplane arrangements and mixed Tate motives

One reason why it is especially interesting, from the point of view of $\mathbb{F}_1$-geometry, to identify necessary and sufficient conditions for hyperplane arrangements to admit an $\mathbb{F}_1$-structure is that these varieties are very general from the point of view of mixed Tate motives.

In fact, one knows that in all flavor of $\mathbb{F}_1$-geometry, the existence of an $\mathbb{F}_1$-structure implies that the motive of the variety is mixed Tate. One can therefore formulate the question of characterizing which mixed Tate motives over $\mathbb{Z}$ admit an $\mathbb{F}_1$-structure.

This first requires a good characterization of mixed Tate motives in terms of appropriate "generators". There are several conjectures to that extent (see [37]), but we focus here especially on the one proposed by Beilinson–Goncharov–Schechtman–Varchenko in [38], which predicts that, in the number field case, all the extensions $\mathrm{Ext}^1(\mathbb{Q}(0), \mathbb{Q}(n))$ in the category of mixed Tate motives are realized in terms of hyperplane arrangements in general position. This was further elaborated, more recently, by Madhav Nori [39], who showed that the construction described in [38] of a graded Hopf algebra of hyperplane arrangements indeed determines a category of mixed Tate motives.

More precisely, one can formulate the conjecture in the following way [38,37].

Conjecture 5.1 (*Beilinson–Goncharov–Schechtman–Varchenko*). *For a number field k, given a choice of hyperplanes L_i, M_i, $i = 0, \dots, n$ in $\mathbb{P}^n$ in general position, defined over k, one obtains a mixed Tate motive*

$$\mathfrak{m}(\mathbb{P}^n \smallsetminus M, L \smallsetminus (L \cap M)) \tag{5.1}$$

whose realizations give the middle dimensional relative cohomology

$$H^n(\mathbb{P}^n \smallsetminus M, L \smallsetminus (L \cap M)).$$

Then the subcategory $\mathcal{C}$ of the category $\mathcal{MT}(k)$ of mixed Tate motives over k generated by the motives of the form (5.1) is in fact all of $\mathcal{MT}(k)$.

Assuming this conjecture, one can then try to characterize which mixed Tate motives over $\mathbb{Z}$ descend to $\mathbb{F}_1$, by identifying conditions under which hyperplane arrangements defined over $\mathbb{Z}$ carry an $\mathbb{F}_1$-structure. Notice that, while the conjecture above is formulated for motives over a number field, as explained in Section 6 of [40], the abelian category of mixed Tate motives over the ring of integers in a number field can be obtained by considering a sub-Hopf-algebra of the Hopf algebra that defines the abelian category of mixed Tate motives over the number field. However, it is unclear whether one can really hope that (5.1) would generate the subcategory of mixed Tate motives over the ring of integers in a number field, as known examples correspond to degenerate configurations of hyperplanes.

5.2. Necessary conditions for $\mathbb{F}_1$-structures

The necessary condition based on the Grothendieck class (which obviously implies the one based on the Euler characteristic) admits a nice interpretation in the case of hyperplane arrangements.

We first recall some standard terminology in hyperplane arrangements, [9,41]. Given a hyperplane arrangement $\mathcal{A} = \{H_j\}_{j=1}^N$ of N hyperplanes in $\mathbb{P}^n$, the central arrangement $\hat{\mathcal{A}} = \{\hat{H}_j\}_{j=1}^N$ is the associated arrangement of hyperplanes in affine space $\mathbb{A}^{n+1}$ through the origin. The characteristic polynomial $\chi_{\hat{\mathcal{A}}}(t)$ of the central arrangement is defined as

$$\chi_{\hat{\mathcal{A}}}(t) = \sum_{x \in \mathcal{L}(\hat{\mathcal{A}})} \mu(x)\, t^{\dim(x)}, \tag{5.2}$$

where $\mathcal{L}(\hat{\mathcal{A}})$ is the poset of the $\cap_{j\in J} \hat{H}_j$, with $J \subseteq \{1, \dots, N\}$, ordered by reverse inclusion, and $\mu(x)$ is the Möbius function of $\mathcal{L}(\hat{\mathcal{A}})$.

