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Painlevé V and time-dependent Jacobi polynomials

Estelle Basor[1,4], Yang Chen[2] and Torsten Ehrhardt[3,5]

[1] American Institute of Mathematics, Palo Alto, CA 94306, USA
[2] Department of Mathematics, Imperial College London, 180 Queen's Gates, London SW7 2BZ, UK
[3] Department of Mathematics, University of California, Santa Cruz, CA 95064, USA

E-mail: ebasor@aimath.org, ychen@imperial.ac.uk and ehrhardt@math.ucsc.edu

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Abstract
In this paper we study the simplest deformation on a sequence of orthogonal polynomials. This in turn induces a deformation on the moment matrix of the polynomials and associated Hankel determinant. We replace the original (or reference) weight $w_0(x)$ (supported on $\mathbb{R}$ or subsets of $\mathbb{R}$) by $w_0(x)\,\mathrm{e}^{-tx}$. It is a well-known fact that under such a deformation the recurrence coefficients denoted as α_n and β_n evolve in t according to the Toda equations, giving rise to the time-dependent orthogonal polynomials and time-dependent determinants, using Sogo's terminology. If w_0 is the normal density e^{-x^2}, $x \in \mathbb{R}$, or the gamma density $x^\alpha\,\mathrm{e}^{-x}$, $x \in \mathbb{R}_+$, $\alpha > -1$, then the initial value problem of the Toda equations can be trivially solved. This is because under elementary scaling and translation the orthogonality relations reduce to the original ones. However, if w_0 is the beta density $(1-x)^\alpha(1+x)^\beta$, $x \in [-1, 1]$, $\alpha, \beta > -1$, the resulting 'time-dependent' Jacobi polynomials will again satisfy a linear second-order ode, but no longer in the Sturm–Liouville form, which is to be expected. This deformation induces an irregular singular point at infinity in addition to three regular singular points of the hypergeometric equation satisfied by the Jacobi polynomials. We will show that the coefficients of this ode, as well as the Hankel determinant, are intimately related to a particular Painlevé V. In particular we show that $\mathsf{p}_1(n,t)$, where $\mathsf{p}_1(n,t)$ is the coefficient of z^{n-1} of the monic orthogonal polynomials associated with the 'time-dependent' Jacobi weight, satisfies, up to a translation in t, the Jimbo–Miwa σ-form of the same P_V; while a recurrence coefficient $\alpha_n(t)$ is up to a translation in t and a linear fractional transformation $\mathrm{P}_V(\alpha^2/2, -\beta^2/2, 2n+1+\alpha+\beta, -1/2)$. These results are found from combining a pair of nonlinear difference equations and a pair of Toda equations. This will in turn allow us to show that a certain Fredholm determinant related to a class of Toeplitz plus Hankel operators has a connection to a Painlevé equation. The case with $\alpha = \beta = -1/2$

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arose from a certain integrable system and this was brought to our attention by A P Veselov.

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1. Introduction

The study of Hankel determinants has seen a flurry of activity in recent years in part due to connections with random matrix theory (RMT). This is because Hankel determinants compute the most fundamental objects studied in RMT. For example, the determinants may represent the partition function for a particular random matrix ensemble or they might be related to the distribution of the largest eigenvalue or they may represent the generating function for a random variable associated with the ensemble. Often there is an associated Painlevé equation that is satisfied by logarithmic derivative of the Hankel determinant with respect to some parameter. This is true, for example, in the Gaussian unitary ensemble and for many other classical cases [34]. Once the Painlevé equation is found, then the Hankel determinant is much better understood. Asymptotics can found via the equation, scalings can be made to find limiting densities and in general the universal nature of the distributions can be analyzed.

Hankel determinants are also fundamental in the study of orthogonal polynomials and the connections between the recursion coefficients for the polynomials and the determinants are well known [33]. In this paper we utilize this relationship to compute the associated Painlevé equations. One advantage of this method is that the steps are quite direct and one derives immediately a second-order Painlevé equation without first encountering a higher order equation and then needing to find a first integral to reduce the order. In addition as a consequence of the method, but not surprisingly, we also show that the highest order non-trivial coefficient of the orthogonal polynomial also satisfies a Painlevé equation.

To be more precise the Hankel determinants of interest are for a weight of the form

$$(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{e}^{-tx}$$

on the interval $[-1, 1]$. Here we take $t \in \mathbb{R}$. We will call this a time-dependent Jacobi weight. Our ultimate goal is to produce a nonlinear second-order differential equation that is satisfied by the logarithmic derivative of $D_n(t)$, where $D_n(t)$ is the determinant of the Hankel matrix generated from the moments of the weight:

$$D_n(t) := \det(\mu_{j+k}(t))_{j,k=0}^{n-1} := \det\left(\int_{-1}^{1} x^{j+k}\,(1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{e}^{-tx}\,\mathrm{d}x\right)_{j,k=0}^{n-1},$$

and we shall initially assume $\alpha, \beta > 0$.

The moments $\mu_k(t)$ can be evaluated as follows:

$$\mu_k(t) = (-1)^k\,\frac{\mathrm{d}^k}{\mathrm{d}t^k}\mu_0(t), \qquad k = 0, 1, 2, \ldots$$

where

$$\mu_0(t) = 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)\,\mathrm{e}^{t}\,M(\beta+1; \alpha+\beta+2; -2t),$$

and $M(a; b; z)$ is the Kummer function with parameters a and b. Because

$$\frac{\mathrm{d}^k}{\mathrm{d}z^k}M(a; b; z) = \frac{(a)_k}{(b)_k}M(a+k; b+k; z),$$

we find

$$\mu_k(t) = 2^{\alpha+\beta+1}\Gamma(\alpha+1)\Gamma(\beta+1)\,\mathrm{e}^t \sum_{r=0}^{k}\binom{k}{r}(-2)^r \times \frac{(\beta+1)_r}{(\alpha+\beta+2)_r}\, M(\beta+r+1;\alpha+\beta+r+2;-2t).$$

It is well known [28] that the Hankel determinant for this time-dependent weight is the partition function for the random matrix ensemble with eigenvalue distribution

$$\prod_{1\leqslant j<k\leqslant n}(x_j-x_k)^2\prod_{l=1}^{n}(1-x_l)^{\alpha}(1+x_l)^{\beta}\,\mathrm{e}^{-tx_l}\,\mathrm{d}x_l\ .$$

When $t = 0$, the partition function is known exactly. However, when the weight is deformed this is not the case and something else needs to be done. In another interpretation the determinant quotient $D_n(t)/D_n(0)$ is the generating function of the random variable or as it is sometimes called, the linear statistics

$$\sum_{j=1}^{n} x_j.$$

For other applications of Hankel determinants and RMT see [28, 32, 33]. For connections to quantum gravity questions, see [20, 21].

Our main result is that $\mathsf{p}_1(n,t)$, where $\mathsf{p}_1(n,t)$ is the coefficient of z^{n-1} of the monic orthogonal polynomials associated with the 'time-dependent' Jacobi weight, with a minor change of variables, satisfies the σ form of a second-order Painlevé V.

