The Patterson Measure: Classics, Variations and Applications

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Abstract

This survey is dedicated to S. J. Patterson's 60th birthday in recognition of his seminal contribution to measurable conformal dynamics and fractal geometry. It focuses on construction principles for conformal measures for Kleinian groups, symbolic dynamics, rational functions and more general dynamical systems, due to Patterson, Bowen-Ruelle, Sullivan, and Denker-Urbański.

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1 The Patterson Measure: Classics

In his pioneering work [75] Patterson laid the foundation for a comprehensive measure theoretical study of limit sets arising from (conformal) dynamical systems. Originally, his main focus was on limit sets of finitely generated Fuchsian groups, with or without parabolic elements. We begin this survey by reviewing his construction and some of its consequences in the slightly more general situation of a Kleinian group. The starting point of this construction is that to each Kleinian group G one can associate the Poincaré series $\mathcal{P}(z,s)$, given by

$$\mathcal{P}(z,s) := \sum_{g\in G} \exp(-sd(z,g(0))),$$

for $s \in \mathbb{R}$, 0 denoting the origin in the $(N+1)$-dimensional hyperbolic space $\mathbb{H}$ (throughout, we always use the Poincaré ball model for $\mathbb{H}$), z an element of $\mathbb{H}$, and where d denotes the hyperbolic metric. The abzissa of convergence $\delta = \delta(G)$ of this series is called the Poincaré exponent of G. It is *a priori* not clear if $\mathcal{P}(z,s)$ converges or diverges for $s=\delta$, and accordingly, G is called of δ-divergence type if $\mathcal{P}(z,\delta)$ diverges, and of δ-convergence type otherwise. Patterson made use of this critical behaviour of $\mathcal{P}(z,s)$ at $s=\delta$ in order to build measures supported on the limit set $L(G)$ of G, that is, the set of accumulation points of the orbit $G(0)$, as follows. In order to incorporate also the δ-convergence type case, he first chooses a sequence (s_j) tending to δ from above, and then carefully crafts a slowly varying function φ such that the modified Poincaré series

$$\mathcal{P}_\varphi(z,s) := \sum_{g\in G} \varphi(d(z,g(0)))\exp(-sd(z,g(0)))$$

still has abzissa of convergence equal to δ, but diverges for $s=\delta$. With this slight alteration of the classical Poincaré series, he then defines discrete measures μ_{z,s_j} by putting weights on the orbit points in $G(0)$ according to

$$\mu_{z,s_j}(g(0)) = \frac{\varphi(d(z,g(0)))\exp(-s_j d(z,g(0)))}{\mathcal{P}_\varphi(z,s_j)}.$$

Due to the divergence of the modified Poincaré series at δ, each weak accumulation point of the resulting sequence $\left(\mu_{z,s_j}\right)$ of measures is clearly supported on $L(G)$, and each of these so obtained limit measures is what one nowadays calls a Patterson measure. One of the success stories of these measures is that if G is geometrically finite, that is, each element of $L(G)$ is either a radial limit point or else is the fixed point of some parabolic element of G, then there exists a unique measure class containing all these measures. In other words, in this situation a weak accumulation point μ_z of the sequence $\left(\mu_{z,s_j}\right)$ does not depend on the particular chosen sequence (s_j). Moreover, in this geometrically finite situation it turns out that G is of δ-divergence type. Let us now concentrate on this particular situation for a moment, that is, let us assume that G is geometrically finite. Then, a crucial property of the family $\{\mu_z : z\in\mathbb{H}\}$ is that it is δ–harmonic, meaning that for arbitrary $z,w\in\mathbb{H}$ we have, for each $x\in L(G)$,

$$\frac{d\mu_z}{d\mu_w}(x) = \exp(\delta b_x(z,w)),$$

where $b_x(z,w)$ denotes the signed hyperbolic distance of z to w at x, obtained by measuring the hyperbolic distance $d_x(z,w)$ between the two horocycles at x, one containing z and the other containing w, and then taking the negative of this distance if w is contained in the horoball bounded by the horocycle through z, and letting it be equal to this hyperbolic distance otherwise. Note that $d_x(z,w)$ is a Busemann function and $b_x(z,w)$ coincides with $\log(P(z,x)/P(w,x))$, for $P(\cdot,\cdot)$ denoting the Poisson kernel in $\mathbb{H}$. Let us also remark that here the wording *δ–harmonic* points towards another remarkable success story of the concept "Patterson measure", namely, its close connection to spectral theory on the manifold associated with G. More precisely, we have that the function ϕ_0, given by

$$\phi_0(z) := \int_{\partial\mathbb{H}} P(z,x)^\delta \, d\mu_0(x),$$

is a G–invariant eigenfunction of the Laplace–Beltrami operator associated with the (smallest) eigenvalue $\delta(N-\delta)$. Moreover, ϕ_0 is always square–integrable on the convex core of $\mathbb{H}/G$, defined by forming first the convex hull of the limit set in $\mathbb{H}$, then taking a unit neighbourhood of this convex hull, and finally quotienting out G.

In order to gain more geometric insight into δ-harmonicity, it is convenient to consider the measure $\mu_{\gamma(0)}$, for some arbitrary $\gamma \in G$. A straightforward computation gives that $\mu_{\gamma(0)} = \mu_0 \circ \gamma^{-1}$, and hence, the δ-harmonicity implies that

$$\frac{d\,(\mu_0 \circ \gamma^{-1})}{d\,\mu_0}(x) = P(\gamma(0),x)^\delta, \text{ for all } \gamma \in G. \tag{1}$$

This property of the Patterson measure μ_0 is nowadays called δ-conformality. Sullivan ([111]) was the first to recognise the geometric strength of this property, which we now briefly comment on. Let s_x denote the hyperbolic ray between $0 \in \mathbb{H}$ and $x \in \partial\mathbb{H}$, and let x_t denote the point on s_x at hyperbolic distance t from the origin. Let $B_c(x_t) \subset \mathbb{H}$ denote the $(N+1)$-dimensional hyperbolic disc centred at x_t of hyperbolic radius $c>0$, and let $\Pi : \mathbb{H} \to \partial\mathbb{H}$ denote the shadow–projection given by $\Pi(C) := \{x \in \partial\mathbb{H} : s_x \cap C \neq \emptyset\}$. Also, if x_t lies in one of the cusps associated with the parabolic fixed points of G, let $r(x_t)$ denote the rank of the parabolic fixed point associated with that cusp, otherwise, put $r(x_t)$ equal to δ. Combining the δ-conformality of μ_0 and the geometry of the limit set of the geometrically finite Kleinian group G, one obtains the following generalised Sullivan shadow lemma ([111] [113] [110]):

$$\mu_0(\Pi(B_c(x_t))) \asymp |\Pi(B_c(x_t)))|_E^\delta \cdot \exp((r(x_t)-\delta)d(x_t,G(0))),$$

