

PARETO OPTIMIZATION IN CATEGORIES

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ABSTRACT. We propose a model of Pareto optimization (multi-objective programming) in the context of a categorical theory of resources. We describe how to adapt multi-objective swarm intelligence algorithms to this categorical formulation.

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1. INTRODUCTION

Pareto optimization (or multi-objective programming) refers to a class of problems where several simultaneous objective functions (objective valuations), usually valued in cones inside real Euclidean spaces, need to be optimized simultaneously. Since they are subject to constraints, the optimization cannot be achieved simply by individually maximizing each function. A Pareto optimal solution (in general non-unique) is a solution where none of the objective functions can be improved without worsening some of the others. More precisely, a possible solution S_1 is said to Pareto dominate another solution S_2 if all the objective valuations f_i satisfy $f_i(S_1) \geq f_i(S_2)$ and for at least one of them the inequality is strict. Pareto optimal solutions are those that

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are not Pareto dominated by any other. The *Pareto frontier* is the set of all Pareto optimal solutions.

While Pareto optimization is a very useful approach in describing optimization problems that cannot be reduced to a single scalar function, the use of functions in Euclidean spaces is still often approached through a process of aggregative “scalarization” that considers weighted combinations of the different objective functions to reproduce a scalar optimization problem for a single real valued function.

Our goal here is to develop a setting for Pareto optimization that is entirely independent of real valued functions and is formulated in terms of the “mathematical theory of resources” (in the sense of [3] and [6]) in a categorical framework. In the setting we develop here, objective functions are replaced by objective *functors*, and the Pareto frontier and Pareto optimization are entirely describable in categorical terms. The categorical setting should not be surprising, as it is easy to see that the typical universal properties in category theory can be expressed in the form of optimization problems, so that category theory is indeed a natural setting for an abstract formulation of optimization problems.

We will formulate here our Pareto optimization setting in terms of assignments of resources to a (finite) set. Thus, the solutions of our optimization problem will be *summing functors* from a category of subsets of a finite set (which we generally think of as subsystems of a given system) to a symmetric monoidal category of resources. One can refine this setting by considering, as in [8], assignments of resources to a network (directed graph) using an appropriate notion of “network summing functors”. This choice of the source category is not necessary and can be replaced by other categories. We choose this setting because the present paper is part of a larger ongoing study of dynamical assignments of resources to networks in a categorical framework, [8], [9].

The objective functors, in turn, will be functors from the category of summing functors to other categories containing target goal objects. These serve the purpose of measuring whether a given assignment of resources to the system suffice to achieve the desired goals.

We introduce the Pareto frontier in terms of an optimization on “convertibility of resources” in a categorical sense, and we present a formulation of the Pareto frontier as a category of essential preimages of colimits of a class of diagrams.

In the usual setting of Pareto optimization, multi-objective programming can be formulated in terms of a *swarm intelligence* algorithm. The goal of the swarm particles in this approach is to find solutions as close as possible to the Pareto frontier and as diverse as possible, mapping out different regions of the Pareto frontier. This is achieved by considering a virtual swarm of “particles” that moves according to some dynamical rules across the landscape of possibilities (the configuration space). The structure of the swarm intelligence algorithm can be summarized as follows:

- (1) the swarm is initialized by a random distribution of positions and momenta drawn with uniform measure over configuration space;

- (2) each individual particle in the swarm can memorize its best solution up to the present time;
- (3) each particle in the swarm tends to search near its best position obtained so far;
- (4) each particle can see the positions of the other particles of the swarm at the same time and evaluate the best position achieved by the swarm at that moment;
- (5) each individual particle tends to move towards the best position achieved within the swarm at that time.

This idea is formalized by update rules (in discrete time) for positions and velocities of the swarm particles, where the velocities are updated by a rule of the form

$$V_i(n+1) = \lambda_3 V_i(n) + \lambda_1 G_1(X_i(n) - X_{i,best}(n)) + \lambda_2 G_2(X_i(n) - X_{best}(n))$$

where the G_i are Gaussians, the λ_i are tunable parameters, the $X_{i,best}(n)$ is the best position of the i -th particle in its previous history, that is, among the set of positions $\{X_i(0), \dots, X_i(n)\}$, while $X_{best}(n)$ is the best position among all the N swarm particles $\{X_1(n), \dots, X_N(n)\}$ at the given time n . The positions are then simply updated by the rule

$$X_i(n+1) = X_i(n) + V_i(n+1).$$

Under good conditions (see the discussion in [11], [12]), for large swarm size N and sufficiently many iterations n , the positions of the swarm draw out the Pareto frontier.

We show that a direct probabilistic analog of this swarm intelligence algorithm can be developed in the categorical framework. A single particle model identifies the Pareto frontier, but the resulting probability distribution is very spread out so it does not provide an efficient algorithm.

2. PARETO FRONTIER IN CATEGORIES

2.1. Categories of resources. As in [3] and [6], *resources* are modelled by a symmetric monoidal category $(\mathcal{C}, \circ, \otimes, \mathbb{I})$ (which we will also be writing “additively” as $(\mathcal{C}, \oplus, 0)$). Objects $A \in \text{Obj}(\mathcal{C})$ represent resources, the monoidal operation $A \otimes B$ represents the combination of resources, with the unit object \mathbb{I} of the monoidal structure representing the empty resource. Morphisms $f : A \rightarrow B$ in $\text{Mor}_{\mathcal{C}}(A, B)$ describe possible processes of conversion of resource A into resource B . Thus, the convertibility of resources is expressed by the condition $\text{Mor}_{\mathcal{C}}(A, B) \neq \emptyset$.

One associates to a category $(\mathcal{C}, \circ, \otimes, \mathbb{I})$ of resources a preordered abelian semigroup $(R, +, \succeq, 0)$ on the set R of isomorphism classes of objects in $\text{Obj}(\mathcal{C})$ with $A + B$ the class of $A \otimes B$ with unit 0 given by the class of the unit object \mathbb{I} and with $A \succeq B$ iff $\text{Mor}_{\mathcal{C}}(A, B) \neq \emptyset$. Measuring semigroups are abelian semigroups with partial ordering and with a semigroup homomorphism $M : (R, +) \rightarrow (S, *)$ with $M(A) \geq M(B)$ in S when $A \succeq B$ in R . It is shown in [6] that they satisfy $\rho_{A \rightarrow B} \cdot M(B) \leq M(A)$ with respect to the maximal conversion rate

$$\rho_{A \rightarrow B} := \sup \left\{ \frac{m}{n} \mid n \cdot A \succeq m \cdot B, m, n \in \mathbb{N} \right\}.$$

2.2. Summing functors. A summing functor is a *consistent assignment* of resources of type \mathcal{C} to *all subsystems* of a given system so that a combination of independent subsystems corresponds to combined resources. The notion of summing functors was first introduced by Segal in [13], in the homotopy theory setting of Gamma-spaces, in the case where \mathcal{C} is a category with sum and zero-object, and was extended by Thomason in [14], [15] to the more general case where \mathcal{C} is a symmetric monoidal category. We formulate it here for finite sets rather than for finite pointed sets as in the original setting.

Let $(\mathcal{C}, \oplus, 0)$ be a symmetric monoidal category, written in additive notation. Let S be a finite set and let $\mathcal{P}(S)$ denote the category with objects the subsets $A \subseteq S$ and morphisms the inclusions $j : A \subseteq A'$. A functor $\Phi_S : \mathcal{P}(S) \rightarrow \mathcal{C}$ is a *summing functor* if

$$\Phi_S(A \cup A') = \Phi_S(A) \oplus \Phi_S(A') \quad \text{when} \quad A \cap A' = \emptyset$$

and $\Phi_S(\emptyset)$ is the monoidal unit 0 of \mathcal{C} .

Let $\Sigma_{\mathcal{C}}(S)$ be the category of summing functors $\Phi_S : \mathcal{P}(S) \rightarrow \mathcal{C}$, with morphisms given by the *invertible* natural transformations. This category describes all the possible assignments of resources of type \mathcal{C} to the subsystems of S , with all the possible equivalences between such assignments.

A summing functor $\Phi_S : \mathcal{P}(S) \rightarrow \mathcal{C}$ is completely determined by values $\Phi_S(x) := \Phi_S(\{x\})$ for $x \in S$, and the category $\Sigma_{\mathcal{C}}(S)$ of summing functors is equivalent to the category $\hat{\mathcal{C}}^n$, where $n = \#S$ and where $\hat{\mathcal{C}}$ is the category with same objects as \mathcal{C} and the invertible morphisms of \mathcal{C} .