Theorem. *Let*

$$\sigma(t) = \frac{1}{2}t\mathsf{p}_1(n,t/2) - \frac{nt}{2} + n(n+\beta). \tag{1.1}$$

Then σ satisfies the second-order nonlinear ode

$$(t\sigma'')^2 = [\sigma - t\sigma' + (2n+\alpha+\beta)\sigma']^2 + 4[\sigma - n(n+\beta) - t\sigma'][(\sigma')^2 - \alpha\sigma']. \tag{1.2}$$

Since, as we will also show,

$$\mathsf{p}_1(n,t) = \frac{\mathrm{d}}{\mathrm{d}t}\log D_n(t),$$

$D_n(t)$ is related in a simple way to the τ-function of the P_V.

The paper is divided into sections as follows. In the next section we reproduce known results that we call coupled Toda equations. We include them in this paper as reference and the Toda equations are summarized in theorem 1. The reader familiar with these equations may easily skip this section. In section 3 we consider ladder operators and derive fundamental equations that are the basis for everything that follows. They give us coupled difference equations and coupled Riccati equations in certain auxiliary quantities denoted as $r_n(t)$ and $R_n(t)$.

The Ricatti equations allow us to find a nonlinear second-order differential equation that is satisfied by the recurrence coefficient $\alpha_n(t)$. A rational change of variable applied to $\alpha_n(t)$ is then a solution to a Painlevé V in standard form and can be found in section 4.

In section 5 we identity the function which satisfies the continuous and discrete σ-form of our P_V. As mentioned above, we show that the σ-function of Jimbo, Miwa and Okamoto is given by

$$\sigma(t) = \frac{t}{2}\mathsf{p}_1(n,t/2) - \frac{n}{2}t + n(n+\beta).$$

In section 6 we show how the Hankel determinants can also be expressed as determinants of finite Toeplitz plus Hankel matrices. This section does not depend on the others and the derived identities are of independent interest. For some special cases, these latter determinants are known exactly and hence so are our Hankel determinants. Thus we, in a round about way, produce second-order differential equations that have solutions that are logarithmic derivatives of Fredholm determinants. This should not come as a great surprise as this is a common occurrence in random matrix theory for the classical ensembles.

Finally using known results for the Fredholm determinants we are able, in some special cases, to write down asymptotic expansions for these determinants and make some predictions about higher order terms.

2. Preliminaries: notations and time evolution

The purpose of this section is to derive two coupled Toda equations that involve the recursion coefficients of the time-dependent Jacobi polynomials. This is not a new result. The rather more general Toda-hierarchy can be found, for example, in [23, 29, 35]. Ours corresponds to the first of the hierarchy. See [24] for a discussion of this in relation to Sato's theory. See also [1] for the 'multi-time' approach to matrix models.

We include the necessary computations here for completeness sake and to set the notations to be used throughout this paper.

To begin we consider a sequence of polynomials $\{P_i(x)\}$ orthogonal with respect to the weight $w_0(x)\,\mathrm{e}^{-tx}$ on $[-1, 1]$. The weight w_0 will be known as the 'reference' weight. The orthogonality condition is

$$\int_{-1}^{1} P_i(x)P_j(x)w_0(x)\,\mathrm{e}^{-tx}\,\mathrm{d}x = h_i(t)\delta_{i,j}, \tag{2.1}$$

and the t dependence through e^{-tx} induces t dependence on the coefficients. We normalize our monic polynomials as

$$P_n(z) = z^n + \mathsf{p}_1(n,t)z^{n-1} + \cdots + P_n(0), \tag{2.2}$$

although sometime we do not display the t dependence of coefficients of z^{n-1}.

An immediate consequence of the orthogonality condition is the three terms recurrence relation

$$zP_n(z) = P_{n+1}(z) + \alpha_n P_n(z) + \beta_n P_{n-1}(z) \tag{2.3}$$

with the initial conditions

$$P_0(z) = 1, \qquad \beta_0 P_{-1}(z) = 0. \tag{2.4}$$

An easy consequence of the recurrence relation is

$$\alpha_n(t) = \mathsf{p}_1(n,t) - \mathsf{p}_1(n+1,t), \tag{2.5}$$

and a telescopic sum of the above equation (bearing in mind that $\mathsf{p}_1(0,t) = 0$) leaves

$$-\sum_{j=0}^{n-1}\alpha_j(t) = \mathsf{p}_1(n,t). \tag{2.6}$$

Please note that in the above and in what follows we use the notation $\alpha_n(t)$, $\beta_n(t)$ (α_n, β_n) for the recurrence coefficients since this is the standard notation. The use of the also standard α and β parameters for the Jacobi weights should not cause confusion since the Jacobi weight parameters will never be subscripted with n and the recurrence coefficients always will be.

First let us discuss the derivatives of $\alpha_n(t)$ and $\beta_n(t)$ with respect to t, as this yields the simplest equations, where we keep w_0 quite general, as long as the moments

$$\mu_i(t) := \int_{-1}^{1} x^i w_0(x)\, \mathrm{e}^{-tx}\, \mathrm{d}x, \qquad i = 0, 1, \ldots \tag{2.7}$$

exist. Taking a derivative of h_n with respect to t

$$h_n'(t) = -\int_{-1}^{1} w_0(x)\, \mathrm{e}^{-tx} x P_n^2(x)\, \mathrm{d}x = -\alpha_n h_n, \tag{2.8}$$

i.e.

$$(\log h_n)' = -\alpha_n, \tag{2.9}$$

and since $\beta_n = h_n/h_{n-1}$, we have the first Toda equation

$$\beta_n' = (\alpha_{n-1} - \alpha_n)\beta_n. \tag{2.10}$$

We define $D_n(t)$ to be the Hankel determinant

$$D_n(t) = \det(\mu_{i+j}(t))_{i,j=0}^{n-1}. \tag{2.11}$$

It is well known that $D_n(t) = \prod_{i=0}^{n-1} h_i(t)$. This yields in view of (2.9) that

$$\frac{\mathrm{d}}{\mathrm{d}t} \log D_n(t) = -\sum_{j=0}^{n-1} \alpha_j(t) = \mathsf{p}_1(n,t).$$

Also,

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-1}^{1} P_n P_{n-1} w_0\, \mathrm{e}^{-tx}\, \mathrm{d}x \\ &= -\int_{-1}^{1} x P_n P_{n-1} w_0\, \mathrm{e}^{-tx}\, \mathrm{d}x + h_{n-1} \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{p}_1(n,t) \\ &= -h_n + h_{n-1} \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{p}_1(n,t), \end{aligned}$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathsf{p}_1(n,t) = \beta_n(t). \tag{2.12}$$

But since $\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1)$, we have the second Toda equation

$$\alpha_n' = \beta_n - \beta_{n+1}. \tag{2.13}$$

To summarize we have the following theorem.