Lemma 5.2. *Let $\mathcal{A} = \{H_j\}_{j=1}^N$ be a hyperplane arrangement with $A = \cup_{j=1}^N H_j$. Then the Grothendieck class satisfies $[A] = \sum_{k\geq 0} a_k \mathbb{T}^k$, with $a_k \geq 0$, if and only if the Möbius function $\mu(x)$ of $\mathcal{L}(\hat{\mathcal{A}})$ satisfies, for all $k \geq 1$*

$$\sum_{x:\, \dim(x)\geq k} \mu(x) \binom{\dim(x)}{k} \leq \binom{n+1}{k}. \tag{5.3}$$

Proof. This is a direct consequence of Theorem 1.1 of [9], according to which the class in the Grothendieck ring of the arrangement complement satisfies

$$[\mathbb{P}^n \smallsetminus A] = \frac{\chi_{\hat{\mathcal{A}}}(\mathbb{L})}{\mathbb{L}-1}.$$

This means that

$$[A] = [\mathbb{P}^n] - \frac{\chi_{\hat{\mathcal{A}}}(\mathbb{L})}{\mathbb{L}-1} = \frac{(1+\mathbb{T})^{n+1} - 1 - \chi_{\hat{\mathcal{A}}}(\mathbb{T}+1)}{\mathbb{T}}.$$

Thus, the condition that $[A] = \sum_{k\geq 0} a_k \mathbb{T}^k$, with $a_k \geq 0$ can be formulated as the stated condition, using the fact that

$$\chi_{\hat{\mathcal{A}}}(t+1) = \sum_{x\in\mathcal{L}(\hat{\mathcal{A}})} \mu(x)\,(t+1)^{\dim(x)} = \sum_{x\in\mathcal{L}(\hat{\mathcal{A}})} \mu(x) \sum_{r=0}^{\dim(x)} \binom{\dim(x)}{r} t^r,$$

so that the above can be written equivalently as

$$[A] = \frac{1}{\mathbb{T}} \sum_{k=1}^{n+1} \left(\binom{n+1}{k} - \sum_{x:\,\dim(x)\geq k} \mu(x) \binom{\dim(x)}{k} \right) \mathbb{T}^k,$$

from which (5.3) follows. □

Notice that the positivity condition described here for the coefficients a_k of the class $[A] = \sum_{k\geq 0} a_k \mathbb{T}^k$ is the same as the condition given in Proposition 6.1 of [9] for the positivity of the coefficients of the Chern–Schwartz–MacPherson class $c_{SM}(A)$. In fact, the term $x = 0$ in the sum on the left-hand-side of (5.3) has $\mu(0) = 1$, hence it cancels the term on the right-hand-side of (5.3), so that the same condition can be expressed as the condition that all the coefficients of the polynomial $-\sum_{x\neq 0} \mu(x)(t+1)^{\dim(x)}$ are non-negative, which is the one given in Proposition 6.1 of [9] for the CSM class. There is an *a priori* reason for this coincidence, namely the result of Proposition 2.2 of [25], which shows that, for all subvarieties of projective spaces that are obtained as unions, intersections, complements, differences of linearly embedded subspaces, the CSM class can be obtained from the class in the Grothendieck ring by formally replacing each power $\mathbb{T}^r$ with the class in the Chow group of a linear subspace $\mathbb{P}^r$. We return to discuss the relevance of CSM classes to questions about $\mathbb{F}_1$-geometry in Section 6 below.

Question 5.3. *Is there a combinatorial condition on hyperplane arrangements that is* sufficient *for the existence of an affine torification on the arrangement or on its complement?*

Coming back to the examples motivated by varieties arising in perturbative quantum field theory, one can formulate the following more specific question.