When the finite set S is replaced by a finite directed graph G , various notions of “network summing functors” can be considered that generalize the setting above. We refer the reader to [8] for a more detailed discussion. For the purposes of this paper we just discuss the case of categories of summing functors $\Sigma_{\mathcal{C}}(S)$ as above. The generalization to networks is straightforward.

2.3. Objective valuation functors. Let S be a finite set as above, with $\Sigma_{\mathcal{C}}(S)$ the category of summing functors for resources of type \mathcal{C} . A *valuation system* $(F, X) = (F_{\alpha}, X_{\alpha})_{\alpha \in \mathcal{I}}$ consists of a finite family $\{\mathcal{V}_{\alpha}\}_{\alpha \in \mathcal{I}}$ of categories that describe possible objectives for optimization, with functors $F_{\alpha} : \Sigma_{\mathcal{C}}(S) \rightarrow \mathcal{V}_{\alpha}$ (valuations) and objects $X_{\alpha} \in \text{Obj}(\mathcal{V}_{\alpha})$ (goals). Valuation functors may factor through the target category of resources \mathcal{C} , but we do not assume that this is necessarily the case. Valuation functors $F_{\alpha} : \Sigma_{\mathcal{C}}(S) \rightarrow \mathcal{V}_{\alpha}$ are in general *not* fully faithful.

A summing functor $\Phi \in \Sigma_{\mathcal{C}}(S)$ is *F-minorized* by another $\Psi \in \Sigma_{\mathcal{C}}(S)$ if

$$\text{Hom}_{\mathcal{V}_{\alpha}}(F_{\alpha}(\Phi), F_{\alpha}(\Psi)) \neq \emptyset \quad \forall \alpha \in \mathcal{I}.$$

It is *strictly F-minorized* if the above holds and there exists some $\alpha \in \mathcal{I}$ with $F_{\alpha}(\Phi)$ and $F_{\alpha}(\Psi)$ not isomorphic in \mathcal{V}_{α} (hence Φ is not isomorphic to Ψ). We can define *F-majorization* in a similar way.

The *F-minorization* condition above means that $F_{\alpha}(\Psi)$ is obtainable from $F_{\alpha}(\Phi)$ through an admissible “conversion of resources” in the category \mathcal{V}_{α} .

A summing functor $\Phi \in \Sigma_{\mathcal{C}}(S)$ is (F, X) -*minorized* by another $\Psi \in \Sigma_{\mathcal{C}}(S)$ if for all $\alpha \in \mathcal{I}$

$$\begin{aligned} \text{Hom}_{\mathcal{V}_\alpha}(F_\alpha(\Phi), F_\alpha(\Psi)) &\neq \emptyset, \\ \text{Hom}_{\mathcal{V}_\alpha}(F_\alpha(\Phi), X_\alpha) &\neq \emptyset, \\ \text{Hom}_{\mathcal{V}_\alpha}(F_\alpha(\Psi), X_\alpha) &\neq \emptyset, \end{aligned}$$

with a strict minorization if, for some α , $F_\alpha(\Phi)$ and $F_\alpha(\Psi)$ are not isomorphic. This means that $F_\alpha(\Psi)$ is obtainable from $F_\alpha(\Phi)$, while both are good enough to obtain the goals X_α .

We then define the Pareto frontier in the following way. An assignment of resources $\Phi \in \Sigma_{\mathcal{C}}(S)$ is on the (F, X) -*Pareto upper frontier* if

$$\text{Hom}_{\mathcal{V}_\alpha}(F_\alpha(\Phi), X_\alpha) \neq \emptyset \quad \forall \alpha \in \mathcal{I}$$

but there is no $\Psi \in \Sigma_{\mathcal{C}}(S)$ not isomorphic to Φ that is a strict (F, X) -minorization of Φ .

The terminology ‘‘upper frontier’’ is used here to indicate an optimization over valuations that lie ‘‘above the goals’’. An analogous notion of lower frontier can be defined with the condition $\text{Hom}_{\mathcal{V}_\alpha}(X_\alpha, F_\alpha(\Phi)) \neq \emptyset$ and (F, X) -majorizations.

2.4. Categorical Pareto frontier. We give here a description of the Pareto frontier as a category.

2.4.1. Preorders and diagrams. A preorder \preceq on a set J is a relation that is transitive ($x \preceq y$ and $y \preceq z \Rightarrow x \preceq z$) and reflexive ($x \preceq x$ for all x). A preorder (J, \preceq) is a directed set if $J \neq \emptyset$ and for all $x, y \in J$ there is a $z \in J$ with $x \preceq z$ and $y \preceq z$. Every finite subset $\{x_1, \dots, x_n\}$ of a directed set (J, \preceq) has an upper bound, that is, an element z such that $x_i \preceq z$ for all $i = 1, \dots, n$. A preorder (J, \preceq) is a thin category with objects $x \in J$ and a single morphism $x \rightarrow y$ when $x \preceq y$. A directed set is a filtered thin category (all finite diagrams have a cocone).

Let $\Sigma_{\mathcal{C}}^{adm}(S)$ denote the full subcategory of $\Sigma_{\mathcal{C}}(S)$ of (F, X) -admissible summing functors, with objects those $\Phi \in \text{Obj}(\Sigma_{\mathcal{C}}(S))$ such that

$$\text{Hom}_{\mathcal{V}_\alpha}(F_\alpha(\Phi), X_\alpha) \neq \emptyset,$$

for all $\alpha \in \mathcal{I}$. On $\text{Obj}(\Sigma_{\mathcal{C}}^{adm}(S))$ consider the preorder relations $\Psi \preceq_\alpha \Phi$ iff

$$\text{Hom}_{\mathcal{V}_\alpha}(F_\alpha(\Phi), F_\alpha(\Psi)) \neq \emptyset.$$

We write $(\mathcal{J}_\alpha, \preceq_\alpha)$ for the thin category with objects $\text{Obj}(\Sigma_{\mathcal{C}}^{adm}(S))$ and a single morphism $\Psi \rightarrow \Phi$ iff $\Psi \preceq_\alpha \Phi$.

Let \mathcal{D}_α denote the category with objects $Y \in \text{Obj}(\mathcal{V}_\alpha)$ that are isomorphic $Y \simeq F_\alpha(\Phi)$ in \mathcal{V}_α , for some $\Phi \in \Sigma_{\mathcal{C}}^{adm}(S)$, and with morphisms $\varphi = (u, v, v')$ that form a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{u} & Y' \\ & \searrow v & \swarrow v' \\ & & X_\alpha \end{array}$$

We can then view minorizations as diagrams $D : \mathcal{J}_\alpha^{op} \rightarrow \mathcal{D}_\alpha$ of the form

$$(2.1) \quad \begin{array}{ccc} F_\alpha(\Phi) & \longrightarrow & F_\alpha(\Psi) \\ & \searrow & \swarrow \\ & X_\alpha & \end{array}$$

Thus, we are interested in considering inverse systems, namely diagrams $D : \mathcal{J}_\alpha^{op} \rightarrow \mathcal{D}_\alpha$, and their colimits in \mathcal{D}_α

$$L_\alpha(D) := \operatorname{colim}_{\mathcal{J}_\alpha^{op}} D.$$

We consider the following class of diagrams.

Let $\operatorname{Diagr}_\alpha$ be the class of diagrams $D : \mathcal{J}_\alpha^{op} \rightarrow \mathcal{D}_\alpha$ with shape

$$(2.2) \quad \bullet \rightarrow \bullet \rightarrow \cdots \bullet \rightarrow$$

either finite or infinite, with the following properties:

- (1) if the diagram is finite of length n then the object $F_\alpha(\Phi_n)$ in the terminal position must satisfy the condition that, for any admissible Ψ ,

$$\operatorname{Hom}(F_\alpha(\Phi_n), F_\alpha(\Psi)) \neq \emptyset \Rightarrow F_\alpha(\Phi_n) \simeq F_\alpha(\Psi),$$

- (2) any two consecutive terms $F_\alpha(\Phi_i) \rightarrow F_\alpha(\Phi_{i+1})$ in the diagrams are non-isomorphic, $F_\alpha(\Phi_i) \not\simeq F_\alpha(\Phi_{i+1})$.