for all $x \in L(G)$ and $t > 0$, for some fixed sufficiently large $c > 0$, and where $|\cdot|_E$ denotes the diameter with respect to the chordal metric in $\partial\mathbb{H}$. Note that in the latter formula the "fluctuation term" $\exp((r(x_t) - \delta)d(x_t, G(0)))$ can obviously also be written in terms of the eigenfunction ϕ_0 of the Laplace–Beltrami operator. Besides, this gives a clear indication towards why the Patterson measure admits the interpretation as a "correspondence principle" which provides a stable bridge between geometry and spectral theory. However, one of the most important consequences of the generalised Sullivan shadow lemma is, that it allows us to use the Patterson measure as a striking geometric tool for deriving significant geometric insights into the fractal nature of the limit set $L(G)$. For instance, it immediately follows that if G has no parabolic elements, then μ_0 coincides, up to a multiplicative constant, with the δ-dimensional Hausdorff measure on $L(G)$. Hence, in this case, the Hausdorff dimension of $L(G)$ is equal to δ. To extend this to the case in which there are parabolic elements, one first establishes the following generalisation of a classical theorem of Khintchine in metrical Diophantine approximations ([55]). The proof in [110] uses the generalised Sullivan shadow lemma and the techniques of Khintchine's classical result (for further results on metrical Diophantine approximations in connection with the Patterson measure see e.g. [42] [76] [96] [97] [98] [99] [100] [101] [102] [103] [105], or the survey article [104]).

$$\limsup_{t\to\infty} \frac{d(x_t, G(0))}{\log t} = (2\delta(G) - r_{max})^{-1}, \text{ for } \mu_0\text{-almost all } x \in L(G).$$

Here, r_{max} denotes the maximal rank of the parabolic fixed points of G. By combining this with the generalised Sullivan shadow lemma, an immediate application of the mass distribution principle gives that even when G has parabolic elements, we still have that δ is equal to the Hausdorff dimension of $L(G)$. Moreover, these observations immediately show that μ_0 is related to the δ-dimensional Hausdorff measure H_δ and packing measure P_δ as follows. For ease of exposition, the following table assumes that G acts on hyperbolic 3-space.

	$0 < \delta < 1$	$\delta = 1$	$1 < \delta < 2$
no cusps	$\mu_0 \asymp H_\delta \asymp P_\delta$	$\mu_0 \asymp H_1 \asymp P_1$	$\mu_0 \asymp H_\delta \asymp P_\delta$
$r_{max} = 1$	$\mu_0 \asymp P_\delta, H_\delta = 0$	$\mu_0 \asymp H_1 \asymp P_1$	$\mu_0 \asymp H_\delta, P_\delta = \infty$
$r_{min} = 2$	n.a.	n.a.	$\mu_0 \asymp P_\delta, H_\delta = 0$
$r_{min} = 1, r_{max} = 2$	n.a.	n.a.	$H_\delta = 0,\ P_\delta = \infty$

Moreover, as was shown in [105], again by applying the generalised Sullivan shadow lemma for the Patterson measure, we additionally have that δ is equal to the box-counting dimension of $L(G)$. At this point it should also be mentioned that in [8] and [103] it was shown that in fact *every* non-elementary Kleinian group G has the property that its exponent of convergence δ is equal to the Hausdorff dimension of its uniformly radial limit set, that is, the subset of the radial limit set consisting of those limit points $x \in L(G)$ for which there exists $c > 0$ such that $d(x_t, G(0)) < c$, for all $t > 0$. The proof of this rather general result is based on an elementary geometrisation of the Poincaré series and does not use any Patterson measure theory (see also [103]). These fractal geometric interpretations of the exponent of convergence are complemented by its dynamical significance. Namely, one finds that the square integrability of the eigenfunction ϕ_0 on the convex core of $\mathbb{H}/G$ implies that the invariant measure for the geodesic flow on $\mathbb{H}/G$ associated with the Patterson measure has finite total mass ([113]). Using this, one then obtains that δ is equal to the measure-theoretic entropy of the geodesic flow. In particular, if there are no cusps, one can define a topological entropy for the invariant set of geodesics with both endpoints in the limit set, and this topological entropy also turns out to be equal to the critical exponent δ ([111]). It is worth mentioning that in this geometrically finite situation the invariant measure for the geodesic flow is not only of finite total mass and ergodic, but it is also mixing and even Bernoulli ([90]). In fact, these strong properties of the geodesic flow have been exploited intensively in the literature to derive various interesting aspects of the limit set. For instance, the marginal measure of the Patterson–Sullivan measure $|x-y|^{-2\delta} d\mu_0(x) d\mu_0(y)$, obtained by disintegration of the first coordinate, leads to a measure which is invariant under the Bowen–Series map. This allows us to bring standard (finite and infinite) ergodic theory into play. As an example of the effectiveness of this connection, we mention the recent result (see [53] in these Proceedings) that for a geometrically finite Kleinian group G with parabolic elements we have that, with $|\cdot|$ denoting the word metric,

$$\sum_{\substack{g \in G \\ |g| \le n}} \exp(-\delta d(0, g(0))) = O(n^{2\delta - r_{max}}).$$

For Kleinian groups which are not geometrically finite the Patterson measure theory is less well developed, although various promising first steps have been undertaken. Here, an interesting class is provided by finitely generated, geometrically infinite Kleinian groups acting on hyperbolic 3–space $\mathbb{H}^3$ whose limit set is not equal to the whole boundary $\partial\mathbb{H}^3$. For these groups it had been conjectured for almost 40 years that the area of their limit sets is always equal to zero. This conjecture was named after Ahlfors and was eventually reduced to the so–called tameness–conjecture, a conjecture which was only very recently confirmed in [5] and [18]. Given the nature of this conjecture, it is perhaps not too surprising that the concept "Patterson measure" also made vital contributions to its solution.

For infinitely generated Kleinian groups, so far only the beginnings of a substantial theory have been elaborated. As Patterson showed in [77], there exist infinitely generated groups whose exponent of convergence is strictly less than the Hausdorff dimension of their limit set. Kleinian groups with this property were named in [37] as discrepancy groups. Also, an interesting class of infinitely generated Kleinian groups is provided by normal subgroups N of geometrically finite Kleinian groups G. For these groups one always has that $L(N) = L(G)$ and $\delta(N) \geq \delta(G)/2$ (see [37]), and this inequality is in fact sharp, as was shown very recently in [11]. Moreover, by a result of Brooks in [14], one has that if G acts on hyperbolic n–space such that $\delta(G) > n/2$, then

$$N \text{ is a discrepancy group if and only if } G/N \text{ is non–amenable.}$$

This result is complemented by beautiful applications of the Patterson measure theory in [86] and [87], where it was shown for the Fuchsian case that if $G/N \cong \mathbb{Z}^k$, and hence $\delta = \delta(N) = \delta(G)$, since $\mathbb{Z}^k$ is clearly amenable, then

$$N \text{ is of } \delta\text{–divergence type } \Leftrightarrow \begin{cases} k \in \{1,2\} & \text{if } G \text{ has no parabolic elements} \\ k = 1 & \text{if } G \text{ has parabolic elements.} \end{cases}$$

Finally, we mention the related work of [2] which considers the special situation of the Riemann surface $\mathbb{C} \setminus \mathbb{Z}$ uniformized by a Fuchsian group N which is a normal subgroup of the subgroup G of index 6 of the modular group $PSL_2(\mathbb{Z})$ uniformizing the three-fold punctured sphere. There it was shown that the Poincaré series $\mathcal{P}(z, s)$ associated with N has abscissa of convergence $\delta(N) = 1$ and that it has a logarithmic singularity at $s = 1$ (for further results of this type see e.g. [4] [79] [80] [65] [72]). This result of [2] is obtained by showing that the associated geodesic flow has a factor which is Gibbs-Markov ([3]) and by using a local limit theorem of Cauchy type.