Theorem 1. *The recursion coefficients $\alpha_n(t)$ and $\beta_n(t)$ satisfy the coupled Toda equations*

$$\beta_n' = (\alpha_{n-1} - \alpha_n)\beta_n, \tag{2.14}$$

$$\alpha_n' = \beta_n - \beta_{n+1}. \tag{2.15}$$

It is also worth pointing out that in view of (2.11) we have the obvious Toda molecule equation [32]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \log D_n(t) = \frac{\mathrm{d}}{\mathrm{d}t} \mathsf{p}_1(n,t) = \beta_n(t) = \frac{D_{n+1}(t) D_{n-1}(t)}{D_n^2(t)}.$$

3. Ladder operators, compatibility conditions and difference equations

In this section we give an account for a recursive algorithm for the determination of the recurrence coefficients α_n, β_n based on a pair of ladder operators and the associated supplementary conditions. The main result is contained in theorem 3. The ladder operators describe below can be thought of as formulas that allow one to increase or decrease the index n on the orthogonal polynomials. Such operators have been derived by various authors over many years. Here we provide a brief guide to the relevant literature. See example [7–11, 13, 15, 20, 21, 26]. In fact Magnus in [26] traced this back to Laguerre. We find the form of the ladder operators set out below convenient to use.

For a sufficiently well-behaved weight (see [15] for a precise statement) of the form $w(x) = \mathrm{e}^{-\mathsf{v}(x)}$ the ladder operators, or as they are also called, lowering and raising operators are

$$P_n'(z) = -B_n(z)P_n(z) + \beta_n A_n(z)P_{n-1}(z), \tag{3.1}$$

$$P_{n-1}'(z) = [B_n(z) + \mathsf{v}'(z)]P_{n-1}(z) - A_{n-1}(z)P_n(z), \tag{3.2}$$

where

$$A_n(z) := \frac{1}{h_n}\int_{-1}^{1} \frac{\mathsf{v}'(z) - \mathsf{v}'(y)}{z-y} P_n^2(y)w(y)\,\mathrm{d}y, \tag{3.3}$$

$$B_n(z) := \frac{1}{h_{n-1}}\int_{-1}^{1} \frac{\mathsf{v}'(z) - \mathsf{v}'(y)}{z-y} P_n(y)P_{n-1}(y)w(y)\,\mathrm{d}y. \tag{3.4}$$

Here we have assumed that $w(\pm 1) = 0$. Additional terms would have to be included in the definitions of $A_n(z)$ and $B_n(z)$ if $w(\pm 1) \neq 0$. See [13] and [15].

A direct calculation produces two fundamental supplementary (compatibility) conditions valid for all z;

$$B_{n+1}(z) + B_n(z) = (z-\alpha_n)A_n(z) - \mathsf{v}'(z) \tag{S_1}$$

$$1 + (z-\alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1}A_{n+1}(z) - \beta_n A_{n-1}(z). \tag{S_2}$$

We note here that (S_1) and (S_2) have been applied to random matrix theory in [34]. It turns out that there is an equation which gives better insight into the α_n and β_n if (S_1) and (S_2) are suitably combined. See [17].

Multiplying (S_2) by $A_n(z)$, we see that the rhs. of the resulting equation is a first-order difference, while the lhs., with $(z-\alpha_n)$ replaced by $B_{n+1}(z) + B_n(z) + \mathsf{v}'(z)$, is a first-order difference plus $A_n(z)$. Taking a telescope sum together with the appropriate 'initial condition',

$$B_0(z) = A_{-1}(z) = 0,$$

produces

$$B_n^2(z) + \mathsf{v}'(z)B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z)A_{n-1}(z). \tag{S_2'}$$

This last equation will be highly useful in what follows. The reason that it is so useful is, roughly, because the logarithm of a determinant is a sum and in order to get information about the determinant we need to reduce the sum to something that depends on only a fixed number of indices, rather than a sum from 0 to n. This will be the ultimate outcome of S_2'.

Equations (S_1), (S_2) and (S_2') were also stated in [26], albeit in a different form. See also [22]. The method described below is similar to that of [17] and [12].

If w_0 is modified by the multiplication of 'singular' factors such as $|x-t|^a$ or $a+bH(x-t)$, where H is the unit step function, then the ladder operator relations, (S_1), (S_2) and (S_2'), remain valid with appropriate adjustments. See [4, 18, 19].

Let $\Psi(z) = P_n(z)$. Eliminating $P_{n-1}(z)$ from the raising and lowering operators gives

$$\Psi''(z) - \left(\mathsf{v}'(z) + \frac{A_n'(z)}{A_n(z)}\right)\Psi'(z) + \left(B_n'(z) - B_n(z)\frac{A_n'(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z)\right)\Psi(z) = 0, \tag{3.5}$$

where we have used (S_2') to simplify the coefficient of Ψ in (3.5).

For the problem at hand, $w(x)$ is the 'time-dependent' Jacobi weight, and now we must suppose that $\alpha > 0$ and $\beta > 0$ so that our weight is suitably well behaved. Then

$$w(x) := (1-x)^\alpha(1+x)^\beta\, \mathrm{e}^{-tx}, \qquad x \in [-1, 1], \tag{3.6}$$

$$\mathsf{v}(z) := -\alpha\log(1-z) - \beta\log(1+z) + tz,$$

$$\mathsf{v}'(z) = -\frac{\alpha}{z-1} - \frac{\beta}{z+1} + t,$$

$$\frac{\mathsf{v}'(z) - \mathsf{v}'(y)}{z-y} = \frac{\alpha}{(y-1)(z-1)} + \frac{\beta}{(y+1)(z+1)}. \tag{3.7}$$

Substituting these into the definitions of $A_n(z)$ and $B_n(z)$ and integrating by parts produces

$$A_n(z) = -\frac{R_n(t)}{z-1} + \frac{t+R_n(t)}{z+1}, \qquad B_n(z) = -\frac{r_n(t)}{z-1} + \frac{r_n(t)-n}{z+1},$$

where

$$R_n(t) := \frac{\alpha}{h_n}\int_{-1}^{1}\frac{P_n^2(y)}{1-y}\,(1-y)^\alpha(1+y)^\beta\,\mathrm{e}^{-ty}\,\mathrm{d}y,$$

$$r_n(t) := \frac{\alpha}{h_{n-1}}\int_{-1}^{1}\frac{P_n(y)P_{n-1}(y)}{1-y}\,(1-y)^\alpha(1+y)^\beta\,\mathrm{e}^{-ty}\,\mathrm{d}y.$$

Substituting the expressions for $A_n(z)$ and $B_n(z)$ into (S_1) and (S_2'), which are identities in z, and equating the residues of the poles at $z = \pm 1$, we find four distinct difference equations and one which importantly performs the summation $\sum_j R_j$:

$$-(r_{n+1} + r_n) = \alpha - R_n(1-\alpha_n) \tag{3.8}$$

$$r_{n+1} + r_n = 2n + 1 + \beta - (R_n + t)(1+\alpha_n) \tag{3.9}$$

$$r_n^2 + \alpha r_n = \beta_n R_n R_{n-1} \tag{3.10}$$

$$(r_n - n)^2 - \beta(r_n - n) = \beta_n(R_n + t)(R_{n-1} + t) \tag{3.11}$$

$$\left(\frac{\beta-\alpha}{2}\right)r_n + \frac{\alpha n}{2} - t\,r_n - r_n(r_n - n) - \sum_{j=0}^{n-1} R_j = -\frac{\beta_n}{2}[R_n(R_{n-1}+t) + (R_n+t)R_{n-1}]. \tag{3.12}$$

We now manipulate equations (3.8)–(3.12) with the aim of expressing the recurrence coefficients $\alpha_n,\ \beta_n$ in terms of $r_n,\ R_n$, and of course $n,\ t$. Adding (3.8) and (3.9) yields

$$2R_n = 2n + \alpha + \beta + 1 - t - t\alpha_n; \tag{3.13}$$

thus, α_n is 'easily' expressed in terms of R_n.