Question 5.4. *Do the hyperplane arrangements $\mathcal{A}_\Gamma$ associated to Feynman graphs admit an affine torification? Are there natural conditions on the graphs, or interesting families of graphs for which this is the case?*

Notice that, in general, one has examples of hyperplane arrangements that do not satisfy the necessary condition on the Euler characteristic (Example 6.4 of [9]), so in the case of the arrangements $\mathcal{A}_\Gamma$ associated to Feynman graphs one seeks conditions on the graph that might relate to the existence of a torification of the arrangement.

5.3. Arrangements, wonderful compactifications, and torifications

Given a hyperplane arrangement $\mathcal{A}$, besides the varieties A and $\mathbb{P}^n \smallsetminus A$, there is another class of varieties that are interesting to consider, which are the wonderful compactifications, in the sense of [20], of the complement $\mathbb{P}^n \smallsetminus A$. We will denote them here by $\overline{\mathrm{Conf}}_{\mathcal{A}}(\mathbb{P}^n)$, by analogy to the notation used above for the graph configuration spaces. Notice that the spaces $\overline{\mathrm{Conf}}_\Gamma(X)$ describe above do not correspond to hyperplane arrangements, because the linear spaces Δ_e are complete intersections $\Delta_e = \{x_{s(e)} = x_{t(e)}\} = \cap_{k=1}^{D}\{x_{s(e),k} = x_{t(e),k}\}$, hence they are of codimension D in $X^{\#V(\Gamma)}$. However, an argument similar to the one used in Section 4.5 to obtain torifications of $\overline{\mathrm{Conf}}_\Gamma(X)$ can be developed for the wonderful compactifications $\overline{\mathrm{Conf}}_{\mathcal{A}}(\mathbb{P}^n)$ of [20], for hyperplane arrangements, leading to conditions on the existence of $\mathbb{F}_1$-structures on these varieties.

6. Algebro-geometric Feynman rules and embedded $\mathbb{F}_1$-structures

In [27], a polynomial invariant of the graph hypersurfaces $\mathbb{A}^n \smallsetminus \hat{X}_\Gamma$ is defined using the Chern–Schwartz–MacPherson characteristic classes of singular varieties. The polynomial $C_\Gamma(T)$ satisfies the "algebro-geometric Feynman rule" property of being multiplicative over disjoint union of graphs, $C_{\Gamma_1\cup\Gamma_2}(T) = C_{\Gamma_1}(T)C_{\Gamma_2}(T)$. Moreover, it contains the Euler characteristic of the *projective* graph hypersurface complement $\chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$ as the coefficient of the degree one term. Thus, it can be thought of as a generalization of this Euler characteristic that satisfies the multiplicative property, which fails for $\chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma)$ itself.

We recall here some basic properties of characteristic classes of singular varieties and the definition of the invariant $C_\Gamma(T)$. We then give an interpretation of $C_\Gamma(T)$ in terms of Euler characteristic using a result of Aluffi [10], and we explain its relevance to the $\mathbb{F}_1$-geometry point of view.

6.1. Chern classes of singular varieties

We recall here briefly a few facts about characteristic classes of singular varieties that we need to use in the following. For a detailed introduction to the subject, we refer the reader to [42].

Let V be a variety over $\mathbb{K}$. For any closed subvariety S we define $\mathbf{1}_S$ to be the function from V to $\mathbb{Z}$ that is 1 on S and 0 on $V \setminus S$. Let $\mathcal{C} : \mathcal{V}_{\mathbb{K}}^p \to Ab$ be the functor from $\mathbb{K}$-varieties and proper maps to abelian groups where $\mathcal{C}(V)$ is the abelian group of $\mathbb{Z}$-linear combinations $\sum m_S \mathbf{1}_S$ over S closed subvarieties. These are called constructible functions. For $f : V \to W$, $\mathcal{C}(f)(\mathbf{1}_S)$ is defined by

$$\mathcal{C}(f)(\mathbf{1}_S)(p) = \chi(f^{-1}(p) \cap S),$$

and extending by linearity.