2 Gibbs Measures

Rokhlin's seminal paper [88] on the foundations of measure theory, dynamical systems and ergodic theory is fundamental for our further discussion of Patterson measures and conformality of measures of the type as in equation (1). Let $R : \Omega_1 \to \Omega_2$ be a measurable, countable-to-one map between two Lebesgue spaces $(\Omega_i, \Sigma_i, \mu_i)$ $(i = 1, 2)$ ([19]), where Σ_i and μ_i denote some Borel fields and measures. If R is nonsingular[1] the Jacobian J_R of R exists, meaning that, for all $E \in \Sigma_1$ such that $R|_E$ is invertible, we have that

$$\mu_2(R(E)) = \int_E J_R d\mu_1. \tag{2}$$

By our assumptions, the images $R(E)$ are always measurable, in fact, throughout this section all functions and sets considered will always be assumed to be measurable. Also, note that J_R is uniquely defined, μ_1-almost everywhere. Moreover, since R is countable-to-one, the Jacobian J_R gives rise to the transfer operator $\mathcal{L}_J = \mathcal{L}_{J_R}$, given by

$$\mathcal{L}_J f(x) = \sum_{R(y)=x} f(y)/J(y), \tag{3}$$

for all measurable functions $f : \Omega_1 \to \mathbb{R}$ for which the right hand side in (3) is well defined. (For example, the latter always holds for f bounded and R finite-to-one, and it holds, more generally, if $\|\mathcal{L}_J 1\|_\infty < \infty$.) For this type of function, we then have that (2) is equivalent to

$$\int f d\mu_1 = \int \mathcal{L}_J f d\mu_2. \tag{4}$$

Note that this identity can also be written in terms of the "dual operator" $\mathcal{L}_J^*$, which maps μ_2 to μ_1. If the two Lebesgue spaces agree and are equal to some Ω, then $R : \Omega \to \Omega$ is a nonsingular transformation of the Lebesgue space Ω, and in this situation we have that $\mu = \mu_1$ is a fixed point of the dual $\mathcal{L}_J^*$.

The δ-conformality of the Patterson measure in (1) can be viewed as determining the Jacobian for the transformations in the Kleinian group G.

[1] R is said to be nonsingular with respect to μ_1 and μ_2, if for each measurable set $E \subset \Omega_2$ one has that $\mu_1(R^{-1}(E)) = 0$ if and only if $\mu_2(E) = 0$.

Hence, the Patterson construction in Section 1 solves the problem of finding a measure whose Jacobian equals a certain power of the derivative of these transformations. This naturally leads to the following question: For a given measurable function ϕ and a transformation T, when does there exist a probability measure with Jacobian equal to e^{ϕ}? It turns out that typical conditions on ϕ and T are certain kinds of conformality as well as some specific geometric and/or analytic properties. Nowadays, this type of question is well addressed, but in the mid 70's the work in [75] paved the way for these developments (see the following sections). Here, it should also be mentioned that, parallel to this development, the theory of Gibbs measures evolved ([12] [122]), solving the analogue question for subshifts of finite type.

Consider a compact metric space (Ω, d) and a continuous finite-to-one transformation $T : \Omega \to \Omega$. For a given continuous function $\phi : \Omega \to \mathbb{R}$, let us first identify nonsingular measures m_ϕ for which (2) is satisfied with $J_T = e^{\phi}$. A good example for this situation is given by a differentiable map T of the unit interval into itself, where the Lebesgue measure satisfies the equality (2), with ϕ being equal to the logarithm of the modulus of the derivative of T.

For an expanding, open map $T : \Omega \to \Omega$ and a nonnegative continuous function ϕ, Ruelle's Perron-Frobenius Theorem ([12]) guarantees the existence of a measure μ satisfying

$$\mu(T(A)) = \lambda \int_A e^{\phi} d\mu, \tag{5}$$

for some $\lambda > 0$ and for each $A \in \Sigma$ for which $T|_A$ is invertible. Each measure so obtained is called a Gibbs measure for the potential function ϕ. This type of measure represents a special case of conformal measures. An open, expanding map on a compact metric space is called R–expanding, where R refers to Ruelle. This includes subshifts of finite type (or topological Markov chains), for which the Ruelle Theorem was originally proven. In fact, such a R–expanding map has the property that the number of pre-images of all points is locally constant. Consequently, for a given $\phi \in C(X)$, the Perron–Frobenius operator (or equally, the transfer operator) $\mathcal{L}_\phi$ acts on the space $C(X)$, and is given by

$$\mathcal{L}_\phi f(x) = \sum_{y:T(y)=x} f(y) \exp(-\phi(y)).$$

In this situation we then have that the map $m \mapsto \mathcal{L}_\phi^* m / m(\mathcal{L}_\phi 1)$ has a fixed point m_ϕ. The measure m_ϕ is a Gibbs measure whose Jacobian is equal to $\lambda \cdot e^\phi$, for $\lambda = m_\phi(\mathcal{L}_\phi 1)$. The logarithm of the eigenvalue λ is called the pressure $P(T, -\phi)$ of $-\phi$.

The following Bowen-Ruelle-Perron-Frobenius Theorem summarised the main results in this area.

Theorem 2.1 *([12]) Let (Ω, T) be a topologically mixing, R–expanding dynamical system. For each Hölder continuous function $\phi : \Omega \to \mathbb{R}$, there exists a probability measure m_ϕ and a positive Hölder continuous function h such that the following hold.*

1. $\mathcal{L}_\phi^* m_\phi = \exp(P(T, \phi)) m_\phi$;
2. $\mathcal{L}_\phi h = \exp(P(T, \phi)) h$;
3. $\mathcal{L}_\phi^n f - \int f h dm_\phi$ *decreases in norm exponentially fast.*

One immediately verifies that the measure $\widetilde{m}_\phi$, given by $d\widetilde{m}_\phi = h \cdot dm_\phi$, is T–invariant, and hence, $\widetilde{m}_\phi$ is often also referred to as the invariant Gibbs measure. In fact, as the name already suggests, the existence of this type of Gibbs measures is closely related to the thermodynamic formalism for discrete time dynamical systems.

Note that the existence of m_ϕ has been derived in [52], whereas some first results in this direction were already obtained in [74]. Alternative proofs of the Bowen-Ruelle-Perron-Frobenius Theorem use, for instance, the Hilbert metric in connection with positive cones (see [38] and section 5) or, for the statement in *2.*, the Theorem of Ionescu-Tulcea and Marinescu (see [50]). Also, note that the original version of this theorem was given in terms of subshifts of finite type. In fact, an R–expanding transformation admits a Markov partition, and therefore, the associated coding space is a subshift of finite type. Nevertheless, the theorem can also be proven directly in terms of R–expanding maps on compact metric spaces.