Subtracting (3.8) from (3.9) gives

$$r_{n+1} + r_n = n + \frac{\beta - \alpha + 1 - t}{2} - \left(\frac{t}{2} + R_n\right)\alpha_n. \tag{3.14}$$

Eliminating $\beta_n R_n R_{n-1}$ from (3.10) and (3.11) we find

$$n(n+\beta) - (2n+\alpha+\beta)r_n = \beta_n[t^2 + t(R_{n-1} + R_n)]. \tag{3.15}$$

Now with the aid of (3.10), replacing $\beta_n R_{n-1}$ by $\left(r_n^2 + \alpha r_n\right)/R_n$ in (3.15) we find

$$t(t+R_n)\beta_n = n(n+\beta) - (2n+\alpha+\beta)r_n - \frac{t}{R_n}\left(r_n^2 + \alpha r_n\right), \tag{3.16}$$

and this expresses β_n in terms of $R_n,\ r_n,\ n,\ t$. It is important to note the absence of $R_{n\pm1}$ and $r_{n\pm1}$. The reader will note that the above manipulations prove that we have expressed α_n and β_n the recurrence coefficients in terms of auxiliary quantities r_n and R_n.

This is summarized in the following.

Theorem 2. *With r_n, R_n, α_n and β_n as defined above and with $\alpha, \beta > 0$*

$$t\alpha_n = 2n + 1 + \alpha + \beta - t - 2R_n, \tag{3.17}$$

$$t(t+R_n)\beta_n = n(n+\beta) - (2n+\alpha+\beta)r_n - \frac{t}{R_n}\left(r_n^2 + \alpha r_n\right). \tag{3.18}$$

In what follows we will find two Riccati equations, one in r_n with coefficients involving R_n and another with the roles of r_n and R_n reversed. We will first show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log D_n(t) = n - 2r_n - \beta_n t = -\sum_{j=0}^{n-1}\alpha_j(t) = \mathsf{p}_1(n,t). \tag{3.19}$$

Equation (3.19) can be derived as follows. Replace n by j in (3.17) and sum over j from 0 to $n-1$ to obtain

$$\sum_{j=0}^{n-1} R_j = \frac{n(n-1)}{2} + \frac{n(\alpha+\beta+1-t)}{2} - \frac{t}{2}\sum_{j=0}^{n-1}\alpha_j.$$

Now we can obtain from (3.12) another expression for $\sum_j R_j$,

$$\begin{aligned}\sum_{j=0}^{n-1} R_j &= \left(\frac{\beta-\alpha}{2}\right) r_n + \frac{\alpha n}{2} - t r_n - r_n^2 + n r_n + \frac{t\beta_n}{2}(R_n + R_{n-1}) + r_n^2 + \alpha r_n \\ &= \left(\frac{\alpha+\beta}{2} + n - t\right) r_n + \frac{\alpha n}{2} + \frac{\beta_n t}{2}(R_n + R_{n-1}) \\ &= \frac{n(n+\alpha+\beta)}{2} - t\, r_n - \frac{\beta_n t^2}{2},\end{aligned} \tag{3.20}$$

where we have eliminated $t\beta_n(R_n + R_{n-1})$ using (3.15). These last two equations yield

$$\mathsf{p}_1(n,t) = -\sum_{j=0}^{n-1}\alpha_j = n - 2r_n - t\beta_n.$$

From this we also deduce that (see (2.5)),

$$\alpha_n = 2(r_{n+1} - r_n) + t(\beta_{n+1} - \beta_n) - 1. \tag{3.21}$$

Hence, in view of (2.12),

$$\alpha_n = 2(r_{n+1} - r_n) - t\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} - 1; \tag{3.22}$$

we find

$$t\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \alpha_n + 1 = 2\left(n + \frac{\beta - \alpha + 1 - t}{2} - \left(\frac{t}{2} + R_n\right)\alpha_n - 2r_n\right),$$

where r_{n+1} has been eliminated from (3.14) and (3.22). Finally,

$$t\frac{\mathrm{d}\alpha_n}{\mathrm{d}t} + \alpha_n = 2n + \beta - \alpha - t - (t + 2R_n)\alpha_n - 4r_n,$$

or replacing α_n in favor of R_n from (3.17)

$$t\frac{\mathrm{d}R_n}{\mathrm{d}t} = \alpha t + (2n + 1 + \alpha + \beta - 2t)R_n - 2R_n^2 + 2tr_n. \tag{3.23}$$

This is a Riccati equation in R_n, but with r_n appearing linearly. Now since

$$-\sum_{j=0}^{n-1}\alpha_j(t) = \mathsf{p}_1(n,t) \qquad \text{and} \qquad \mathsf{p}_1'(n,t) = \beta_n(t),$$

see (2.6) and (2.11), we find, upon taking a derivative of (3.19) with respect to t,

$$-2\frac{\mathrm{d}r_n}{\mathrm{d}t} - \beta_n - t\frac{\mathrm{d}\beta_n}{\mathrm{d}t} = \beta_n,$$

or

$$-\frac{\mathrm{d}r_n}{\mathrm{d}t} = \beta_n + \frac{t}{2}(\alpha_{n-1} - \alpha_n)\beta_n, \tag{3.24}$$

where we have replaced $\beta_n'(t)$ by $(\alpha_{n-1} - \alpha_n)\beta_n$ with the aid of the first Toda equation, (2.13). A simple computation with (3.13) gives

$$\frac{t}{2}(\alpha_{n-1} - \alpha_n) = -1 + R_n - R_{n-1};$$

hence,

$$\begin{aligned}-\frac{\mathrm{d}r_n}{\mathrm{d}t} &= \beta_n(R_n - R_{n-1})\\ &= \beta_n R_n - \frac{r_n^2 + \alpha r_n}{R_n}\\ &= \frac{R_n}{t(t + R_n)}\left[n(n+\beta) - (2n + \alpha + \beta)r_n - \frac{t}{R_n}\left(r_n^2 + \alpha r_n\right)\right] - \frac{r_n^2 + \alpha r_n}{R_n},\end{aligned} \tag{3.25}$$

which is a Riccati equation in r_n. We summarize in the following theorem.

Theorem 3. *The quantities r_n and R_n satisfy the coupled Riccati equations:*

$$t\frac{\mathrm{d}R_n}{\mathrm{d}t} = \alpha t + (2n + 1 + \alpha + \beta - 2t)R_n - 2R_n^2 + 2r_n t, \tag{3.26}$$

$$-\frac{\mathrm{d}r_n}{\mathrm{d}t} = \frac{R_n}{t(t + R_n)}\left[n(n+\beta) - (2n + \alpha + \beta)r_n - \frac{t}{R_n}\left(r_n^2 + \alpha r_n\right)\right] - \frac{r_n^2 + \alpha r_n}{R_n}. \tag{3.27}$$

We end this section by pointing out that the above equations not only produce differential equations in our various unknown quantities, but also a pair of coupled nonlinear first-order difference equations in R_n and r_n. If we substitute β_n into (3.10), we obtain the following result.