Grothendieck and Deligne conjectured the existence of a map c_* from $\mathcal{C}(V)$ to the homology of V satisfying the following properties for any constructible functions α and β on V and any $f : V \to W$:

(1) $c_*(\mathcal{C}(f)(\alpha)) = f_* c_*(\alpha)$
(2) $c_*(\alpha + \beta) = c_*(\alpha) + c_*(\beta)$
(3) $c_*(\mathbf{1}_V) = c(V) \cap [V]$, when V is smooth,

where $c(V)$ denotes the usual Chern class of V and $c(V) \cap [V]$ is the image of $c(V)$ in homology under Poincare duality. Any such c_*, if it exists, is unique. Schwartz and MacPherson independently constructed such a c_*, showing existence. (In fact, the classes defined by M.H. Schwartz were introduced prior to the functorial formulation, and later proved to satisfy the properties above by Brasselet and Schwartz.) One defines $c_{SM}(V)$ for any (not necessarily smooth) V to be $c_*(\mathbf{1}_V)$, see [42,43].

6.2. The CSM Feynman rule

We recall here the definition and properties of the invariant $C_\Gamma(T)$ constructed in [27].

Given a locally closed subset $\hat{X}$ of $\mathbb{P}^N$, one can write the CSM class in the Chow group $A_*(\mathbb{P}^N)$ as

$$c_*(\mathbf{1}_{\hat{X}}) = a_0[\mathbb{P}^0] + \cdots a_N[\mathbb{P}^N]. \tag{6.1}$$

One then introduces an associated polynomial of the form

$$G_{\hat{X}}(T) := a_0 + a_1 T + \cdots + a_N T^N, \tag{6.2}$$

obtained by formally replacing the class $[\mathbb{P}^k]$ in $A_*(\mathbb{P}^N)$ with the variable T^k. This polynomial can be equivalently written as $G_{\hat{X}}(T) = \sum_{k\geq 0} c_k T^k$, where the coefficient c_k is the degree of the k-dimensional piece of the CSM class of $\hat{X}$. In particular $G_{\hat{X}}(0) = \chi(\hat{X})$. In the case where $\hat{X} = \mathbb{A}^N$, we have $G_{\mathbb{A}^N}(T) = (T+1)^N$.

One then defines the polynomial $C_\Gamma(T)$ as the difference

$$C_\Gamma(T) = G_{\mathbb{A}^n}(T) - G_{\hat{X}_\Gamma}(T), \tag{6.3}$$

for $n = \#E(\Gamma)$. This is a monic polynomial of degree n. Here the affine graph hypersurface $\hat{X}_\Gamma \subset \mathbb{A}^n$, which is the affine cone over the projective $X_\Gamma \subset \mathbb{P}^{n-1}$ is seen as embedded in $\mathbb{P}^n$. It is proved in [27] that if $\Gamma = \Gamma_1 \cup \Gamma_2$ is a disjoint union of graphs, then the polynomial behaves multiplicatively,

$$C_\Gamma(T) = C_{\Gamma_1}(T) C_{\Gamma_2}(T), \tag{6.4}$$

and it contains the Euler characteristic of the projective hypersurface complement as a coefficient,

$$C'_\Gamma(0) = \chi(\mathbb{P}^{n-1} \smallsetminus X_\Gamma). \tag{6.5}$$

The multiplicative property (6.4) means that $C : \mathcal{H} \to \mathbb{Z}[T]$ is a *ring homomorphism* from the Hopf algebra of graphs to the ring of polynomial. This is the minimal algebraic requirement needed for a "Feynman rule", in the setting of Hopf algebra-based renormalization, see [27]. For a more detailed discussion of this multiplicative property, see also [10].

6.3. Chern classes and Euler characteristics

Let $\hat{X} \subset \mathbb{P}^N$ be an embedded subvariety and set

$$\chi_k := \chi(X \cap H_1 \cap \cdots \cap H_k), \tag{6.6}$$

where χ is the topological Euler characteristic and $H_1, \dots, H_k$ are general hyperplanes in $\mathbb{P}^N$. One also defines

$$\chi_{\hat{X}}(T) = \sum_{k \geq 0} \chi_k T^k. \tag{6.7}$$

The value at zero is the usual topological Euler characteristic, $\chi_{\hat{X}}(0) = \chi(\hat{X})$.