These are finite or infinite diagrams in \mathcal{D}_α of the form

$$\begin{array}{ccccc} F_\alpha(\Phi_1) & \longrightarrow & \cdots & \longrightarrow & F_\alpha(\Phi_n) & \cdots \\ & \searrow & & \swarrow & & \\ & & \downarrow & & & \\ & & X_\alpha & & & \end{array}$$

where all the horizontal arrows are strict minorizations.

Let $\mathcal{L}_\alpha = \{L_\alpha(D) \mid D \in \operatorname{Diagr}_\alpha\}$ be the colimits of the diagrams in $\operatorname{Diagr}_\alpha$ (when they exist in \mathcal{D}_α). Note that, in the case of a finite diagram of shape $\bullet \rightarrow \cdots \rightarrow \bullet$ the colimit is isomorphic to the last term. So the interesting case is that of infinite diagrams.

2.4.2. The Pareto frontier category. Given a valuation functor $F_\alpha : \Sigma_{\mathcal{C}}(S) \rightarrow \mathcal{V}_\alpha$ and the class \mathcal{L}_α of objects of \mathcal{D}_α given by colimits of diagrams in $\operatorname{Diagr}_\alpha$ as above, consider the full subcategory $F_\alpha^{-1}(\mathcal{L}_\alpha)$ of $\Sigma_{\mathcal{C}}(S)$ with objects

$$\operatorname{Obj}(F_\alpha^{-1}(\mathcal{L}_\alpha)) = \{\Phi \in \operatorname{Obj}(\Sigma_{\mathcal{C}}^{adm}(S)) \mid F_\alpha(\Phi) \simeq L_\alpha(D) \text{ for some } D \in \operatorname{Diagr}_\alpha\}.$$

Given the finite collection $F = \{F_\alpha\}_{\alpha \in \mathcal{I}}$ of valuation functors, and the collection of objects $\mathcal{L} = \cup_\alpha \mathcal{L}_\alpha$ we similarly define the subcategory $\Sigma_{\mathcal{C}}^{(F, \mathcal{L})}(S) \subset \Sigma_{\mathcal{C}}^{adm}(S)$ as the full subcategory with objects given by

$$\operatorname{Obj}(\Sigma_{\mathcal{C}}^{(F, \mathcal{L})}(S)) = \cap_\alpha \operatorname{Obj}(F_\alpha^{-1}(\mathcal{L}_\alpha))$$

We can then extend our previous definition of the Pareto frontier in the following way. We define the *Pareto frontier category* to be the category $\Sigma_{\mathcal{C}}^{(F, \mathcal{L})}(S)$ obtained above. Namely, an object $\Phi \in \Sigma_{\mathcal{C}}^{adm}(S)$ is on the (upper) Pareto frontier with respect to the system (F, X) of valuations and goals, iff Φ is in $\Sigma_{\mathcal{C}}^{(F, \mathcal{L})}(S)$, with \mathcal{L}_{α} the colimits of diagrams in Diagr_{α} (whenever these colimits exist in \mathcal{D}_{α}).

Note that, for all the finite diagrams, this reproduces by construction the Pareto frontier as we described it above, since the essential preimages of the colimits in this case are exactly those admissible Φ that have no strict minorization, namely

$$\text{Hom}(F_{\alpha}(\Phi), F_{\alpha}(\Psi)) \neq \emptyset \Rightarrow F_{\alpha}(\Phi) \simeq F_{\alpha}(\Psi).$$

The difference here is that we include the colimits of the infinite sequences of strict minorizations, if these colimits exist in \mathcal{D}_{α} , and we describe the Pareto frontier as a category rather than a set/class.

3. PROBABILISTIC PARTICLES

In this section we discuss a direct analog, in our categorical setting, of the usual swarm intelligence algorithm for multi-objective optimization, and we show that this simple generalization does not suffice to identify the Pareto frontier.

3.1. Probabilistic categories. Let \mathcal{FP} be the category of finite probabilities, with objects (X, P) consisting of a finite set X with a probability measure P , and morphisms $S \in \text{Hom}_{\mathcal{FP}}((X, P), (Y, Q))$ given by stochastic $(\#Y \times \#X)$ -matrices S , namely matrices with $S_{yx} \geq 0$, for all $x \in X$, $y \in Y$ and $\sum_{y \in Y} S_{yx} = 1$ for all $x \in X$, such that the probability measures are related by $Q = SP$.

As shown in [10], given a category \mathcal{C} , one can construct a probabilistic version \mathcal{PC} , which can be viewed as a wreath product $\mathcal{FP} \wr \mathcal{C}$ of the category \mathcal{C} with the category \mathcal{FP} of finite probabilities.

The objects of \mathcal{PC} and formal finite convex combinations

$$\Lambda C = \sum_i \lambda_i C_i$$

with $\Lambda = (\lambda_i)$ a finite probability and $C_i \in \text{Obj}(\mathcal{C})$.

Morphisms in $\text{Hom}_{\mathcal{PC}}(\Lambda C, \Lambda' C')$ are pairs $(S, f) : \Lambda C \rightarrow \Lambda' C'$ with S a stochastic matrix with $S\Lambda = \Lambda'$, and $f = \{f_{ab,r}\}$ finite collection of morphisms $f_{ab,r} : C_b \rightarrow C'_a$ with assigned probabilities μ_r^{ab} . These probabilities satisfy $\sum_r \mu_r^{ab} = S_{ab}$.

Morphisms (S, f) in \mathcal{PC} can be interpreted as ways of mapping C_b to C'_a by randomly choosing a morphism from the set $\{f_{ab,r}\}$, with probability μ_r^{ab} of choosing $f_{ab,r}$.

We have the following simple characterization of isomorphic objects in probabilistic categories.

Lemma 3.1. *Two objects $\Lambda X = \sum_i \lambda_i^n X_i$ and $\Lambda' X' = \sum_{j=1}^m \lambda'_j X'_j$ in a probabilistic category \mathcal{PC} are isomorphic if and only if $n = m$ with $X_i \simeq X'_{\sigma(i)}$ (isomorphic in \mathcal{C}) for some permutation σ and $\lambda_i = \lambda'_{\sigma(i)}$.*

Proof. The isomorphism $\Lambda X \simeq \Lambda' X'$ means that there is an invertible morphism $(S, f) : \Lambda X \rightarrow \Lambda' X'$. In particular S with $S\Lambda = \Lambda'$ must be a stochastic matrix with stochastic inverse, hence we have $n = m$ and S is necessarily a permutation matrix. Thus, $\lambda_i = \lambda'_{\sigma(i)}$ for a permutation σ . The collection of morphisms $f = \{f_{ij,r}\}$ then have probabilities $\mu_{ij,r}$ satisfying $\sum_r \mu_{ij,r} = S_{ij}$ hence they can be nonzero only for $j = \sigma(i)$, with $\sum_r \mu_{i\sigma(i),r} = 1$, and $f_{i,\sigma(i),r} : X_i \rightarrow X'_{\sigma(i)}$ an isomorphism. \square

For \mathcal{C} a small category, it is also natural to assume that an object $\sum_i \lambda_i C_i$ of \mathcal{PC} where $C_i = C$ for all i would be the same as the object C with probability $\Lambda = \{1\}$. However, it is better to just require, more generally, that, whenever $C_i \simeq C$, the objects $\sum_i \lambda_i C_i$ and C are isomorphic objects. This can be achieved by a localization of the category \mathcal{PC} .

Lemma 3.2. *Let \mathcal{W} be the class of morphisms in \mathcal{PC} of the form $\varphi : \sum_i \lambda_i C_i \rightarrow C$ with $\varphi = (S, f)$ with an $n \times (n+k)$ stochastic matrix S of the form*

$$S = \left(\begin{array}{ccc|ccc|ccc} 1 & 0 & \cdots & 0 & 0 & 0 \cdots 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & 0 & 0 & 0 \cdots 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \cdots 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 1 & 1 \cdots 1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \cdots 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \cdots 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 0 & 0 & 0 \cdots 0 & 0 & 0 & \cdots & 1 \end{array} \right)$$

$\underbrace{\hspace{10em}}_{\ell} \quad \underbrace{\hspace{10em}}_k \quad \underbrace{\hspace{10em}}_{n-\ell}$

and where the set f of morphisms consists of the identity id_{C_i} for $i = 1, \dots, \ell$ and $i = \ell + k + 1, \dots, n + k$, and of isomorphisms $f_i : C_i \rightarrow C$, for $i = \ell + 1, \ell + k$, all of them occurring with probability 1. Then the localization $\mathcal{PC}[\mathcal{W}^{-1}]$ implements the equivalence relation described above.