Theorem 4. *The quantities r_n and R_n satisfy the coupled difference equations*

$$2t(r_{n+1}+r_n)=4R_n^2+2R_n(2t-2n-1-\alpha-\beta)-2\alpha t \tag{3.28}$$

$$n(n+\beta)-(2n+\alpha+\beta)r_n=\left(r_n^2+\alpha r_n\right)\left(\frac{t^2}{R_nR_{n-1}}+\frac{t}{R_n}+\frac{t}{R_{n-1}}\right), \tag{3.29}$$

together with the 'initial' conditions

$$r_0=0, \tag{3.30}$$

$$R_0=\frac{\alpha+\beta+1}{2}\frac{M(1+\beta;\alpha+\beta+1;-2t)}{M(1+\beta;\alpha+\beta+2;-2t)}. \tag{3.31}$$

The initial condition for R_0 can be found by direct integration. Observe that the representation of R_0 in terms of a ratio of the Kummer functions allows for the analytic continuation of $\alpha,\ \beta$ down to $\alpha=\beta=-1/2$, due to the relation

$$\lim_{b\to 0} b\,M(a;b;z)=a\,zM(a+1,2,z).$$

Hence,

$$\lim_{\alpha\to-1/2,\beta\to-1/2} R_0(t)=\frac{t}{2}\left(\frac{I_1(2t)}{I_0(2t)}-1\right).$$

Indeed, by formally continuing β so that $\beta+1=-k,\ k=0,1,2\ldots,$ we find

$$R_0(t)=\frac{\alpha}{2}\frac{L_k^{(\alpha-1-k)}(-2t)}{L_k^{(\alpha-k)}(-2t)}$$

expressed as the ratio of Laguerre polynomials of degree k. It is clear that iterating (3.28) and (3.29) with the above R_0 and $r_0=0$ will generate rational solutions (in the variable t) of our P_V derived in section 4. It is interesting to note that R_0 is also a rational function of α and t; therefore, for the values of the parameters stated above our P_V is a rational function in α and t.

Also note that the above equations define the quantities r_n and R_n for all $\alpha>-1$ and $\beta>-1$. To verify our answers we return to the pure Jacobi case and let $t=0$, then (3.13) gives

$$R_n=n+\frac{\alpha+\beta+1}{2},$$

and is consistent with (3.20) at $t=0$. Now equating (3.10) and (3.11) at $t=0$ gives

$$r_n=\frac{n(n+\beta)}{\alpha+\beta+2n},$$

and

$$\beta_n=\frac{r_n^2+\alpha r_n}{R_nR_{n-1}}=\frac{4n(n+\alpha)(n+\beta)(n+\alpha+\beta)}{(2n+\alpha+\beta+1)(2n+\alpha+\beta-1)(2n+\alpha+\beta)^2}.$$

With the R_n given above, we find α_n from (3.13) at $t=0$,

$$\alpha_n=\frac{\beta^2-\alpha^2}{(2n+\alpha+\beta)(2n+\alpha+\beta+2)}.$$

These are in agreement with those of [15].

4. Identification of P_V

The idea is to eliminate $r_n(t)$ from our coupled Ricatti equations to produce a second-order ode in $R_n(t)$. This is straightforward and messy and we omit the details. A further change of variable,

$$R_n(t) = -\frac{t}{1-y(t)},$$

leaves, after some simplification,

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{t} + \mathrm{Rat}(y,t),$$

where the last term is a particular rational function of two variables defined as follows:

$$\mathrm{Rat}(y,t) = -\frac{2y(y+1)}{y-1} + \frac{2(2n+1+\alpha+\beta)y}{t} + \frac{(y-1)^2}{t^2}\left[\frac{\alpha^2}{2}y - \frac{\beta^2}{2y}\right].$$

Therefore,

$$y(t) := 1 + \frac{t}{R_n(t)}$$

satisfies

$$y'' = \frac{3y-1}{2y(y-1)}(y')^2 - \frac{y'}{t} + 2(2n+1+\alpha+\beta)\frac{y}{t} - 2\frac{y(y+1)}{y-1} + \frac{(y-1)^2}{t^2}\left[\frac{\alpha^2}{2}y - \frac{\beta^2/2}{y}\right],$$

which is almost a P_V. To fit the above into a P_V, we make the replacement $t \to t/2$ followed by

$$y(t/2) = Y(t),$$

and find

$$Y'' = \frac{3Y-1}{2Y(Y-1)}(Y')^2 - \frac{Y'}{t} + \frac{(Y-1)^2}{t^2}\left[\frac{\alpha^2}{2}Y - \frac{\beta^2/2}{Y}\right] + (2n+1+\alpha+\beta)\frac{Y}{t} - \frac{1}{2}\frac{Y(Y+1)}{Y-1}, \tag{4.1}$$

which is

$$\mathrm{P}_V(\alpha^2/2, -\beta^2/2, 2n+1+\alpha+\beta, d = -1/2).$$

The initial conditions are

$$Y(0) = 1, \qquad Y'(0) = \frac{1}{2n+\alpha+\beta+1}.$$

It is well known that there is a Hamiltonian associated with P_V. To identify it, we substitute

$$R_n(t) := -tq \tag{4.2}$$

$$r_n(t) := -pq(q-1) + \rho q, \tag{4.3}$$

where $p = p(t)$, $q = q(t)$ into (3.19), and choose ρ so that the resulting expression is a polynomial in p and q. There are two possible ρ: $\rho = n$ and $\rho = n + \beta$.

Case I. $\rho = n$

$$t\mathsf{p}_1(n,t) + n(n+\alpha+\beta) - nt = p(p+2t)q(q-1) - 2ntq + \beta\, pq + \alpha\, p(q-1)\,. \tag{4.4}$$

Case II. $\rho = n + \beta$

$$t\mathsf{p}_1(n,t) + n(n+\alpha+\beta) - \alpha\beta - nt = p(p+2t)q(q-1) + 2(\beta-n)qt + \beta pq + \alpha p(q-1). \tag{4.5}$$

Replacing t by $t/2$ we see that the lhs of (4.35) and (4.36) are the two Hamiltonians $t\mathsf{H}_1$ and $t\mathsf{H}_2$ for our P_V. The Hamiltonian as presented in Okamoto [30] (see also [31]) is

$$t\mathsf{H} = p(p+t)q(q-1) + \alpha_2 qt - \alpha_3 pq - \alpha_1 p(q-1),$$

where

$$a = \frac{\alpha_1^2}{2}, \qquad b = -\frac{\alpha_3^2}{2}, \qquad c = \alpha_0 - \alpha_2, \qquad d = -\frac{1}{2}, \qquad \alpha_0 = 1 - \alpha_1 - \alpha_2 - \alpha_3.$$