It was proved by Aluffi [10] that the polynomial $G_{\hat{X}}(T)$ defined in (6.2) above can be expressed in terms of the polynomial $\chi_{\hat{X}}(T)$ of (6.7), and conversely, according to the formulas

$$\chi_{\hat{X}}(T) = \frac{T\, G_{\hat{X}}(T-1) - G_{\hat{X}}(0)}{T-1} \quad \text{and} \quad G_{\hat{X}}(T) = \frac{T\, \chi_{\hat{X}}(T+1) + \chi_{\hat{X}}(0)}{T+1}. \tag{6.8}$$

This result means that the Chern–Schwartz–MacPherson class can be computed completely in terms of Euler characteristics. Notice, however, that the Euler characteristics $\chi_k = \chi(\hat{X} \cap H_1 \cap \cdots \cap H_k)$ depend not only on the variety $\hat{X}$ itself, but on the way in which it is embedded into $\mathbb{P}^N$. This corresponds to the fact, already discussed at length in [27], that the CSM class does not factor through the Grothendieck ring of varieties but through a refinement, a Grothendieck ring of immersed conical varieties.

The relations (6.5) and (6.8) also give

$$\chi(X_\Gamma) = \chi_{\hat{X}_\Gamma}(1) - \chi_{\hat{X}_\Gamma}(0) = \sum_{k \geq 1} \chi_k(\hat{X}_\Gamma). \tag{6.9}$$

In fact, we have

$$\begin{aligned} C'_\Gamma(T) &= G'_{\mathbb{A}^n}(T) - G'_{\hat{X}_\Gamma}(T) \\ &= n(T+1)^{n-1} - \frac{(\chi_{\hat{X}_\Gamma}(T+1) + T\chi'_{\hat{X}_\Gamma}(T+1))(T+1) - T\chi_{\hat{X}_\Gamma}(T+1) - \chi_{\hat{X}_\Gamma}(0)}{(T+1)^2} \end{aligned}$$

so that, using (6.5), we get $C'_\Gamma(0) = n - (\chi_{\hat{X}_\Gamma}(1) - \chi_{\hat{X}_\Gamma}(0)) = n - \chi(X_\Gamma)$.

6.4. Embedded $\mathbb{F}_1$-structures

The observation above leads us to consider a different, relative, notion of $\mathbb{F}_1$-structure, which takes into account not only the variety itself but regards it as an embedded subvariety.

We propose the following definition of an embedded $\mathbb{F}_1$-structure.

Definition 6.1. Let $\hat{X} \subset \mathbb{P}^N$ be an embedded subvariety, defined over $\mathbb{Z}$, which has the structure of an $\mathbb{F}_1$-variety (it admits an affine torification). Then $\hat{X}$ has an embedded $\mathbb{F}_1$-structure if, in addition, the varieties $\hat{X} \cap H_1 \cap \cdots \cap H_k$, where $H_1, \dots, H_k$ are general hyperplanes, are also $\mathbb{F}_1$-varieties, that is, they also admit affine torifications.

While we focus here on torifications, the notion of embedded $\mathbb{F}_1$-structure is in general stronger than the property that $\hat{X}$ is an $\mathbb{F}_1$-variety, regardless of which of the currently available notions of $\mathbb{F}_1$-geometry one decides to adopt.

Remark 6.2. In general, one should not expect that, if $\hat{X} \subset \mathbb{P}^N$ is an $\mathbb{F}_1$-variety, then a hyperplane section $\hat{X} \cap H$ would also be. For example, an elliptic curve (which is not a Tate motive, hence not an $\mathbb{F}_1$-variety) can be realized as a hyperplane section of a $\mathbb{P}^2$ embedded in $\mathbb{P}^9$. However, it makes sense to look for examples of varieties satisfying this property among those that are obtained by unions, intersections and complements of linear subspaces, such as hyperplane arrangements.