Finally, let us remark that the method above can be extended to systems which are neither open nor expanding. For instance, the potential function ϕ may have properties which only requires T to be expanding along certain orbits. A typical condition of this type is that the pressure function at ϕ exceeds $\sup(\phi)$, where the supremum is taken over the state space Ω. This situation arises, for instance, if T is a rational map on the Riemann sphere (see e. g. [33] and [27], or [47] for the case of a map of the interval). In this case we still have that $\mathcal{L}_\phi$ acts on the set of continuous functions, and the proof of the existence of the invariant Gibbs measure then uses that for a Hölder continuous potential function ϕ most of the branches are contracting and that the contributions of other branches are negligible, due to the boundedness condition on ϕ. In fact, this approach turns out to be somehow characteristic for certain non-uniformly hyperbolic systems.

3 Sullivan's Conformal Measure

As already mentioned at the beginning, originally one of the main motivations for the construction of the Patterson measure was to study fractal geometric properties of limit sets of Fuchsian groups. The analogue of Patterson's construction for Julia sets of either hyperbolic or parabolic rational maps was first noticed by Sullivan in [112]. Recall that a rational function $R : S^2 \to S^2$ is called hyperbolic if its Julia set does not contain any critical or rationally indifferent (parabolic) periodic points, whereas R is called parabolic if its Julia set contains a parabolic periodic point, but does not contain any critical point. Here, the key observation is that in these expansive cases the Julia set can be considered as being the "limit set" of the action of the rational map on its Fatou component. The elaboration of this analogue between Fuchsian groups and rational maps in [112] has led to what is nowadays called Sullivan's dictionary (for some further chapters of this dictionary, see e.g. [112] [109] [108] [107] [106]).

The idea of a conformal measure for a rational map R appeared first in [112], Theorem 3, where the existence of a conformal measure for the function $|R'|^t$, for some $t \in \mathbb{R}$, was established. Moreover, in the same paper Sullivan showed that this measure is unique in the hyperbolic case. In fact, in this case one easily verifies that $\delta = \inf\{t > 0 : \text{a } t\text{-conformal measure exists}\}$ coincides with the Hausdorff dimension h of the Julia set. Sullivan's construction modifies the Patterson measure construction, and his method was later extended in [28] to more general classes of transformations.

Recall that the starting point of Patterson's and Sullivan's construction is to consider powers ϕ^t of the exponential of some potential function $\log \phi$, for t greater than a certain critical value, and then to proceed by letting t decrease to this critical value. However, in the case of expanding rational maps it is much simpler to use the theory of Gibbs measures, as explained in Section 2.

One immediately verifies that there always exists a Gibbs measure m_t for $\phi^t = |R'|^t$, for some $t \geq 0$ (this follows from the discussion in section 2). Since

$$m_t(R(A)) = \lambda_t \int_A |R'|^t dm_t,$$

where as before $\log \lambda_t = P(R, -t \log |R'|)$, we have that the measure m_t is conformal if and only if $P(R, -t \log |R'|) = 0$. If R is expanding, it is easy to see that the pressure function is continuous and strictly decreasing, for $t \geq 0$. In particular, we have that $P(R, 0)$ $(= \log deg(R))$ is equal to the topological entropy (see [66]) and that $P(R, -t \log |R'|) \to -\infty$, for t tending to infinity. This implies that there exists a unique t for which the pressure function vanishes. In fact, this is precisely the content of the Bowen–Manning–McCluskey formula ([68], [13]). Using this observation, it can then be shown that the so obtained t is equal to the Hausdorff dimension of the Julia set of R, a result due to Sullivan in [112] (see [13] for related earlier results on dynamical and geometric dimensions). Note that Sullivan's construction employs Patterson's approach, replacing the orbit under the Fuchsian group by the set of pre-images under R of some point in the Fatou set which accumulates at the Julia set. This approach can be viewed as some kind of "external construction" (see [40]).

For more general rational functions, it is necessary to gain better control over the eigenvalues of the transfer operator. This can hardly be done by the type of functional analytic argument given above. However, for a parabolic rational map one still finds that there exists a unique non–atomic ergodic conformal measure with exponent equal to the Hausdorff dimension of the Julia set. Although there still exists such a conformal measure, in this situation one finds that every other ergodic conformal measure is concentrated on the orbit of the parabolic points (see [32] [34]). While the construction is still straightforward in this parabolic case, other cases of rational functions have to be treated with refined methods and require certain "internal constructions", of which we now recall a few (see also Section 4).

One of these methods is Urbański's KV-method, which considers invariant subsets of the Julia set whose closures do not contain any critical point. Given that these sets exhaust the Julia set densely, this method allows us to construct measures which converge weakly to the conformal measure in question. Here, the main work consists in showing that the obtained limit measure has no atoms at the critical orbits. This is achieved by employing a certain type of tightness argument. In a similar fashion to that outlined above, the construction leads to a conformal measure with a minimal exponent (see [31] [83]). Although it is still an open problem to decide if this measure is overall non-atomic, one nevertheless has that the minimal exponent is equal to the dynamical dimension of the system.

Another method is the constructive method of [29], which applies in the case of subexpanding rational functions and in the case of rational functions satisfying the Collet-Eckmann condition. It also applies to rational maps which satisfy the following summability condition of [40] and [84]:

$$\sum_{n=1}^{\infty} |(R^n)'(R^{n_c}(c))|^{-\alpha} < \infty,$$

for some $\alpha \geq 0$, for all critical points c in the Julia set, and for some $n_c \in \mathbb{N}$. In this case the existence of a non–atomic conformal measure is guaranteed, given that the Julia set does not contain parabolic points and given that $\alpha < h/(h+\mu)$, where h denotes the Hausdorff dimension of the Julia set and μ the maximal multiplicity of the critical points in the Julia set.

Finally, let us also mention that for a general rational map we have that the dynamical dimension of its Julia set coincides with the minimal t for which a t-conformal measure exists ([31] [83]).

The following theorem summarises the discussion above.

Theorem 3.1 *Let R be a rational map of the Riemann sphere, and let h denote the Hausdorff dimension of its Julia set $J(R)$. Then there exists a non–atomic h-conformal probability measure m on $J(R)$, given that one of the following conditions hold:*
(1) ([112]) R is hyperbolic. In this case m is the unique t-conformal measure, for all $t \in \mathbb{R}$.
(2) ([32]) R is parabolic. In this case m is the unique non–atomic t-conformal measure, for all $t \in \mathbb{R}$.
(3) ([29]) R is subexpanding (of Misiurewicz type). In this case m is the unique non–atomic h-conformal measure.
(4) ([118]) If $J(R)$ does not contain any recurrent critical points of R, then m is the unique h-conformal measure. Moreover, m is ergodic and conservative.
(5) ([40],[84]) R satisfies the above summability condition. In this case m is the unique non–atomic h-conformal measure.
(6) ([6]) R is a Feigenbaum map for which the area of $J(R)$ vanishes. In

this case m is the unique h-conformal measure and there exists a non–atomic t-conformal measure, for each $t \geq h$.