Comparing with our $t\mathsf{H}_1$, while keeping in mind that the t is in fact $t/2$, we find

$$\begin{gathered} \alpha_2 = -n, \qquad \alpha_3 = -\beta, \qquad \alpha_1 = -\alpha, \qquad \alpha_0 = n+1+\alpha+\beta, \\ a = \frac{\alpha^2}{2}, \qquad b = -\frac{\beta^2}{2}, \qquad c = 2n+1+\alpha+\beta. \end{gathered} \tag{4.6}$$

Comparing with our $t\mathsf{H}_2$, we find

$$\begin{gathered} \alpha_2 = \beta - n, \qquad \alpha_3 = -\beta, \qquad \alpha_1 = -\alpha, \qquad \alpha_0 = n+1+\alpha \\ a = \frac{\alpha^2}{2}, \qquad b = -\frac{\beta^2}{2}, \qquad c = 2n+1+\alpha+\beta. \end{gathered} \tag{4.7}$$

Hence, both H_1 and H_2 generate our P_V, where

$$Y(t) = 1 - \frac{1}{q(t/2)}.$$

5. The continuous and discrete σ-form of P_V

Recall from section 2 that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log D_n(t) = \mathsf{p}_1(n,t),$$

and

$$\mathsf{p}_1'(n,t) = \beta_n(t). \tag{5.1}$$

Now we come to the continuous σ-form of P_V satisfied by $\mathsf{p}_1(n,t)$, with n fixed and t being the variable. The idea is to express β_n, r_n, r_n' in terms of $\mathsf{p}_1(n,t)$ and its derivatives with respect to t. Let us begin with (3.19),

$$\begin{aligned} \mathsf{p}_1(n,t) &= n - 2r_n - t\beta_n \\ &= n - 2r_n - t\mathsf{p}_1'(n,t). \end{aligned} \tag{5.2}$$

From the last equality of (5.2) we have

$$r_n(t) = \frac{1}{2}\left\{n - \frac{\mathrm{d}}{\mathrm{d}t}[t\mathsf{p}_1(n,t)]\right\}. \tag{5.3}$$

Under some minor rearrangements, equations (3.18) and (3.25) become

$$t\beta_n R_n + \frac{t}{R_n}\left(r_n^2 + \alpha r_n\right) = n(n+\beta) - (2n+\alpha+\beta)r_n - t^2\beta_n \tag{5.4}$$

$$-t\beta_n R_n + \frac{t}{R_n}(r_n^2 + \alpha r_n) = tr_n', \tag{5.5}$$

respectively, which is a system of linear equations in $1/R_n$ and R_n. Solving for $1/R_n$ and R_n we find

$$\frac{2t}{R_n}\left(r_n^2+\alpha r_n\right) = tr_n' + n(n+\beta) - (2n+\alpha+\beta)r_n - t^2\beta_n \tag{5.6}$$

$$2t\beta_n R_n = -tr_n' + n(n+\beta) - (2n+\alpha+\beta)r_n - t^2\beta_n. \tag{5.7}$$

Taking the product of the above we arrive at an identity free of R_n:

$$4t^2\beta_n\left(r_n^2+\alpha r_n\right) = [n(n+\beta) - (2n+\alpha+\beta)r_n - t^2\beta_n]^2 - (tr_n')^2. \tag{5.8}$$

To identify the σ-function of Jimbo and Miwa [25], we replace t by $t/2$ so that

$$R_n(t/2) = -\frac{t}{2(1-Y(t))},$$

and substitute the above in (3.26) in the variable $t/2$. After a little simplification we find

$$t\frac{\mathrm{d}Y}{\mathrm{d}t} = tY - 2r_n(t/2)(1-Y)^2 - (Y-1)(\alpha Y + 2n + \beta).$$

Comparing this with the first equation of (C.40) of [25], we have

$$z(t) = -r_n(t/2),$$

$$\frac{\theta_0 - \theta_1 + \theta_\infty}{2} = \alpha,$$

$$\frac{3\theta_0 + \theta_1 + \theta_\infty}{2} = -2n - \beta,$$

and consequently

$$1 - \theta_0 - \theta_1 = 2n + 1 + \alpha + \beta = c,$$

consistent with the parameter c of our P_V. Furthermore, comparing (4.32) with (C.41) of [25], we find a possible identification

$$\alpha = \frac{\theta_0 - \theta_1 + \theta_\infty}{2} \qquad \beta = \frac{\theta_0 - \theta_1 - \theta_\infty}{2},$$

and consequently

$$\theta_0 = -n, \qquad \theta_1 = -n - \alpha - \beta, \qquad \theta_\infty = \alpha - \beta.$$

But since

$$\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t) = z(t) = -r_n(t/2),$$

and bearing in mind (5.3), we have upon integration and fixing a constant,

$$\sigma(t) = \frac{1}{2}t\mathsf{p}_1(n, t/2) - \frac{nt}{2} + n(n+\beta). \tag{5.9}$$

The σ-form of our P_V is essentially a second-order nonlinear ode satisfied by $\mathsf{p}_1(n,t)$, and reads

$$(t\sigma'')^2 = [\sigma - t\sigma' + (2n+\alpha+\beta)\sigma']^2 + 4[\sigma - n(n+\beta) - t\sigma'][(\sigma')^2 - \alpha\sigma'], \tag{5.10}$$

with the initial conditions

$$\sigma(0) = n(n+\beta), \qquad \sigma'(0) = -r_n(0) = -\frac{n(n+\beta)}{\alpha+\beta+2n}.$$

After some calculations we find that (5.10) is in fact the Jimbo–Miwa σ-form (C.45) with

$$\nu_0 = 0, \qquad \nu_1 = -\alpha, \qquad \nu_2 = n, \qquad \nu_3 = n + \beta. \tag{5.11}$$

To obtain (5.10) we first replace t by $t/2$ in (5.8), and substitute

$$r_n(t/2) = -\sigma'(t)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} r_n(t/2) = -\sigma''(t)$$
$$\beta_n(t/2) = 2\frac{\mathrm{d}}{\mathrm{d}t}\mathsf{p}_1(n, t/2) = 4\frac{\mathrm{d}}{\mathrm{d}t}\frac{[\sigma(t) + nt/2 - n(n+\beta)]}{t}$$

into (5.8) at $t/2$. Furthermore, since $\mathsf{p}_1(n,t) = (\log D_n(t))'$, we have

$$D_n(t) = D_n(0)\exp\left[\int_0^t \frac{\sigma(2s) - n(n+\beta) + ns}{s}\mathrm{d}s\right],$$

where $D_n(0)$ given by (1.6) of [3].