The simple necessary condition on the Euler characteristic for the existence of an $\mathbb{F}_1$-structure translates immediately into a condition expressible in terms of Chern classes for the existence of an embedded $\mathbb{F}_1$-structure.

Lemma 6.3. *If $\hat{X} \subset \mathbb{P}^N$ has an embedded $\mathbb{F}_1$-structure, then all the coefficients of the polynomial $(T+1)\, G_{\hat{X}}(T)$ are non-negative.*

Proof. The first requirement for an embedded $\mathbb{F}_1$ structure is that the variety $\hat{X}$ is an $\mathbb{F}_1$-variety. This implies that $\hat{X}$ is polynomially countable and with non-negative Euler characteristic, as we discussed in the previous sections. Thus, one obtains the necessary condition $G_{\hat{X}}(0) \geq 0$. Moreover, the requirement that the hyperplane sections $\hat{X} \cap H_1 \cap \cdots \cap H_k$ are also $\mathbb{F}_1$-varieties implies that the Euler characteristics $\chi_k = \chi(\hat{X} \cap H_1 \cap \cdots \cap H_k)$ should also be non-negative, and by (6.8) this implies that all the other coefficients of the polynomial $(T+1)\, G_{\hat{X}}(T)$ should also be non-negative. □

We then obtain the following condition for the graph hypersurfaces, based on the positivity condition for the CSM classes.

Corollary 6.4. *A necessary condition for the graph hypersurface complement $\mathbb{A}^n \setminus \hat{X}_\Gamma$ to have the structure of an embedded $\mathbb{F}_1$-variety is the positivity of all the coefficients of the polynomial $(T+1)\, C_\Gamma(T)$.*

Proof. This follows immediately from Lemma 6.3 by noticing that

$$C_\Gamma(T) = G_{\mathbb{A}^n}(T) - G_{\hat{X}_\Gamma}(T) = G_{\mathbb{A}^n \smallsetminus \hat{X}_\Gamma}(T). \quad \square$$

If the coefficients of the polynomial $G_{\hat{X}}(T)$ are themselves non-negative, then so are those of $(T+1)\, G_{\hat{X}}(T)$. Thus, effectivity of the CSM classes implies that the necessary condition of Lemma 6.3 is satisfied.

The question of the positivity of the coefficients of the CSM class of graph hypersurfaces was first raised in [25], where it is conjectured that this condition may always be satisfied. However, Aluffi has recently shown that there are examples of graph hypersurfaces, for sufficiently large graphs, where the positivity condition may fail, [44]. Thus, the necessary condition for the existence of embedded $\mathbb{F}_1$-structure may not be always satisfied for graph hypersurfaces.

It is important to stress the fact that the condition of Lemma 6.3 is certainly not sufficient, even when combined with the other necessary condition that the varieties $\hat{X} \cap H_1 \cap \cdots \cap H_k$ are polynomially countable (Tate motives). For example, it is known from the results of [12,45,28], that the CSM classes of many Schubert varieties satisfy the positivity condition of Lemma 6.3, but Schubert varieties are not necessarily $\mathbb{F}_1$-varieties: for example, in the torified-schemes formulation of $\mathbb{F}_1$-geometry developed in [7], Grassmannians and Schubert varieties are torified varieties but not necessarily endowed with an affine torification, see the discussion in Example 1.15 of [7]. We will return to discuss the role of torifications in Section 7.

6.5. A q-deformed CSM class

Let $\hat{X} \subset \mathbb{P}^N$, defined over $\mathbb{Z}$, be a polynomially countable subvariety with the property that its hyperplane sections are also polynomially countable. This condition allows for the definition of a q-deformed CSM-class, whose limit as $q \to 1$ gives back the invariant $G_{\hat{X}}(T)$.