Proof. The category $\mathcal{PC}[\mathcal{W}^{-1}]$ is the localization of \mathcal{PC} at \mathcal{W} . Isomorphisms in $\mathcal{PC}[\mathcal{W}^{-1}]$ are arbitrary compositions of the isomorphisms of Lemma 3.1 and morphisms in \mathcal{W} and their formal inverses. Thus, in this category we have isomorphisms between an object $\sum_{i=1}^m \lambda_i C_i$ where the C_i for a subset $i \in I \subset \{1, \dots, m\}$ of indices are all isomorphic to the same object C and the object $(\sum_{i \in I} \lambda_i) C + \sum_{i \in I^c} \lambda_i C_i$. \square

3.1.1. *Probabilistic categories of functors.* Of particular interest here is the case where the category \mathcal{C} is a category of functors $\mathcal{C} = \text{Func}(\mathcal{D}, \mathcal{D}')$. In this case, when we form the probabilistic category $\mathcal{PFunc}(\mathcal{D}, \mathcal{D}')$, we want to interpret an object $\Lambda F = \sum_i \lambda_i F_i$ in $\mathcal{PFunc}(\mathcal{D}, \mathcal{D}')$ as a functor that to an object X in $\text{Obj}(\mathcal{D})$ assigns $F_i(X) \in \text{Obj}(\mathcal{D}')$ with probability λ_i . In order to make this heuristics precise, so that we

can use it in defining swarm dynamics in categories, we need the following simple statements.

Lemma 3.3. *Functors $F : \mathcal{C} \rightarrow \mathcal{C}'$ extend to functors $F : \mathcal{PC} \rightarrow \mathcal{PC}'$ mapping an object $\Lambda C = \sum_i \lambda_i C_i$ of \mathcal{PC} to the object $\Lambda F(C) = \sum_i \lambda_i F(C_i)$ of \mathcal{PC}' and a morphism (S, f) with $f = \{f_{ab,r}\}$ and probabilities $\mu_{ab,r}$ to the morphism $F(S, f) = (S, F(f))$ with $F(f) = \{F(f_{ab,r})\}$ with probabilities $\mu_{ab,r}$.*

This follows directly from the definition. Moreover, there is a functor between the probabilistic category of functors $\mathcal{PFunc}(\mathcal{D}, \mathcal{D}')$ and the category of functors between the probabilistic categories $\text{Func}(\mathcal{PD}, \mathcal{PD}')$.

Lemma 3.4. *There is a faithful functor $\mathcal{PFunc}(\mathcal{D}, \mathcal{D}') \rightarrow \text{Func}(\mathcal{PD}, \mathcal{PD}')$.*

Proof. Let $\Lambda F = \sum_i \lambda_i F_i$ be an object in the probabilistic category $\mathcal{PFunc}(\mathcal{D}, \mathcal{D}')$. We first show that ΛF defines a functor in $\text{Func}(\mathcal{PD}, \mathcal{PD}')$. For $\Omega X = \sum_a \omega_a X_a \in \text{Obj}(\mathcal{PD})$ we take $\Lambda F(\Omega X) = \sum_{i,a} \lambda_i \omega_a F_i(X_a) =: \Lambda \Omega F(X)$ in $\text{Obj}(\mathcal{PD}')$, where $(\Lambda \Omega)_{i,a} = \lambda_i \omega_a$. For $(S, f) : \Omega X \rightarrow \Omega' X'$ in $\text{Hom}_{\mathcal{PD}}(\Omega X, \Omega' X')$, with $f = \{f_{ab,r}\}$ with probabilities $\mu_{ab,r}$, we set $\Lambda F(S, f) : \Lambda \Omega F(X) \rightarrow \Lambda \Omega' F(X')$ with $\Lambda F(S, f) = (S', f')$, where $S'_{ij,ab} = \delta_{ij} S_{ab}$, so that $S' \Lambda \Omega = \Lambda \Omega'$, and $f' = \{F_i(f_{ab,r}) \cdot \delta_{ij}\}$ with probabilities $\mu_{ab,r}$. Consider then a morphism (R, α) in $\text{Hom}_{\mathcal{PFunc}(\mathcal{D}, \mathcal{D}')}(\Lambda F, \Lambda' F')$, with a stochastic matrix R with $R \Lambda = \Lambda'$ and $\alpha = \{\alpha_{ij,s}\}$ with probabilities $\nu_{ij,s}$ with $\sum_s \nu_{ij,s} = R_{ij}$, and with $\alpha_{ij,s} : F_i \rightarrow F'_j$ a collection of natural transformations between functors in $\text{Func}(\mathcal{D}, \mathcal{D}')$. Then (R, α) defines a natural transformation of functors in $\text{Func}(\mathcal{PD}, \mathcal{PD}')$, by taking, for each object $\Omega X \in \text{Obj}(\mathcal{PD})$ assigns the morphism in \mathcal{PD}'

$$(R, \alpha)|_{\Omega X} : \Lambda F(\Omega X) \rightarrow \Lambda' F'(\Omega X)$$

with stochastic matrix R and with the collection $\{\alpha_{ij,s}|_{X_a}\}$ with probabilities $\nu_{ij,s}$, where $\alpha_{ij,s}|_{X_a} : F_i(X_a) \rightarrow F'_j(X_a)$ is the morphism in $\text{Hom}_{\mathcal{D}'}(F_i(X_a), F'_j(X_a))$ specified by the natural transformation $\alpha_{ij,s}$. The morphism (R, α) in $\mathcal{PFunc}(\mathcal{D}, \mathcal{D}')$ uniquely specifies this natural transformation. \square

We can interpret the difference between viewing an object $\Lambda F = \sum_i \lambda_i F_i$ with $F_i \in \text{Obj}(\mathcal{E}(\mathcal{C}))$ and Λ a probability distribution as objects of $\mathcal{P}(\mathcal{E}(\mathcal{C}))$ or (through the functor of Lemma 3.4) as objects in $\mathcal{E}(\mathcal{P}(\mathcal{C}))$ as, respectively, the *probabilistic* and *deterministic* interpretations of ΛF .

3.1.2. Other probabilistic conditions. We assume in this section that the category \mathcal{C} or resources, where summing functors $\Phi \in \Sigma_{\mathcal{C}}(S)$ take values, is a *small* category endowed with a *probability distribution* \mathbb{P} on the set $\text{Obj}(\mathcal{C})$. This probability can be seen as modeling the relative abundance or scarcity of resources.

Through the identification of summing functors in $\Sigma_{\mathcal{C}}(S)$ with objects in $\hat{\mathcal{C}}^n$, with $n = \#S$, we then obtain an induced probability, which we also denote by \mathbb{P} , on $\text{Obj}(\Sigma_{\mathcal{C}}(S))$.

Consider, as above, the subcategory $\Sigma_{\mathcal{C}}^{adm}(S)$. The condition

$$\mathbb{P}(\text{Obj}(\Sigma_{\mathcal{C}}^{adm}(S))) > 0$$

ensures that the set (F, X) of goals and valuations is not incompatible with the availability of resources of type \mathcal{C} .

Let $\mathcal{M}_{(X,F)}^{adm}(\Phi) \subset \text{Obj}(\Sigma_{\mathcal{C}}^{adm}(S))$ denote the set of all strict (F, X) -minorizations of $\Phi \in \Sigma_{\mathcal{C}}^{adm}(S)$. The condition that Φ is on the Pareto frontier is then that $\mathcal{M}_{(X,F)}^{adm}(\Phi) = \emptyset$. If the measure \mathbb{P} has no non-empty sets of measure zero, then $\lambda(\Phi) := \mathbb{P}(\mathcal{M}_{(X,F)}^{adm}(\Phi)) = 0$ iff Φ is on the Pareto frontier.

3.2. Single particle. For the dynamics of a single particle, we initialize at time zero by drawing an object Φ_0 from $\text{Obj}(\Sigma_{\mathcal{C}}^{adm}(S))$ uniformly at random with respect to the probability measure \mathbb{P} .