In order to complete this list, let us also mention that Prado has shown in [82] that for certain infinitely renormalizable quadratic polynomials (originally introduced in [67]), the equality $h = \inf\{t : \exists \text{ a } t\text{-conformal measure}\}$ still holds. The ergodicity problem for the conformal measure of quadratic polynomials is treated in [81] and then extended further in [48].

An interesting new approach for obtaining the existence of conformal measures is developed by Kaimanovich and Lyubich. They study conformal streams which are defined on laminations of conformal structures. This setting is very much in the spirit of our discussion of bundle maps in section 5. For further details concerning the construction of conformal streams and its application to rational functions we refer to [51]. Moreover, note that the theory of conformal measures has also been elaborated for semigroups of rational functions (see [115] [116] [114]).

Up to now, the classification of conformal measures has not been completed. Clearly, since the space of conformal measures is compact with respect to the weak topology, we always have that there exists a conformal measure of minimal exponent. However, this measure can be either non–atomic, or purely atomic, or even a mixture of both of these types. This follows by convexity of the space of conformal measures (cf. [40]). At this point, it should be remarked that [9] contains an interesting result which clarifies under which conditions on the critical and parabolic points one has that a conformal measure is non–atomic. Also, let us remark that an important aspect when studying conformal measures is provided by the attempts to describe the essential support of a conformal measure in greater detail (see [15] [49] [71] [26] [85]). Of course, the set of radial limit points marks the starting point for this journey.

There are various further fundamental results on the fine structure of Julia sets which have been obtained via conformal measures. For instance, conformal measures led to the striking result that the Hausdorff dimension of the Julia set of parabolic maps of the Riemann sphere lies strictly between $p/(p+1)$ and 2 (see [1]), where p denotes the maximum of the number of petals to be found at parabolic points of the underlying rational map. Also, conformal measures have proven to be a powerful tool in studies of continuity and analyticity of the Hausdorff-dimension-function on families of rational maps ([125] [36]).

Recently, the existence of Sullivan's conformal measures has also been established for meromorphic functions ([63]). The following theorem summarises some of the most important cases.

Theorem 3.2 *Let T be a meromorphic function on $\mathbb{C}$, and let F be the projection of T onto $\{z \in \mathbb{C} : -\pi < Re(z) \leq \pi\}$. With h_T (resp. h_F) denoting the Hausdorff dimension of $J(T)$ (resp. $J(F)$), in each of the following cases we have that there exists a h_T-conformal (resp. h_F-conformal) measure.*
(1) ([57]) T is a transcendental function of the form $T(z) = R(\exp(z))$, where R is a non-constant rational function whose set of singularities consists of finitely many critical values and the two asymptotic values $R(0)$ and $R(\infty)$. Moreover, the critical values of T are contained in $J(T)$ and are eventually mapped to infinity, and the asymptotic values are assumed to have orbits bounded away from $J(T)$. In this case, there exists $t < h_F$ such that if $t > 1$ then there is only one t-conformal measure. Also, the h_F-conformal measure is ergodic, conservative and vanishes on the complement of the set of radial limit points. In particular, this h_F-conformal measure lifts to a σ-finite h_F-conformal measure for T.
(2) ([61] [59]) T is either elliptic and non-recurrent or weakly non-recurrent[2]. We then have that the h_T-conformal measure is non–atomic, ergodic and conservative, and it is unique as a non–atomic t-conformal measure.
(3) ([119] [121]) T is either exponential[3] and hyperbolic or super-growing[4]. Here, if $t > 1$ then the h_F-conformal measure is ergodic, conservative and unique as a t-conformal measure for F. Also, this conformal measure lifts to a σ-finite h_F-conformal measure for T.
(4) ([120]) T is given by $T(z) = \exp(z - 1)$ (parabolic). Here, the h_F-conformal measure is non–atomic, ergodic and conservative. Also, for $t > 1$ it is the unique non–atomic t-conformal measure for F, and if $t \neq h$ then there exist discrete t-conformal measures for F, whereas no such discrete t-conformal measure for F exists for $t = h$. Again, this conformal measure lifts to a σ-finite h_F-conformal measure for T.
(5) ([94]) T is given by $T(z) = R(\exp(z))$, where R is a non-constant rational function with an asymptotic value which eventually maps to infinity.

[2]the ω limit sets of critical points in the Fatou set are attracting or parabolic cycles and the ω limit set of critical points c in the Julia set are compact in $\mathbb{C} \setminus \{c\}$ (resp. $T^n(c) = \infty$, for some $n \geq 1$).

[3]i.e. of the form $T(z) = \lambda \exp(z)$.

[4]the sequence of real parts α_n (resp. the absolute value) of $T^n(0)$ is exponentially increasing, that is, $\alpha_{n+1} \geq c \exp \alpha_n$, for all $n \in \mathbb{N}$ and for some $c > 0$.

Here, the h_F-conformal measure is non–atomic, conservative and ergodic, where h_F denotes the Hausdorff dimension of the radial Julia set of F. Also, this measure is unique as a h_F-conformal measure, and it lifts to a σ-finite h_F-conformal measure for T.

The proofs of these statements follow the general construction method which will be described in the next section. Furthermore, the proofs use the well–known standard method of extending a finite conformal measure for an induced transformation to the full dynamics (see e.g. [35]). Note that [58] gives a finer analysis of the geometric measures appearing in part (5) of the previous theorem. Furthermore, we would like to mention the work in [62] and [60], where one finds a discussion of the relations between different geometric measures. Also, fractal geometric properties of conformal dynamical systems are surveyed in [117] (see also the surveys in [78] and [104]).

4 Conformal Measures for Transformations

As mentioned before, the Patterson-Sullivan construction relies on approximations by discrete measures supported on points outside the limit set, and hence can be viewed as some kind of "external construction". In contrast to this, we are now going to describe an "internal construction", which uses orbits inside the limit set. The basic idea of this construction principle is inspired by the original Patterson measure construction in [75], and also by the method used for deriving equilibrium measures in the proof of the variational principle for the pressure function ([73]). Note that the method does not use powers of some potential function, instead, it mimics the general construction of Gibbs measures, and one is then left to check the vanishing of the pressure function.

Throughout this section let (X, d) be a compact metric space, equipped with the Borel σ-field $\mathcal{F}$. Also, let $T : X \to X$ be a continuous map for which the set $\mathcal{S}(T)$ of singular points $x \in X$ (that is, T is either not open at x or non–invertible in some neighbourhood of x) is finite. Furthermore, let $f : X \to \mathbb{R}$ be a given continuous function, and let $(E_n : n \in \mathbb{N})$ be a fixed sequence of finite subsets of X.

Recall that for a sequence of real numbers $(a_n : n \in \mathbb{N})$, the number $c = \limsup_{n\to\infty} a_n/n$ is called the transition parameter of that sequence.