Definition 6.5. Let $\hat{X} \subset \mathbb{P}^N$, defined over $\mathbb{Z}$, be such that $\hat{X} \cap H_1 \cap \cdots \cap H_k$ is polynomially countable, for general hyperplanes H_j. Then the q-deformed CSM class is given by

$$G_{\hat{X}}(q,T) := \frac{TN_{\hat{X}}(q,T) + N_{\hat{X}}(q)}{T+1}, \tag{6.10}$$

where the polynomial $N_{\hat{X}}(q,T)$ is defined as

$$N_{\hat{X}}(q,T) = \sum_{k\geq 0} N_{\hat{X}\cap H_1\cap\cdots\cap H_k}(q) T^k, \tag{6.11}$$

with $N_{\hat{X}\cap H_1\cap\cdots\cap H_k}(q)$ the polynomial interpolating the number of points over $\mathbb{F}_q$ of the mod p reduction of $\hat{X} \cap H_1 \cap \cdots \cap H_k$, and $N_{\hat{X}}(q) = N_{\hat{X}}(q,0) = \#\hat{X}_p(\mathbb{F}_q)$, with $\hat{X}_p$ the mod p reduction of $\hat{X}$.

While the counting of points over finite fields is defined for $q = p^r$ a prime power, the polynomially countable hypothesis implies that all the $N_{\hat{X}\cap H_1\cap\cdots\cap H_k}(q)$ are polynomials, defined for any real or complex value of q. Thus, it makes sense to take a limit when $q \to 1$. By construction, this limit of $G_{\hat{X}}(q,T)$ gives back the usual CSM class, in the form of the polynomial $G_{\hat{X}}(T)$.

7. Regular torifications and Chern classes

In [11] an explicit way is obtained to compute a generalization of the CSM class $c_{SM}(X)$ for any variety X, which reduces to the usual CSM class when X is complete. This is called the pro-Chern class, and is also denoted by c_{SM}. It can be constructed in terms of a notion of good closure and a pro-Chow functor. Namely, let U be a smooth variety with closure $\overline{U}$ such that $\overline{U}$ is a *good* closure, that is smooth with $\overline{U} \setminus U$ a normal crossings divisor. Let $\{U\} = c(\Omega^1_{\overline{U}}(\overline{U} \setminus U)^\vee) \cap [\overline{U}]$ where c denotes the usual Chern class of a bundle and $\Omega^1_{\overline{U}}(\overline{U} \setminus U)$ are the differentials with logarithmic poles on $\overline{U} \setminus U$. Then $c_{SM}(U) = \{U\}$. Let X be any variety and suppose $\mathbf{1}_X = \sum m_U \mathbf{1}_U$ where U are smooth, locally closed subvarieties and m_U are integers. Then the class $c_{SM}(X)$ is expressed as a finite sum $c_{SM}(X) = \sum m_U i_{U*}\{U\}$, where $i_U : U \to X$ is the inclusion of U into X and more generally $c_{SM}(\alpha) = \sum m_U i_{U*}\{U\}$, whenever $\alpha = \sum m_U \mathbf{1}_U$ is a constructible function on X. Theorem 3.3 of [11] shows that this is independent of compatible good closures. That is, if $\overline{U}$ is the closure of U in X with inclusion i and $\overline{U}'$ is a good closure of U with inclusion j, such that there exists a morphism $\pi : \overline{U} \to \overline{U}'$ with $j = \pi \circ i$, then both these closures give the same class $\{U\}$ up to passing to the pro-Chow homology of X.

7.1. Chern class for toric varieties and positivity

An interesting application, given in [11], of this construction of CSM classes is a new simple derivation of the Ehlers formula for the Chern classes of toric varieties, which in particular shows that positivity holds, since the Chern class reduces to a sum of fundamental classes of closures of torus orbits.

More precisely, it is proved in Theorem 4.2 of [11] that the pro-CSM class of a toric variety X can be written as

$$c_{SM}(X) = \sum_{O \in X/T} c(\Omega^1_{\bar{O}}(\log D)^\vee) \cap [\bar{O}] = \sum_{O \in X/T} [\bar{O}], \tag{7.1}$$

where O ranges over the orbits of the action of the torus T on X, with $[\bar{O}]$ the fundamental class of the orbit closure in the pro-Chow homology of X, and where $D = \bar{O} \smallsetminus O$. The second equality depends on the well known fact [46, p. 87] that $\Omega^1_{\bar{O}}(\log D)$ is trivial in this case.