The dynamics then proceeds by making new random steps and comparing them (“velocities” are here regarded as probabilistic jumps to a new position). Thus, at the first step (time $t = 1$) a new draw of an element Φ_1 is performed. With probability $\lambda_0 = \mathbb{P}(\mathcal{M}_{(X,F)}^{adm}(\Phi_0))$ this new point improves the position with respect to Φ_0 , being a strict minorization of Φ_0 . If it is not (with probability $1 - \lambda_0$) then one keeps the same position Φ_0 . This means that the result of the first step is an object in the probabilistic category $\mathcal{P}\Sigma_{\mathcal{C}}^{adm}(S)$ of the form

$$(\Lambda\Phi)_1 := (1 - \lambda_0)\Phi_0 + \lambda_0\Phi_1.$$

At the second step (time $t = 2$), one makes another random draw Φ_2 . With $\lambda_1 = \mathbb{P}(\mathcal{M}_{(X,F)}^{adm}(\Phi_1))$, one then obtains a new object in the probabilistic category of the form

$$\begin{aligned} (\Lambda\Phi)_2 &= (1 - \lambda_0)((1 - \lambda_0)\Phi_0 + \lambda_0\Phi_2) + \lambda_0((1 - \lambda_1)\Phi_1 + \lambda_1\Phi_2) \\ &= (1 - \lambda_0)^2\Phi_0 + \lambda_0(1 - \lambda_1)\Phi_1 + \lambda_0(1 - (\lambda_0 - \lambda_1))\Phi_2, \end{aligned}$$

that describes all the possible relative positions of Φ_0, Φ_1, Φ_2 with the respective probabilities.

An inductive argument shows that one obtains the following behavior of this single particle case.

Proposition 3.5. *After n steps the outcome is an object of $\mathcal{P}\Sigma_{\mathcal{C}}^{adm}(S)$ of the form*

$$(\Lambda\Phi)_n = \sum_{k=0}^n c_n^k \Phi_k$$

with the probability $\Lambda_n = (c_n^k)_{k=0}^n$ satisfying the recursion (with $c_0^0 = 1$)

$$(3.1) \quad \begin{cases} c_n^k = c_k^k (1 - \lambda_k)^{n-k} & 0 \leq k \leq n-1 \\ c_n^n = \sum_{k=0}^{n-1} \lambda_k (1 - \lambda_k)^{n-1-k} c_k^k \end{cases}.$$

Proof. We have $c_0^0 = 1$ with $c_1^0 = 1 - \lambda_0$ and $c_1^1 = \lambda_0$ as above. At the $(n + 1)$ -st step we are comparing the new draw Φ_{n+1} with each of the previous Φ_k : it will be better than Φ_k (a strict minorization) with probability λ_k and not better with probability $1 - \lambda_k$. When applied to the previous $(\Lambda\Phi)_n$ we then obtain a new $(\Lambda\Phi)_{n+1}$ of the form

$$c_n^0((1 - \lambda_0)\Phi_0 + \lambda_0\Phi_{n+1}) + \cdots + c_n^n((1 - \lambda_n)\Phi_n + \lambda_n\Phi_{n+1}),$$

which gives $c_{n+1}^0 = (1 - \lambda_0)^{n+1}$, $c_{n+1}^1 = \lambda_0(1 - \lambda_0)^n$, $c_{n+1}^2 = c_2^2(1 - \lambda_2)^{n-1}$, \dots , $c_{n+1}^n = c_n^n(1 - \lambda_n)$, and

$$c_{n+1}^{n+1} = c_n^0\lambda_0 + c_n^1\lambda_1 + \cdots + c_n^n\lambda_n.$$

The first relations give $c_{n+1}^k = c_n^k(1 - \lambda_k)^{n+1-k}$, while the last one combined with this gives the second recursive relation of the statement. The recursion directly implies that the normalization $\sum_k c_n^k = 1$ holds. \square

We have the following easy reformulation of the recursion (3.1).

Corollary 3.6. *The recursive relation for the probabilities $c_n = (c_n^k)_{k=1}^n$ is implemented by $c_{n+1} = S_n c_n$, with the $(n + 1) \times n$ stochastic matrix*

$$S_n = \begin{pmatrix} 1 - \lambda_0 & 0 & 0 & \cdots & 0 \\ 0 & 1 - \lambda_1 & 0 & \cdots & 0 \\ 0 & 0 & 1 - \lambda_2 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 - \lambda_n \\ \lambda_0 & \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{pmatrix}.$$

All the coefficients c_n^k are polynomials in the λ_i so they depend on Φ_0, \dots, Φ_n . The coefficient c_n^k is the probability of having Φ_k as the ‘‘best position’’ of the particle during the first n steps. More precisely, the coefficient c_n^k measures the probability that, among the first draws $\{\Phi_0, \dots, \Phi_n\}$ there is a subsequence of k strict minorizations ending with Φ_k .

Simple numerical examples with different choices of a sequence $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \cdots$ show that this probability distribution $\Lambda_n = (c_n^k)_{k=0}^n$ can be very spread out: it peaks somewhere in k , but not always at the end term and can be very non-concentrated.

Lemma 3.7. *For $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_n \cdots$, the coefficients c_n^n satisfy the estimate*

$$(3.2) \quad c_k^k(1 - \lambda_0)^{n-k} \leq c_n^n \leq c_k^k$$

for all $k = 0, \dots, n - 1$.

Proof. By the recursion (3.1) we have

$$c_n^n = \sum_{k=0}^{n-2} c_k^k(1 - \lambda_k)^{n-k} \lambda_k + c_{n-1}^{n-1} \lambda_{n-1} \leq \left(\sum_{k=0}^{n-2} c_k^k(1 - \lambda_k)^{n-1-k} \lambda_k \right) (1 - \lambda_{n-1}) + c_{n-1}^{n-1} \lambda_{n-1} = c_{n-1}^{n-1}$$

while similarly we also get $c_{n-1}^{n-1}(1 - \lambda_0) \leq c_n^n$. Iterating these estimates we get (3.2). \square

(2) A collection of M cocones in \mathcal{D}_α given by commutative diagrams

$$(3.5) \quad \begin{array}{ccccccc} F_\alpha(\Phi_0) & \xrightarrow{\varphi_0} & F_\alpha(\Phi_{\ell_1}) & \xrightarrow{\varphi_1} & \cdots & \longrightarrow & F_\alpha(\Phi_{\ell_k}) & \xrightarrow{\varphi_k} & \cdots \\ & & \searrow^{f_{0,r}} & \searrow^{f_{1,r}} & & & \swarrow_{f_{n,r}} & & \\ & & & & & & & & Y_{\alpha,r} \end{array}$$

for $r = 1, \dots, M$, together with a sequence of $M \times n$ stochastic matrices \tilde{S}_n with columns equal to the uniform distribution $1/M$ on the set $r \in \{1, \dots, M\}$, induces a cocone in \mathcal{PD}_α with commutative diagrams

$$(3.6) \quad \begin{array}{ccc} \cdots \longrightarrow \sum_{k=0}^n c_n^k F_\alpha(\Phi_k) & \xrightarrow[(\tilde{S}_n, \underline{f}_n)]{(S_n, \underline{\varphi}_n)} & \sum_{k=0}^{n+1} c_{n+1}^k F_\alpha(\Phi_k) \longrightarrow \cdots \\ & \searrow & \downarrow (\tilde{S}_{n+1}, \underline{f}_{n+1}) \\ & & \tilde{\Lambda} Y_\alpha \end{array}$$

where $\tilde{\Lambda} Y_\alpha = \sum_{r=1}^M \tilde{\lambda}_r Y_{\alpha,r}$ with $\tilde{\Lambda} = (\tilde{\lambda}_j = 1/M)$ the uniform probability distribution $1/M$. The set of morphisms \underline{f}_n consists of the $f_{k,r} : F_\alpha(\Phi_k) \rightarrow Y_{\alpha,r}$, for $k = 0, \dots, n$, each occurring with probability $1/M$.

Proof. (1) This follows from the fact that the maps $\varphi_{k,n+1} = \varphi_n \circ \cdots \circ \varphi_k$ satisfy the properties of a directed system $\varphi_{n+2,m+1} \circ \varphi_{k,n+1} = \varphi_{k,m+1}$ and $\varphi_{k,k} = \text{id}_{F_\alpha(\Phi_k)}$.