Clearly, the value of c is uniquely determined by the fact that it is the abzissa of convergence of the series $\sum_{n\in\mathbb{N}} \exp(a_n - ns)$. For $s = c$, this series may or may not converge. Similarly to [75] (see also Section 1), an elementary argument shows that there exists a slowly varying sequence $(b_n : n \in \mathbb{N})$ of positive reals such that

$$\sum_{n=1}^{\infty} b_n \exp(a_n - ns) \begin{cases} \text{converges} & \text{for } s > c \\ \text{diverges} & \text{for } s \leq c. \end{cases} \tag{6}$$

The construction principle.
Define $a_n = \log \sum_{x\in E_n} \exp S_n f(x)$, where $S_n f = \sum_{0\leq k<n} f \circ T^k$, and let c be the transition parameter of the sequence $(a_n : n \in \mathbb{N})$. Also, let $(b_n : n \in \mathbb{N})$ be a slowly varying sequence satisfying (6). For each $s > c$, we then define the normalised measure

$$m_s = \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x\in E_n} b_n \exp(S_n f(x) - ns)\delta_x, \tag{7}$$

where M_s is a normalising constant, and where δ_x denotes the Dirac measure at the point $x \in X$. A straightforward calculation then shows that, for $A \in \mathcal{F}$ such that $T|_A$ is invertible,

$$\begin{aligned} m_s(TA) \quad = \quad & \int_A \exp(c - f) dm_s + \mathrm{O}(s - c) \qquad\qquad (8) \\ & -\frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x\in A\cap(E_{n+1}\Delta T^{-1}E_n)} b_n \exp(S_n f(T(x)) - ns). \end{aligned}$$

For $s \searrow c$, any weak accumulation point of $\{m_s : s > c\}$ will be called a limit measure associated with f and $(E_n : n \in \mathbb{N})$. In order to find conformal measures among these limit measures, we now have a closer look at the terms in (8). There are two issues to discuss here. First, if A is a set which can be approximated from above by sets A_n for which $T|_{A_n}$ is invertible and for which the limit measure of their boundaries vanishes, then the outer sum on the right hand side of (8) converges to the integral with respect to the limit measure. Obviously, this convergence depends on how the mass of m_s is distributed around the singular points. If the limit measure assigns zero measure to these points, the approximation works well. In this case one has to check whether the second summand in (8) tends to zero as $s \searrow c$. The simplest case is that $E_{n+1} = T^{-1}(E_n)$, for all $n \in \mathbb{N}$, and then nothing has to be shown.

This discussion has the following immediate consequences.

Proposition 4.1 *([28]) Let T be an open map, and let m be a limit measure assigning measure zero to the set of periodic critical points. If we have*

$$\lim_{s \searrow c} \frac{1}{M_s} \sum_{n=1}^{\infty} \sum_{x \in E_{n+1} \Delta T^{-1} E_n} b_n \exp(S_n f(T(x)) - ns) = 0,$$

then there exists a $\exp(c - f)$–conformal measure μ. Moreover, if m assigns measure zero to all critical points, then $\mu = m$.

Clearly, the proposition guarantees, in particular, that for an arbitrary rational map R of the Riemann sphere we always have that there exists a $\exp(p - f)$–conformal measure supported on the associated Julia set, for some $p \in \mathbb{R}$.

Also, the above discussion motivates the following weakening of the notion of a conformal measure.

Definition 4.2 *With the notation as above, a Borel probability measure m is called weakly $\exp(c - f)$–conformal, if*

$$m(T(A)) = \int_A \exp(c - f)\, dm$$

for all $A \in \mathcal{F}$ such that $T|_A$ is invertible and $A \cap \mathcal{S}(T) = \emptyset$.

The following proposition shows that these weakly conformal measures do in fact always exist.

Proposition 4.3 *([28]) With the notation as above, we always have that there exists a weakly $\exp(p - f)$–conformal Borel probability measure m, for some $p \in \mathbb{R}$.*

The following theorem addresses the question of how to find the transition parameter c, when constructing a conformal measure by means of the construction principle above. Obviously, the parameter c very much depends on the potential f as well as on the choice of the sequence $(E_n : n \in \mathbb{N})$. In most cases, the sets E_n can be chosen to be maximal separating sets, and then the parameter c is clearly equal to the pressure of ϕ. However, in general, it can be a problem to determine the value of c. The following theorem gives a positive answer for a large class of maps.

Theorem 4.4 *([28]) For each expansive map T we have that there exists a weakly* $\exp(P(T, f) - f)$*–conformal measure m. If, additionally, T is an open map, then m is an* $\exp(P(T, f) - f)$*–conformal measure.*

Note that besides its fruitful applications to rational and meromorphic functions of the complex plane, the above construction principle has also been used successfully for maps of the interval (including circle maps) (see e.g. [28] [16] [46] [45] [44] [43]). In particular, it has been employed to establish the existence of a 1-conformal measure for piecewise continuous transformations of the unit interval, which have neither periodic limit point nor wandering intervals, and which are irreducible at infinity (see [16]). Moreover, conformal measures for higher dimensional real maps appear in [17], and there they are obtained via the transfer operator method.

Currently, it is an active research area to further enlarge the class of transformations for which the existence of conformal measures can be established. This area includes the promising attempts to construct conformal measures on certain characteristic subsets of the limit set, such as on the radial limit set ([26] [49] [85]) or on certain other attractors ([25]). Also, a related area of research aims to elaborate fractal geometry for systems for which weakly conformal measures exist (see e.g. [31]).

We end this section by giving two further examples of systems for which the theory of conformal measures has proven to be rather successful. The first of these is the case of expanding maps of the interval. Here, Hofbauer was one of the leading architects during the development of the general theory.

Theorem 4.5 *([45]) Let $T : [0, 1] \to [0, 1]$ be an expanding, piecewise monotone map of the interval which is piecewise Hölder differentiable. Let $A \subset [0, 1]$ have the Darboux property and positive Hausdorff dimension h, and assume that the forward orbit of each element of A does not intersect the endpoints of the monotonicity intervals of T. Then we have that there exists a non–atomic h-conformal measure, which is unique as a t-conformal measure for $t > 0$.*

Also, for expansive $C^{1+\epsilon}$–maps Gelfert and Rams obtained the following result.

Theorem 4.6 *([39]) Let (X, T) be an expansive, transitive $C^{1+\epsilon}$–Markov system whose limit set has Hausdorff dimension equal to h. Then there exists* *a h-conformal measure. In particular, we have that h is the least exponent for which a t-conformal measure exists, and h is also the smallest zero of the pressure function* $P(T, -t \log |T'|)$.

Finally, let us mention that the above construction principle can obviously also be applied to iterated function systems and graph–directed Markov systems. For these dynamical systems, conformal measures are obtained by considering the inverse branches of the transformations coming with these systems. For further details we refer to [70].

5 Gibbs Measures for Bundle Maps

In this section we give an outline of how to extend the concept of a Gibbs measure to bundles of maps over some topological (or measurable) space X (cf. Section 2). For this, let (X, T) be a dynamical system for which the map $T : X \to X$ factorises over some additional dynamical system (Y, S) such that the fibres are non-trivial. Then there exists a map $\pi : X \to Y$ such that $\pi \circ T = S \circ \pi$. We will always assume that π is either continuous (if X is compact) or measurable. A system of this type is called a fibred system. Note that the set of fibred systems includes dynamical systems which are skew products. For ease of exposition, let us mainly discuss the following two cases: (i) (Y, S) is itself a topological dynamical system and π is continuous; (ii) $(Y, \mathcal{B}, P, S)$ is a measurable dynamical system, with P being a probability measure on Y, S an invertible probability preserving transformation, and where π is measurable.