7.2. Regular torifications

In general, one does not expect that a similar result would hold for torified varieties. In fact, in a torification in general there is no torus action, unlike in the case of the toric varieties, and the natural torus action one has on each of the tori T_i of the torification in X does not in general extend to the closure $\bar{T}_i$ in X.

However, there is a better behaved class of torifications, which were considered in Section 6.2 of [7] as a possible way to eliminate the ambiguity, due to the choice of torification, that can lead to different $\mathbb{F}_1$-structures on the same $\mathbb{Z}$-manifold. These are the *regular* torifications.

Definition 7.1 (*López Peña and Lorscheid [7]*)**.** A torification $e_X : T = \amalg_{i\in I} T_i \to X$ is regular if, for all tori $T_i \subset T$, there exists a set $J_i \subseteq I$ such that the closure $\bar{T}_i$ in X is a union of tori in the same torification, $\bar{T}_i = \amalg_{j\in J_i} T_j$.

It is not known, at present, whether all varieties admitting a torification (or an affine torification) necessarily admit a regular one, though this is the case in all the explicit examples studied so far.

Thus, it is interesting to ask whether the argument of Theorem 4.2 of [11] for the computation of CSM classes would extend to the case of regular torifications. Notice that, because all the flag varieties admit a regular torification, this would then automatically prove the conjectured positivity of the CSM classes for such varieties. The question is therefore whether, given a regular torification $e_X : T \to X$, the pro-Chern class

$$c_{SM}(X) = \sum_{T_i \in T} c(\Omega^1_{\bar{T}_i}(\log D_i)^\vee) \cap [\bar{T}_i]$$

can be written as

$$c_{SM}(X) = \sum_{T_i \in T} [\bar{T}_i].$$

This would be the case if we knew that $\Omega^1_{\bar{T}_i}(\log D_i)$ is trivial.

However, this will in general not be the case.

Lemma 7.2. *Let $e : T = \mathbb{G}_m^d \hookrightarrow X$ be an embedding of a torus in a smooth projective variety X such that $D = \bar{T} \smallsetminus T$ is a simple normal crossings divisor, but $\bar{T}$ is not an equivariant compactification of a semi-abelian variety. Then $\Omega^1_{\bar{T}}(\log D)$ is non-trivial.*

Proof. There is a characterization [47,48] of varieties with trivial logarithmic tangent bundle. In particular, by Corollary 1 of [47], if X is a non-singular complete algebraic complex variety and $D \subset X$ a simple normal crossings divisor, then the condition that the logarithmic tangent bundle $\Omega^1_X(\log D)^\vee$ is trivial is equivalent to the existence of a semi-abelian variety A that acts on X with $X \smallsetminus D$ an open orbit. □

Therefore, one can construct examples where the argument of Theorem 4.2 of [11] does not directly extend to regular torifications, by producing an example of a regular torification $e_X : T \to X$, where some of the tori T_i satisfy the condition of Lemma 7.2.

However, there are cases where one may still be able to use a *regular* torification to obtain useful information on the CSM classes. For instance, consider the case of flag varieties. It is known (see Sections 1.3.5 and 6.2 of [7]) that these admit a unique isomorphism class of regular torifications, coming from Schubert cell decomposition (these are in general not affine torifications, though). In [12] an explicit construction of the Chern classes of Schubert varieties is given in terms of the decomposition in Schubert cells, and a computation of the CSM classes of Schubert cells in terms of certain uniquely determined integer coefficients (which are conjectured to be always non-negative). In view of the considerations above, one can ask the following question.

Question 7.3. *Is there a description of the coefficients $\gamma_{\underline{\alpha},\underline{\beta}}$ in the computation of the CSM classes of Schubert varieties in [12] in terms of counting of tori in a (unique) regular torification? Can such an approach be used to prove positivity?*

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