(2) In order to obtain commutative diagrams the sequence of stochastic matrices \tilde{S}_n must satisfy the recursive condition $\tilde{S}_{n+1} \cdot S_n = \tilde{S}_n$, or equivalently $(1 - \lambda_k)(\tilde{S}_{n+1})_{rk} + \lambda_k(\tilde{S}_{n+1})_{r,n+1} = (\tilde{S}_n)_{rk}$, for $k = 1, \dots, n$. This recursive condition $\tilde{S}_{n+1} \cdot S_n = \tilde{S}_n$ is solved by the stochastic matrices \tilde{S}_n with columns given by the uniform distribution $1/M$ on $r \in \{1, \dots, M\}$. Thus, we obtain commutative diagrams

$$\begin{array}{ccc} \cdots \longrightarrow \sum_{k=0}^n c_n^k F_\alpha(\Phi_k) & \xrightarrow[(\tilde{S}_n, \underline{f}_n)]{(S_n, \underline{\varphi}_n)} & \sum_{k=0}^{n+1} c_{n+1}^k F_\alpha(\Phi_k) \longrightarrow \cdots \\ & \searrow & \downarrow (\tilde{S}_{n+1}, \underline{f}_{n+1}) \\ & & \tilde{\Lambda} Y_\alpha \end{array}$$

where $\tilde{\Lambda} Y_\alpha = \sum_r \tilde{\lambda}_r Y_{\alpha,r}$, with the probability measure $\tilde{\Lambda}$ determined by $\tilde{S}_n c_n = \tilde{\Lambda}$, where the left-hand-side is independent of n by the conditions $\tilde{S}_{n+1} \cdot S_n = \tilde{S}_n$ and $S_n c_n = c_{n+1}$. For \tilde{S}_n as above this gives that $\tilde{\Lambda}$ is the uniform distribution $1/M$. This diagram defines a cocone in \mathcal{PD}_α . \square

Corollary 3.10. Consider a diagram (3.3) with colimit $L_\alpha(D)$ in \mathcal{D}_α , and the class of cocone diagrams in \mathcal{PD}_α of the form (3.6) obtained as in Proposition 3.9. This class of cocones, seen in the localization $\mathcal{PD}_\alpha[\mathcal{W}^{-1}]$, has colimit isomorphic to $L_\alpha(D)$.

Proof. For each diagram we have canonical maps from the system to the colimit and from the colimit to the tip of the cocone

$$\begin{array}{ccccccc}
 F_\alpha(\Phi_0) & \xrightarrow{\varphi_0} & F_\alpha(\Phi_{\ell_1}) & \xrightarrow{\varphi_1} & \cdots & \longrightarrow & F_\alpha(\Phi_{\ell_k}) & \xrightarrow{\varphi_k} & \cdots \\
 & \searrow & \searrow & \searrow & & & \searrow & & \\
 & & & & & & L_\alpha(D) & & \\
 & & & & & & \downarrow & & \\
 & & & & & & Y_{\alpha,r} & &
 \end{array}$$

As in Proposition 3.9 this gives induced diagrams in \mathcal{PD}_α with $\tilde{\Lambda}$ the uniform probability distribution,

$$\begin{array}{ccccc}
 \cdots & \longrightarrow & \sum_{k=0}^n c_n^k F_\alpha(\Phi_k) & \xrightarrow{(S_n, \varphi_n)} & \sum_{k=0}^{n+1} c_{n+1}^k F_\alpha(\Phi_k) & \longrightarrow & \cdots \\
 & & \searrow & & \searrow & & \\
 & & & & \tilde{\Lambda} L_\alpha(D) & & \\
 & & & & \downarrow & & \\
 & & & & \tilde{\Lambda} Y_\alpha & &
 \end{array}$$

hence we can identify $\tilde{\Lambda} L_\alpha(D)$ with the colimit of this class of cocones. By viewing these induced diagram in the localization $\mathcal{PD}_\alpha[\mathcal{W}^{-1}]$ we obtain an isomorphism $\tilde{\Lambda} L_\alpha(D) \simeq L_\alpha(D)$. \square

In particular, consider the case of finite diagrams in D of strict minorizations in Diagr_α , with Φ_n on the Pareto frontier,

$$(3.7) \quad \begin{array}{ccccccc}
 F_\alpha(\Phi_0) & \longrightarrow & F_\alpha(\Phi_1) & \longrightarrow & \cdots & \longrightarrow & F_\alpha(\Phi_n) \\
 & & \searrow & & & & \searrow \\
 & & & & & & X_\alpha
 \end{array}$$

where we have $\lambda_0 \geq \lambda_1 \geq \cdots \geq \lambda_{n-1}$ and $\lambda_n = 0$, since Φ_n is on the Pareto frontier, hence $\mathcal{M}_{(X,F)}^{adm}(\Phi_n) = \emptyset$. We obtain in this case, by the same argument above, that the colimit of the induced finite diagrams of the $\sum_k c_m^k F_\alpha(\Phi_k)$ with $0 \leq m \leq n$, is given by the object $\tilde{\Lambda} F_\alpha(\Phi_n)$ with $\tilde{\Lambda}$ the uniform distribution. This is isomorphic to $F_\alpha(\Phi_n)$ in the localization $\mathcal{PD}_\alpha[\mathcal{W}^{-1}]$.

This shows that our probabilistic single particle model still computes the same colimit over a sequence of minorizations in Diagr_α , so it does identify the Pareto frontier, both in the case of finite and of infinite chains of minorizations.

However, because of the fact that the probability distribution $c_n = \{c_n^k\}_{k=1}^n$ tends to be very spread out, the single particle model does not provide an efficient computational method. Moreover, we have not established yet a method of construction

of successive approximations to the Pareto frontier using sequences of random draws $\{\Phi_n\}_{n \geq 0}$ and their associated probabilistic objects $\sum_k c_n^k \Phi_k$. We will address these in the next section.

4. SWARMS IN CATEGORIES

In order to address approximation, we need to assume additional structure on the target categories \mathcal{V}_α of the valuation functors. In particular, we introduce a *scale* parameter in the \mathcal{V}_α so that we can consider these target objects as varying at different scales. We then use the changes of scale to introduce a notion of proximity, in the form of the associated interleaving distance, [1], [2]. This is a way of measuring proximity between \mathcal{V}_α -type resources by checking whether resource conversion can be inverted after a sufficiently small change of scale. This provides a convenient measurement of approximation and convergence to the Pareto frontier, with respect to which one can evaluate possible approximation algorithms. In particular, we will discuss analogs in this setting of the swarm intelligence algorithms for multi-objective programming.

4.1. Scale structure. As described above, we consider an additional *scale* parameter, that we incorporate in the target categories, by replacing the \mathcal{V}_α with categories of functors $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)} = \text{Func}((\mathbb{R}, \leq), \mathcal{V}_\alpha)$ from the thin category (\mathbb{R}, \leq) to the category \mathcal{V}_α . Thus, objects in $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$ are determined by a family $Y_\alpha(s)$ of objects in \mathcal{V}_α , parameterized by $s \in \mathbb{R}$, and a family of morphisms $\varphi_{s \leq s'} : Y(s) \rightarrow Y(s')$. Morphisms $\text{Hom}_{\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}}(Y, Y')$ are natural transformations, determined by collections of morphisms $\varphi(s) : Y(s) \rightarrow Y'(s)$ in \mathcal{V}_α satisfying the compatibility $\varphi(s')\varphi_{s \leq s'} = \varphi_{s \leq s'}\varphi(s)$.

We consider, as before, valuation functors $F_\alpha : \Sigma_{\mathcal{C}}(S) \rightarrow \mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$ and target objects $X_\alpha = ((X_\alpha(s))_{s \in \mathbb{R}}, \varphi_{s \leq s'})$ in $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$. Thus, in this setting the target objects vary with the scale parameter $s \in \mathbb{R}$. Similarly, using the natural identification of functors in $\text{Func}(\Sigma_{\mathcal{C}}(S), \text{Func}((\mathbb{R}, \leq), \mathcal{V}_\alpha))$ with functors in $\text{Func}(\Sigma_{\mathcal{C}}(S) \times (\mathbb{R}, \leq), \mathcal{V}_\alpha)$, we can think of the valuation functors themselves as dependent on a scale factor $F_\alpha = (F_{\alpha, s})_{s \in \mathbb{R}}$ with $F_{\alpha, s} : \Sigma_{\mathcal{C}}(S) \rightarrow \mathcal{V}_\alpha$.

On the category $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$ there are “change of scale” functors, which are a special case of the more general flow/coflow structure we recall in §4.2 below. For each $\epsilon \geq 0$, there are endofunctors $T_\epsilon : \mathcal{V}_\alpha^{(\mathbb{R}, \leq)} \rightarrow \mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$ with $T_0 = \text{id}$ and $T_\epsilon T_{\epsilon'} = T_{\epsilon + \epsilon'}$ with $T_\epsilon X(s) = X(s + \epsilon)$ and $T_\epsilon \varphi_{s \leq s'} = \varphi_{s + \epsilon \leq s' + \epsilon}$.