In the first case, one can define a family $(\mathcal{L}_\phi^{(y)} : y \in Y)$ of transfer operators, given on the space C_y of continuous functions on $\pi^{-1}(\{y\})$ (the image does not necessarily have to be a continuous function), by

$$\mathcal{L}_\phi^{(y)} f(x) = \sum_{\substack{T(z)=x \\ \pi(z)=y}} f(z) e^{-\phi(z)}.$$

If the fibre maps $T_y = T|_{\pi^{-1}(y)}$ are uniformly open and expanding[5], these operators act on the spaces of continuous functions on the fibres. In this situation we have that an analogue of the Bowen-Ruelle-Perron-Frobenius

[5]i.e. there exist $a > 0$ and $\lambda > 1$ such that for all $x, x' \in \pi^{-1}(\{y\})$ we have that $d(x, x') < a$ implies that $d(T(x), T(x')) \geq \lambda d(x, x')$.

Theorem holds. Note that it is not known whether this analogue can be obtained via some fixed point theorem. The currently-known proof uses the method of invariant cones and Hilbert's projective metric (see [7] [22]). More precisely, a conic bundle $(K_y : y \in Y)$ over X is given as follows. For each $y \in Y$, let $K_y \subset C_y$ be the cone defined by

$$K_y = \{f \in C_y : \; f(x_1) \leq \rho(x_1, x_2) f(x_2); \; x_1, x_2 \in \pi^{-1}(\{y\}); \; d(x_1, x_2) < a\},$$

where $\rho(x_1, x_2) = \exp(2\beta(d(x_1, x_2))^\gamma)$, and where β is chosen such that $\beta > \alpha\lambda^\gamma/2(1-\lambda^\gamma)$. Here, $0 < \gamma \leq 1$ denotes the Hölder exponent of the potential function ϕ. One then verifies that $T_y(K_y) \subset K_{S(y)}$ and that the projective diameter of $K_{S(y)}$ is finite and does not depend on y. By using Birkhoff's Theorem ([7]), we then obtain that the fibre maps T_y are contractions. This method of employing Hilbert's projective metric in order to derive conformal measures is due to Ferrero and Schmidt, and we refer to [38] for further details. The following theorem states this so obtained analogue of the Bowen-Ruelle-Perron-Frobenius Theorem for bundle maps.

Theorem 5.1 *([22]) Assume that the fibre maps are uniformly expanding, open and (uniformly) exact*[6]*. For each Hölder continuous function $\phi : X \to \mathbb{R}$, we then have that there exists a unique family $\{\mu_y : y \in Y\}$ of probability measures μ_y on $\pi^{-1}(\{y\})$ and a unique measurable function $\alpha : Y \to \mathbb{R}_{>0}$ such that, for each $A \subset X$ measurable,*

$$\mu_{S(y)}(T(A)) = \alpha(y) \int_A \exp(\phi(x)) \, d\mu_y(x). \tag{9}$$

Moreover, the map $y \mapsto \mu_y$ is continuous with respect to the weak topology.

The family of measures obtained in this theorem represents a generalisation of the concept "Gibbs measure", which also explains why such a family is called a Gibbs family. Note that the strong assumptions of the theorem are necessary in order to guarantee the continuity of the fibre measures. Moreover, note that, under some mild additional assumptions, the function α can be shown to be continuous (and in some cases, it can even be Hölder continuous) ([22]). Since by changing the metric ([20] [30]), each expansive system can be made into an expanding system, one immediately verifies that the

[6]i.e. for $\epsilon > 0$ there exists some $n \geq 1$ such that $T^n(B(x, \epsilon)) \supset \pi^{-1}(\{S^n(\pi(x))\})$

previous theorem can be extended such that it includes fibrewise uniformly expansive systems. The proof of this extension is given in [89].

Let us also mention that typical examples for these fibred systems are provided by Julia sets of skew products for polynomial maps in $\mathbb{C}^d$. For these maps, it is shown in [23] that various outcomes of the usual thermodynamic formalism can be extended to the Gibbs families associated with these maps. This includes the existence of measures of maximal entropy for certain polynomial maps. Note that, alternatively, these measures can also be obtain via pluriharmonic functions.

For more general dynamical systems, the fibre measures do not have to be continuous. In fact, as observed by Bogenschütz and Gundlach, the Hilbert metric also turns out to be a useful tool for investigating the existence of Gibbs families for more general maps. One of the problems which one then usually first encounters is to locate a suitable subset of Y for which the relation in (9) is satisfied. It turns out that here a suitable framework is provided by the concept of a random dynamical system. More precisely, let us assume that the map S is invertible and that (Y, S) is equipped with a σ-algebra $\mathcal{B}$ and an S-preserving ergodic probability measure P. The following "random version" of the Bowen–Ruelle Theorem has been obtained in [10]. Note that in here we have that (9) holds P–almost everywhere. Also, note that in the special case in which S is invertible, we have that each of the operators $\mathcal{L}_\phi(y)$ is nothing else but a restriction of the transfer operator to fibres. Moreover, the theorem uses the concept of a random subshift of finite type. Such a subshift is defined by a bounded random function $l : Y \to \mathbb{N}$ and a random matrix $A(\cdot) = (a_{i,j}(\cdot))$ over Y with entries in $\{0, 1\}$, such that the fibres are given by

$$\pi^{-1}(\{y\}) = \{(x_n)_{n\geq 0} : x_k \leq l(S^k(y)) \text{ and } a_{x_k,x_{k+1}}(S^k(y)) = 1, \forall k \in \mathbb{N}\}.$$

Theorem 5.2 *([10]) Let (X, T) be a random subshift of finite type for which $\|\log \mathcal{L}_\phi\|_\infty \in L_1(P)$, $A(\cdot)$ is uniformly aperiodic and $\phi_{|\pi^{-1}(\{y\})}$ is uniformly Hölder continuous, for each $y \in Y$. Then there exist a random variable λ with $\log \lambda \in L_1(P)$, a positive random function g with $\|\log g\|_\infty \in L_1(P)$, and a family of probability measures μ_y such that the following hold, for all $y \in Y$.*

1. $\mathcal{L}_\phi^{(y)*} \mu_{S(y)} = \lambda(y) \mu_y$;
2. $\mathcal{L}_\phi^{(y)} g = \lambda(y) g$;

3. $\int g d\mu_y = 1$*;*

4. The system has exponential decay of correlation for Hölder continuous functions.

Further results in this direction can be found in [56], [41] and [54]. Note that none of these results makes use of the Patterson construction, but for random countable Markov shifts the construction principle of Section 4 has been successfully applied, and this will be discussed in the following final section of this survey.

6 Gibbs Measure on Non-Compact Spaces

Without the assumption of X being compact, the weak convergence in the Patterson construction needs some additional care in order to overcome the lack of relative compactness of the associated space of probability measures.