We expect that there exists a discrete analog of the continuous σ-form, namely a difference equation in the variable n, satisfied by $\mathsf{p}_1(n,t)$ with t fixed. To simplify notations, we do not display the t dependence. The idea is similar to the continuous case; namely we express β_n, r_n and R_n in terms of $\mathsf{p}_1(n)$ and $\mathsf{p}_1(n \pm 1)$, and substitute these into (3.10), that is,

$$r_n^2 + \alpha\, r_n = \beta_n R_n\, R_{n-1}. \tag{3.10}$$

To begin with, we note that (3.19) is linear in β_n and r_n, which we rewrite as

$$t\beta_n + 2r_n = n - \mathsf{p}_1(n). \tag{5.12}$$

We now find another linear equation in β_n and r_n. First note that $\alpha_n = \mathsf{p}_1(n) - \mathsf{p}_1(n+1)$, and from (3.13) we have

$$2R_n + t = 2n + 1 + \alpha + \beta + t[\mathsf{p}_1(n+1) - \mathsf{p}_1(n)]. \tag{5.13}$$

The sum (5.13) at 'n' and the same but at '$n-1$' leaves

$$R_n + R_{n-1} + t = (2n + \alpha + \beta) + (t/2)[\mathsf{p}_1(n+1) - \mathsf{p}_1(n-1)].$$

Substituting the above into (3.15) results the other linear equation mentioned above:

$$t\beta_n\{2n + \alpha + \beta + (t/2)[\mathsf{p}_1(n+1) - \mathsf{p}_1(n-1)]\} + (2n + \alpha + \beta)r_n = n(n+\beta). \tag{5.14}$$

Solving for $t\beta_n$ and $2r_n$ from the linear system (5.13) and (5.14), we find

$$2r_n = \frac{(t/2)[n - \mathsf{p}_1(n)][\mathsf{p}_1(n+1) - \mathsf{p}_1(n-1)] - n(n+\beta)}{n + (\alpha+\beta)/2 + (t/2)[\mathsf{p}_1(n+1) - \mathsf{p}_1(n-1)]} \tag{5.15}$$

$$t\beta_n = \frac{[n - \mathsf{p}_1(n)][n + (\alpha+\beta)/2] + n(n+\alpha+\beta)}{n + (\alpha+\beta)/2 + (t/2)[\mathsf{p}_1(n+1) - \mathsf{p}_1(n-1)]}, \tag{5.16}$$

and the discrete σ-form results from substituting (5.13), (5.15) and (5.16) into (3.10).

Imagine for a moment that we leave our original problem behind and only consider the functions Y and σ that satisfy the two Painlevé equations (4.32) and (5.10) with the appropriate initial conditions. Then our orthogonal polynomials $P_n(z,t)$ satisfy the linear second-order ode

$$\Psi''(z) + \mathsf{P}(z)\Psi'(z) + \mathsf{Q}(z)\Psi(z) = 0, \tag{5.17}$$

where

$$\mathsf{P}(z) := \frac{1+\alpha}{z-1} + \frac{1+\beta}{z+1} - t - \frac{1}{z - [1 + Y(2t)]/[1 - Y(2t)]} \tag{5.18}$$

$$
\begin{aligned}
\mathsf{Q}(z) :=& -\frac{(1/2)\frac{\mathrm{d}\sigma(2t)}{\mathrm{d}t}}{(z-1)^2} + \frac{n+(1/2)\frac{\mathrm{d}\sigma(2t)}{\mathrm{d}t}}{(z+1)^2} \\
&+ \left[\frac{n+(1/2)\frac{\mathrm{d}\sigma(2t)}{\mathrm{d}t}}{z+1} - \frac{(1/2)\frac{\mathrm{d}\sigma(2t)}{\mathrm{d}t}}{z-1}\right] \\
&\cdot \left[\frac{2/(Y(2t)-1)}{z-1} + \frac{2Y(2t)/(Y(2t)-1)}{z+1} - 1\right] \cdot \frac{1}{z-[Y(2t)+1]/[Y(2t)-1]} \\
&+ \frac{(t^2/2)\frac{\mathrm{d}}{\mathrm{d}t}t^{-1}(\sigma(2t)+nt-n(n+\beta)) - (t/2)\frac{\mathrm{d}\sigma(2t)}{\mathrm{d}t} - n(n+\alpha+\beta)/2}{z-1} \\
&+ \frac{n(n+\alpha+\beta+2t)/2 + (t/2)\frac{\mathrm{d}\sigma(2t)}{\mathrm{d}t} - (t^2/2)\frac{\mathrm{d}}{\mathrm{d}t}t^{-1}(\sigma(2t)-nt-n(n+\beta))}{z+1}.
\end{aligned}
\tag{5.19}
$$

This is a deformation of the classical ode satisfied by the Jacobi polynomials. When $t = 0$, this ode reduces to a hypergeometric equation.

6. Toeplitz and Hankel determinants

In this section we introduce certain matrices that are combinations of finite Toeplitz and Hankel matrices. There are identities that link these matrices directly to the Hankel moment matrices that appear in the first section of this paper and define our quantity $D_n(t)$. We will use these identities in some special cases to get exact formulas for $D_n(t)$ and, as a by-product, find Painlevé-type results for some other interesting determinants. We include the Toeplitz/Hankel computations because as far as we know they are not written down explicitly in this form in any other place. However, case 2 was established already by two of the authors in [5, section 2]. The current derivation follows that in [5].

Given a sequence $\{a_k\}_{k=-\infty}^{\infty}$ of complex numbers, we associate the formal Fourier series

$$
a(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k=-\infty}^{\infty} a_k\,\mathrm{e}^{\mathrm{i}k\theta}, \qquad \mathrm{e}^{\mathrm{i}\theta} \in \mathbb{T}. \tag{6.1}
$$

The $n \times n$ Toeplitz and Hankel matrices with the (Fourier) symbol a are defined by

$$
T_n(a) = (a_{j-k})_{j,k=0}^{n-1}, \qquad H_n(a) = (a_{j+k+1})_{j,k=0}^{n-1}. \tag{6.2}
$$

Usually a represents an L^1-function defined on the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, in which case the numbers a_k are the Fourier coefficients,

$$
a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} a(\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{e}^{-\mathrm{i}k\theta}\,\mathrm{d}\theta, \qquad k \in \mathbb{Z}. \tag{6.3}
$$

Note that while the matrices $H_n(a)$ are classically referred to as Hankel matrices they are not the same as the Hankel moment matrices considered in the previous sections of this paper. To make the connection to Hankel matrices defined by moments, we write

$$
H_n[b] = (b_{j+k})_{j,k=0}^{n-1}, \qquad b_k = \frac{1}{\pi}\int_{-1}^{1} b(x)(2x)^{j+k}\,\mathrm{d}x, \tag{6.4}
$$

where $b(x)$ be an L^1-function defined on $[-1, 1]$. Note the difference in notation in comparison to (2.7) and (2.11). Our goal in this section is to prove four identities. Let $z = \mathrm{e}^{\mathrm{i}\theta}$. Then for each $n \geqslant 1$ the following statements are true.