Note that colimits of functors are evaluated pointwise, so if colimits exist in \mathcal{V}_α then they also exist in $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$. The functors T_ϵ preserve colimits, namely the colimit of the image diagram is the image of the colimit.

4.2. Categories with coflow. We consider here the case where the target categories of the valuation functors are *categories with coflow* in the sense of [4], see also [5], [7]. Examples of categories with flows (and dually coflows) include persistence modules and derived sheaves ([5], [7]). We recall briefly the main properties of categories with flows and coflows that we need to use in the following.

A *flow on a category* \mathcal{V} is a functor $T : [0, \infty) \rightarrow \mathcal{E}(\mathcal{V})$ from the thin category $([0, \infty), \leq)$ to the category $\mathcal{E}(\mathcal{V})$ of endofunctors of \mathcal{V} , with natural transformations $\mu_{\epsilon, \epsilon'} : T_\epsilon T_{\epsilon'} \rightarrow T_{\epsilon + \epsilon'}$ and $\mu_0 : \text{id}_{\mathcal{C}} \rightarrow T_0$, so that $\mu_{0, \epsilon} \mu_0 \text{id}_{T_\epsilon} : T_\epsilon \rightarrow T_0 T_\epsilon \rightarrow T_\epsilon$ and $\mu_{0, \epsilon} \text{id}_{T_\epsilon} \mu_0 : T_\epsilon \rightarrow T_\epsilon T_0 \rightarrow T_\epsilon$ are the identity and $\mu_{\epsilon, \zeta + \delta} \text{id}_{T_\epsilon} \mu_{\zeta, \delta} = \mu_{\epsilon + \zeta, \delta} \mu_{\epsilon, \zeta} \text{Id}_{T_\delta}$ and $T_{\epsilon + \zeta + \delta + \kappa} \mu_{\epsilon, \zeta} = \mu_{\delta, \kappa} T_{\epsilon \leq \delta} T_{\zeta \leq \kappa}$, with $T_{s \leq s'}$ the natural transformation in $\mathcal{E}(\mathcal{V})$ associated to the morphism $s \leq s'$ in $[0, \infty)$, see Definition 1 of [4].

A *coflow on a category* is defined dually. Namely (\mathcal{V}, T) is a category with coflow iff $(\mathcal{V}^{op}, T^{op})$ is a category with flow. (The opposite functor T^{op} acts as T on objects and morphisms, but natural transformations between opposite functors have the reversed direction.) The special case of the ‘‘change of scale’’ functors on a category $\mathcal{V}^{(\mathbb{R}, \leq)}$ described in §4.1 are an example of a *strict flow*, where $T_0 = \text{id}$ and $T_\epsilon T_{\epsilon'} = T_{\epsilon + \epsilon'}$ instead of having natural transformations between them. While we will work here mostly with this special case, we recall here the more general setting, as most of what we will discuss generalizes easily to more general categories with coflows.

The main advantage of a flow or coflow structure on a category \mathcal{V} is that it determines on $X = \text{Obj}(\mathcal{V})$ an *extended-pseudo-metric*. We assume here that X is a set. By extended-pseudo-metric we mean a function $d : X \times X \rightarrow \mathbb{R}_{\geq 0} \cup \{\infty\}$ with

- (1) $d(x, y) \geq 0$ for all $x, y \in X$, with $d(x, y) = 0$ if $x = y$;
- (2) $d(x, y) = d(y, x)$ for all $x, y \in X$;
- (3) $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

So unlike an actual metric d can take value ∞ and can also take value 0 on some pairs of non-coincident points. The extended-pseudo-metric structure determined by a flow or coflow is called the *interleaving distance*. It is defined as follows.

As in [4], [5], we say that two objects A, B in $X = \text{Obj}(\mathcal{V})$ are ϵ -interleaved if there are morphisms $\alpha : A \rightarrow T_\epsilon B$ and $\beta : B \rightarrow T_\epsilon A$ in \mathcal{V} with a commutative diagram

$$\begin{array}{ccccc}
 T_0 A & \longleftarrow & A & & B & \longrightarrow & T_0 B \\
 & & \searrow \alpha & & \swarrow \beta & & \\
 & & T_\epsilon A & & T_\epsilon B & & \\
 & & \searrow T_\epsilon \alpha & & \swarrow T_\epsilon \beta & & \\
 T_{2\epsilon} A & \longleftarrow & T_\epsilon T_\epsilon A & & T_\epsilon T_\epsilon B & \longrightarrow & T_{2\epsilon} B
 \end{array}$$

Then define the *interleaving distance* as

$$d_{(\mathcal{V}, T)}(A, B) := \inf\{\epsilon \geq 0 \mid A \text{ and } B \text{ are } \epsilon\text{-interleaved}\},$$

where the value can be equal to ∞ if no ϵ -interleaving occurs for any $\epsilon \in \mathbb{R}_{\geq 0}$. The distance is zero if A and B are isomorphic.

We also recall the following result of [4].

Proposition 4.1. *Let (\mathcal{V}, T) be a category with a coflow with the following properties:*

- (1) *all diagrams D of shape $\bullet \rightarrow \bullet \rightarrow \bullet \rightarrow \dots$ have colimits in \mathcal{V} ;*
- (2) *for all $\epsilon > 0$, the functor T_ϵ preserves the colimits, namely if $T_\epsilon \text{colim} D \simeq \text{colim} T_\epsilon D$.*

Then $(\text{Obj}(\mathcal{V}), d_{(\mathcal{V}, T)})$ is metrically complete.

4.2.1. *Scale and conversion of resources.* In the categorical theory of resources [3], [6], one interprets morphisms as processes of conversion of resources. Thus, two isomorphic objects represent resources with the property that there is a conversion process from one to the other that is fully reversible. Based on this idea, and on the interleaving distance recalled above, we can describe proximity of resources through the existence of a conversion process that becomes reversible at a different scale.

Namely, if our category of resources is of the form $\mathcal{V}^{(\mathbb{R}, \leq)}$ (or is more generally a category with a coflow that we can think of as implementing changes of scale), we say that resources $A(s)$ and $B(s)$ are ϵ -convertible if there are conversion processes $\alpha_{s, \epsilon} : A(s) \rightarrow B(s + \epsilon)$ and $\beta_{s, \epsilon} : B(s) \rightarrow A(s + \epsilon)$ that form an ϵ -interleaving diagram. We say that $A(s)$ and $B(s)$ are ϵ -close if their interleaving distance $d_{(\mathcal{V}^{(\mathbb{R}, \leq)}, T)}(A, B) \leq \epsilon$, with respect to the change of scale functors T .

One can think of this condition, for example, in a setting where higher scales correspond to a coarsening of the system with fewer averaged out variables, as in a renormalization procedure in statistical physics, for example. In such a setting, one can have a conversion of resources $A(s) \rightarrow B(s)$ at a certain scale s , where $B(s)$ is not sufficient to fully reconstruct $A(s)$ but sufficient to reconstruct an averaged out version at a larger scale $A(s + \epsilon)$. In a different setting, one can instead think of a situation where the system at lower scales is run by simpler variables but at higher scales it involves more complex emergent phenomena, with a conversion at higher scale not being fully reversible, but still sufficient to reconstruct the simpler systems at lower scales. In this second case $s \mapsto s + \epsilon$ zooms in to lower scales rather than to higher scales: we should in general think of s as an order of magnitude and some λ^s with $\lambda > 0$, either smaller or larger than 1, as the actual scale parameter.

Definition 4.2. *In a category of resources of the form $\mathcal{V}^{(\mathbb{R}, \leq)}$, with a scale parameter $s \in \mathbb{R}$, we say that a conversion of resources given by a morphism $\varphi_s : A(s) \rightarrow B(s)$ is ϵ -reversible if there is a $\beta_{s, \epsilon} : B(s) \rightarrow A(s + \epsilon)$ such that $\beta_{s, \epsilon} \circ \varphi_s = T_\epsilon : A(s) \rightarrow A(s + \epsilon)$.*

Remark 4.3. *An ϵ -reversible morphism $\varphi_s : A(s) \rightarrow B(s)$ fits into an ϵ -interleaving diagram obtained from the two commutative diagrams*

$$\begin{array}{ccc} A(s) & \xrightarrow{T_\epsilon \circ \varphi_s} & B(s + \epsilon) & \xrightarrow{\beta_{s, \epsilon}} & A(s + 2\epsilon) \\ & \searrow & \text{---} & \nearrow & \\ & & T_{2\epsilon} & & \end{array}$$

and

$$\begin{array}{ccc} B(s) & \xrightarrow{\beta_{s, \epsilon}} & A(s + \epsilon) & \xrightarrow{T_\epsilon \circ \varphi_{s + \epsilon}} & B(s + 2\epsilon) \\ & \searrow & \text{---} & \nearrow & \\ & & T_{2\epsilon} & & \end{array}$$

4.3. Colimits and approximation. Consider as before a diagram D in Diagr_α consisting of a sequence of minorizations as in (3.3). Now all the objects are in $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$ so we write the scale dependence explicitly

$$(4.1) \quad F_{\alpha,s}(\Phi_0) \xrightarrow{\varphi_{0,s}} F_{\alpha,s}(\Phi_1) \xrightarrow{\varphi_{1,s}} \cdots \rightarrow F_{\alpha,s}(\Phi_n) \xrightarrow{\varphi_{n,s}} \cdots$$

with colimit $L_{\alpha,s}(D)$, for $s \in \mathbb{R}$.