One of the the simplest examples, in which the quality of the whole space X does not play any role, is the following. Suppose that there exists a compact subset of X to which the forward orbit of a generic point under a given transformation $T : X \to X$ returns infinitely often. More specifically, let us assume that the map admits a countable Markov partition and that there exists some compact atom A of this partition such that $A \subset \bigcup_{n\in\mathbb{N}} T^{-n}(A)$. We then consider the induced transformation $T_A : A \to A$, given for each $x \in A$ by

$$T_A(x) = T^{n(x)}(x),$$

where $n(x) = \inf\{k \in \mathbb{N} : T^k(x) \in A\}$. Likewise, for a given potential function ϕ on X, we define the induced potential function ϕ_A by $\phi_A(x) = \phi(x) + ... + \phi(T^{n(x)-1})$. In order to see in which way Gibbs measures for T_A give rise to Gibbs measures for T, let μ be a given Gibbs measure for the transformation T_A and the induced potential function ϕ_A. Then define a measure m by

$$\int f dm = \int \sum_{k=0}^{n(x)-1} f(T^k(x))\, d\mu(x).$$

One immediately verifies that m is a σ-finite Gibbs measure for the potential function ϕ (see [35]).

If in the absence of compactness one still wants to employ any of the general construction principles for conformal measures, discussed in Section

1, 2 and 4, one needs to use the concept of tightness of measures. For instance, for S-uniformal maps of the interval, tightness has been used in [25] to show that there exists a conformal measure concentrated on a dense symbolic subset of the associated limit set. Also, Urbański's KV–method, discussed in Section 3, appears to be very promising here, since it gives rise to conformal measures which are concentrated on non-compact subsets of X (although, strictly speaking, the construction is carried out for a compact space, where limits do of course exist). Moreover, there is ongoing research on the existence of Gibbs measures for countable topological Markov chains. In all of the results obtained thus far, tightness plays a key role. For this non–compact situation, there are various examples in the literature for which the existence of Gibbs measures is discussed. However, the first general result was derived in [69].

In the following theorem we consider a topologically mixing Markov chain X, given by a state space Λ, a map $T : X \to X$, and a transition matrix $\Sigma = (\sigma_{ij})_{i,j\in\Lambda}$. Recall that (X,T) is said to have the big images and big pre-images property, abbreviated as (BIP), if there exist a finite set $\Lambda_0 \subset \Lambda$ of states such that for each $\ell \in \Lambda$ there exist $a, b \in \Lambda_0$ for which

$$\sigma_{a\ell}\sigma_{\ell b} = 1.$$

Note that this property is equivalent to what Mauldin and Urbański call "finitely primitive" ([69]). Also, mark that the property (BIP) is more restrictive than the big image property of [1], which was there used to obtain absolutely continuous invariant measures.

The following theorem is due to Sarig. The proof of the sufficiency part of this theorem can also be found in [69].

Theorem 6.1 *([93]) Let (X,T) be a topologically mixing infinite topological Markov chain, and let $\phi \in C(X)$ have summable variation*[7]*. In this situation we have that the following two statements are equivalent.*
(1) There exists an invariant Gibbs measure for ϕ.
(2) (X,T) has the property (BIP) and the Gurevic pressure $P_G(\phi)$ of ϕ is finite, that is, for some $\ell \in \Lambda$ we have

$$P_G(\phi) := \lim_{n\to\infty} \frac{1}{n} \log \sum_{T^n(x)=x} \mathbb{I}_\ell(x) \exp(\phi(x) + ... + \phi(T^{n-1}(x))) < \infty.$$

[7] i.e. $\sum_{n=1}^{\infty} V_n(\phi) < \infty$, where $V_n(\phi)$ denotes the maximal variation of ϕ over cylinders of length n.

Recently, this result has been partially extended by Stadlbauer in [95] to the case of random countable topological Markov chains. Moreover, for the situation of the theorem with the additional assumptions that a certain random (BIP) holds and that $V_1^y(\phi) < \infty$ for all $y \in Y$, it was shown in [24] that there exists an invariant measure.

Theorem 6.2 *([95]) Let (X, T) be a random topological Markov shift, and let ϕ be a locally fibre Hölder continuous function of index two*[8] *with finite Gurevic pressure. Also, assume that the functions $y \mapsto \sum_{k=1}^{\infty} \kappa(S^{-k}(y))r^k$, $y \mapsto \log \sup\{\mathcal{L}_\phi^{(y)} 1(x) : x \in X_{S(y)}\}$ and $y \mapsto \log \inf\{\mathcal{L}_\phi^{(y)} 1(x) : x \in X_{S(y)}\}$ are P-integrable, and let (X, T, ϕ) be of divergence type*[9]*. Then there exists a measurable function $\alpha : Y \to \mathbb{R}_+$ and a Gibbs family $\{\mu_y : y \in Y\}$ for the potential $P_G(\phi) - \phi$ such that, for all $y \in Y$ and all x in the fibre over y,*

$$\frac{d\mu_{S(y)}}{d\mu_y}(x) = \alpha(y) \exp(P_G(\phi) - \phi(x)).$$

In the work of Sarig in [91] and [92], which is closely related to the thermodynamic formalism, tightness is used to construct Gibbs measures via transfer operator techniques. Contrary to this approach, the results in [95] combine the Patterson measure construction with Crauel's Prohorov Theorem on tightness ([21]). To be more precise, let (X, T) be a random Markov chain over the base $(Y, \mathcal{B}, R, P)$, where P is some fixed probability measure. Then Crauel's theorem states that a sequence of bundle probabilities $\{\mu_y^{(n)} : y \in Y\}$ is relatively compact with respect to the narrow topology if and only if $\{\mu_y^{(n)} : y \in Y\}$ is tight[10]. Here, convergence of the discrete fibre measures $\{\mu_y^{(n)} : y \in Y\}$ towards $\{\mu_y : y \in Y\}$ with respect to the narrow topology means that for all functions f, which are continuous and bounded as functions on fibres, we have that

$$\int \int f d\mu_y^{(n)} \, dP(y) = \int \int f d\mu_y \, dP(y).$$

[8] $V_n^y(\phi) \leq \kappa(y) r^n$ for $n \geq 2$ and $\int \log \kappa \, dP < \infty$.

[9] For a given fixed measurable family $\xi_y \in \pi^{-1}(y)$, we have that the series $\sum_{n: S^n(y) \in Y'} s^n (\mathcal{L}_\phi^{(y)})^n (1)(\xi_{S^n(y)})$ converges for $s < 1$ and diverges for $s = 1$, where Y' is some set of positive measure.

[10] i.e. for all $\epsilon > 0$ there exists a measurable set $K \subset X$ such that $K \cap \pi^{-1}(\{y\})$ is compact, for all $y \in Y$, and $\inf_n \int \mu_y^{(n)}(K) \, dP(y) > 1 - \epsilon$.

The construction in Section 4 can then be carried out fibrewise, showing that there exist weak accumulation points with respect to the narrow topology (see also [24]). It is worth mentioning that, beyond this result, in this situation no further results on the existence of conformal measures seem to be known. Also, thorough investigations of the fractal geometry of such systems are currently still missing.

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