Proposition 4.4. *Given a diagram as in (4.1), if for all $\epsilon > 0$ there is an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ there are maps $v_{n,\epsilon} : F_{\alpha,s}(\Phi_{n+1}) \rightarrow F_{\alpha,s+\epsilon}(\Phi_n)$ that fit into a commutative diagram*

$$\begin{array}{ccc} F_{\alpha,s}(\Phi_n) & \xrightarrow{\varphi_{n,s}} & F_{\alpha,s}(\Phi_{n+1}) \\ T_\epsilon \downarrow & \swarrow v_{n,\epsilon} & \downarrow T_\epsilon \\ F_{\alpha,s+\epsilon}(\Phi_n) & \xrightarrow{\varphi_{n,s+\epsilon}} & F_{\alpha,s+\epsilon}(\Phi_{n+1}) \end{array}$$

then the sequence $\{F_{\alpha,s}(\Phi_k)\}$ converges in the interleaving distance to the colimit $L_{\alpha,s}(D)$ of the diagram (4.1).

Proof. We obtain from the commutative diagram above an ϵ -interleaving diagram as in Remark 4.3, with

$$\begin{array}{ccccc} F_{\alpha,s}(\Phi_n) & \xrightarrow{T_\epsilon \circ \varphi_{n,s}} & F_{\alpha,s+\epsilon}(\Phi_{n+1}) & \xrightarrow{v_{n,\epsilon}} & F_{\alpha,s+2\epsilon}(\Phi_n) \\ & \searrow & \searrow & \searrow & \searrow \\ & & & & T_{2\epsilon} \end{array}$$

and

$$\begin{array}{ccccc} F_{\alpha,s}(\Phi_{n+1}) & \xrightarrow{v_{n,\epsilon}} & F_{\alpha,s+\epsilon}(\Phi_n) & \xrightarrow{\varphi_{n,s+2\epsilon} \circ T_\epsilon} & F_{\alpha,s+2\epsilon}(\Phi_{n+1}) \\ & \searrow & \searrow & \searrow & \searrow \\ & & & & T_{2\epsilon} \end{array}$$

We then have the following commutative diagram, where $\psi_{n,s}$ are the morphisms to the colimit, and the morphisms $\omega_{\alpha,\epsilon} : L_{\alpha,s}(D) \rightarrow F_{\alpha,s+\epsilon}(\Phi_n)$ are uniquely determined by the universal property of the colimit,

$$\begin{array}{ccccc} F_{\alpha,s}(\Phi_n) & \xrightarrow{\varphi_{n,s}} & F_{\alpha,s}(\Phi_{n+1}) & & \\ \downarrow T_\epsilon & \searrow & \downarrow T_\epsilon & \searrow \psi_{n+1,s} & \\ & & & & L_{\alpha,s}(D) \\ & \swarrow v_{n,\epsilon} & \downarrow T_\epsilon & \swarrow \psi_{n,s} & \\ F_{\alpha,s+\epsilon}(\Phi_n) & \xrightarrow{\varphi_{n,s+\epsilon}} & F_{\alpha,s+\epsilon}(\Phi_{n+1}) & & \\ & \swarrow \omega_{\alpha,\epsilon} & & & \end{array}$$

This similarly determines ϵ -interleaving diagrams

$$\begin{array}{ccccc} F_{\alpha,s}(\Phi_n) & \xrightarrow{T_\epsilon \circ \psi_{n,s}} & L_{\alpha,s+\epsilon}(D) & \xrightarrow{\omega_{\alpha,\epsilon}} & F_{\alpha,s+2\epsilon}(\Phi_n) \\ & \searrow & \searrow & \searrow & \searrow \\ & & & & T_{2\epsilon} \end{array}$$

and

$$\begin{array}{ccc}
 L_{\alpha,s}(D) & \xrightarrow{\omega_{\alpha,\epsilon}} F_{\alpha,s+\epsilon}(\Phi_n) & \xrightarrow{\psi^{n,s+2\epsilon} \circ T_\epsilon} L_{\alpha,s+2\epsilon}(D) \\
 & \searrow & \nearrow \\
 & & T_{2\epsilon}
 \end{array}$$

Thus, we obtain that the interleaving distance $d(F_\alpha(\Phi_n), L_\alpha(D)) \leq \epsilon$. \square

4.4. Particle swarm algorithm. We can then consider a possible particle swarm algorithm designed in the following way. We are assuming that, given two objects Y, Y' in a target category $\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}$, the morphisms $\text{Hom}_{\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}}(Y, Y')$ are explicitly known. The algorithm also requires the existence diagrams in Diagr_α with ϵ -reversible strict minorizations for sufficiently large n . We can then map an ϵ -neighborhood of the Pareto frontier with the following swarm algorithm, using a large number n of draws for each particle and a large number N of particles.

- Choose an approximation level $\epsilon > 0$.
- Initialize a swarm of N probabilistic particles, by random draws of their initial positions $\Phi_0^{(1)}, \dots, \Phi_0^{(N)}$ in $\Sigma_{\mathcal{C}}^{\text{adm}}(S)$.
- For each $i = 1, \dots, N$ proceed as in the case of the single particle of §3 through successive draws of positions $\Phi_k^{(i)}$ in $\Sigma_{\mathcal{C}}^{\text{adm}}(S)$, for $k = 1, \dots, n$.
- At each successive draw $\Phi_k^{(i)}$ check if the (F, X) -strict minorization condition $\text{Hom}_{\mathcal{V}_\alpha^{(\mathbb{R}, \leq)}}(F_\alpha(\Phi_\ell^{(i)}), F_\alpha(\Phi_k^{(i)})) \neq \emptyset$ holds for previous draws $\Phi_\ell^{(i)}$, $0 \leq \ell < k$.
- If the strict minorization condition holds, check if the ϵ -reversibility, namely the existence of morphisms as in Proposition 4.4, reversing the minorization direction up to a scale shift.
- If a strict minorization with ϵ -reversibility exists, then the new draw $\Phi_k^{(i)}$ is in an ϵ -neighborhood of the Pareto frontier. The chain of minorizations from $F_\alpha(\Phi_0^{(i)})$ to this $F_\alpha(\Phi_k^{(i)})$ gives a corresponding ϵ -approximation to a diagram in Diagr_α .
- If ϵ -invertibility is not satisfied, select a longest chains of strict minorizations in the sequence $\{F_\alpha(\Phi_0^{(i)}), \dots, F_\alpha(\Phi_n^{(i)})\}$. Let $F_\alpha(\Phi_k^{(i)})$ be the last term of this sequence.
- Search for chains of strict minorizations starting at $F_\alpha(\Phi_k^{(i)})$ among the set $\{F_\alpha(\Phi_k^{(j)})\}_{j=1}^N$ of the other swarm particles positions at the same time.
- For each strict minorization check ϵ -invertibility. Whenever an ϵ -reversible strict minorization is found the corresponding particle position is in an ϵ -neighborhood of the Pareto frontier.
- For all the chains of strict minorizations that do not satisfy ϵ -reversibility among the $F_\alpha(\Phi_k^{(j)})$ continue the search with the new draws $F_\alpha(\Phi_{k+1}^{(j)})$ and repeat the process.

This procedure alternates between new draws $\Phi_{k+1}^{(j)}$ for each particle and searching for best positions for a given particle up to a given time n and comparing positions of different particles $\Phi_k^{(j)}$, $i = 1, \dots, N$ at a fixed time k , as in the case of the original

swarm intelligence algorithm. Multiple runs of the algorithm will identify points in an ϵ -neighborhood of the Pareto frontier.

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