(1) If $a(e^{i\theta}) = b(\cos\theta)(2+2\cos\theta)^{-1/2}(2-2\cos\theta)^{-1/2}$, then

$$\det(T_n(a) - H_n(z^{-1}a)) = \det H_n[b].$$

(2) If $a(e^{i\theta}) = b(\cos\theta)(2+2\cos\theta)^{-1/2}(2-2\cos\theta)^{1/2}$, then

$$\det(T_n(a) + H_n(a)) = \det H_n[b].$$

(3) If $a(e^{i\theta}) = b(\cos\theta)(2+2\cos\theta)^{1/2}(2-2\cos\theta)^{-1/2}$, then

$$\det(T_n(a) - H_n(a)) = \det H_n[b].$$

(4) If $a(e^{i\theta}) = b(\cos\theta)(2+2\cos\theta)^{1/2}(2-2\cos\theta)^{1/2}$, then

$$\tfrac{1}{4}\det(T_n(a) + H_n(za)) = \det H_n[b].$$

In these identities the function a is always even, which means in terms of its Fourier coefficients that $a_k = a_{-k}$. Moreover, in these formulas we assume that $a \in L^1(\mathbb{T})$, which implies that (and in case 4 is equivalent to) $b \in L^1[-1,1]$. We also remark that cases 2 and 3 can be derived from each other by making the substitutions $a(e^{i\theta}) \mapsto a(e^{i(\theta+\pi)})$ and $b(x) \mapsto b(-x)$.

Our interest in the above formulas stems from the circumstance that they allow us to use existing results [6] on the asymptotics of the Toeplitz+Hankel determinants with well-behaved symbols a in order to derive the asymptotics of the Hankel moment determinants.

In the above identities four types of finite symmetric Toeplitz+Hankel matrices as well as a finite Hankel moment matrix occur. These finite matrices can be obtained from their (on-sided) infinite matrix versions by taking the finite sections. It turns out that these infinite matrices are related to each other in a very simply way; namely they can be transformed into one another by multiplying with appropriate upper and lower triangular (infinite) matrices form the left and right. These identities for the infinite matrices will be established in the next theorem (and the remarks afterwards) in most general setting, where we do not assume that the symbols are L^1-functions.

Let us introduce the infinite matrices

$$D_+ = \begin{pmatrix} 1 & & & \\ -1 & 1 & & \\ & -1 & 1 & \\ & & \ddots & \ddots \end{pmatrix}, \qquad D_- = \begin{pmatrix} 1 & & & \\ 1 & 1 & & \\ & 1 & 1 & \\ & & \ddots & \ddots \end{pmatrix}.$$

These are just the well-known Toeplitz operators $D_\pm = T(1 \mp z)$ and their transposes are denoted by $D_\pm^T$. We also need the infinite diagonal matrix $R = \operatorname{diag}\left(\frac{1}{2}, 1, 1, \ldots\right)$.

Theorem 5. *For sequences of numbers* $\{a_n\}_{n=-\infty}^{\infty}$, $\{a_n^+\}_{n=-\infty}^{\infty}$, $\{a_n^-\}_{n=-\infty}^{\infty}$ *and* $\{a_n^\#\}_{n=-\infty}^{\infty}$ *satisfying*

$$a_n = a_{-n}, \qquad a_n^+ = a_{-n}^+, \qquad a_n^- = a_{-n}^-, \qquad a_n^\# = a_{-n}^\#,$$

define

$$\begin{aligned} A &= (a_{j-k} - a_{j+k+2})_{j,k=0}^{\infty} \qquad & A^+ &= \left(a_{j-k}^+ + a_{j+k+1}^+\right)_{j,k=0}^{\infty} \\ A^- &= \left(a_{j-k}^- - a_{j+k+1}^-\right)_{j,k=0}^{\infty} \qquad & A^\# &= (a_{j-k}^\# + a_{j+k}^\#)_{j,k=0}^{\infty}. \end{aligned} \tag{6.5}$$

Then the following holds true.

(1) If $a_k^+ = 2a_k - a_{k-1} - a_{k+1}$, *then* $D_+ A D_+^T = A^+$.

(2) If $a_k^- = 2a_k + a_{k-1} + a_{k+1}$, *then* $D_- A D_-^T = A^-$.

(3) If $a_k^\# = 2a_k^+ + a_{k-1}^+ + a_{k+1}^+$, *then* $D_- A^+ D_-^T = RA^\# R$.
(4) If $a_k^\# = 2a_k^- - a_{k-1}^- - a_{k+1}^-$, *then* $D_+ A^- D_+^T = RA^\# R$.

Moreover, if we define a sequence $\{b_n\}_{n=0}^\infty$ *and an infinite Hankel matrix by*

$$b_n = \frac{1}{2}\sum_{k=0}^{n} a_{n-2k}^\# \binom{n}{k}, \qquad B = (b_{j+k})_{j,k=0}^\infty, \tag{6.6}$$

respectively, then

$$B = S_\# R A^\# R S_\#^T \qquad \text{with} \quad S_\# = \begin{pmatrix} \binom{0}{0} & & & & & \\ 0 & \binom{1}{0} & & & & \\ \binom{2}{1} & 0 & \binom{2}{0} & & & \\ 0 & \binom{3}{1} & 0 & \binom{3}{0} & & \\ \binom{4}{2} & 0 & \binom{4}{1} & 0 & \binom{4}{0} & \\ \vdots & & & & & \ddots \end{pmatrix}. \tag{6.7}$$

Proof. Before we start with the actual proof, we remark that the various products of the infinite matrices make sense in terms of the usual matrix multiplication because the left and right factors are always (infinite) band matrices.

In order to prove the first statement (1) we consider the (j,k)-entries of the following (products of) infinite matrices and compute as follows:

$$\begin{aligned}\left[D_+ A D_+^T\right]_{j,k} &= \begin{cases} (2a_{j-k} - a_{j-k-1} - a_{j-k+1}) - (a_{j+k+2} - 2a_{j+k+1} + a_{j+k}) & \text{if } j,k \geqslant 1 \\ (a_{-k} - a_{-k+1}) - (a_{k+2} - a_{k+1}) & \text{if } j=0, k \geqslant 1 \\ (a_j - a_{j-1}) - (a_{j+2} - a_{j+1}) & \text{if } j \geqslant 1, k=0 \\ a_0 - a_2 & \text{if } j=k=0 \end{cases} \\ &= \begin{cases} a_{j-k}^+ + a_{j+k+1}^+ & \text{if } j,k \geqslant 1 \\ a_{-k}^+ + a_{k+1}^+ & \text{if } j=0, k \geqslant 1 \\ a_j^+ + a_{j+1}^+ & \text{if } j \geqslant 1, k=0 \\ a_0^+ + a_1^+ & \text{if } j=k=0 \end{cases} = [A^+]_{j,k}\end{aligned}$$

Herein we use the fact that $a_k - a_{k-1} - a_{k+2} + a_{k+1} = (2a_k - a_{k-1} - a_{k+1}) + (2a_{k+1} - a_k - a_{k+2})$, a similar identity statement for j, and $a_0 - a_2 = (2a_0 - 2a_1) + (2a_1 - a_0 - a_2)$. Moreover, we use the assumption that $a_n = a_{-n}$.

Similarly, we compute the (j,k)-entry for the product appearing in (2):

$$\begin{aligned}\left[D_- A D_-^T\right]_{j,k} &= \begin{cases} (2a_{j-k} + a_{j-k-1} + a_{j-k+1}) - (a_{j+k+2} + 2a_{j+k+1} + a_{j+k}) & \text{if } j,k \geqslant 1 \\ (a_{-k} + a_{-k+1}) - (a_{k+2} + a_{k+1}) & \text{if } j=0, k \geqslant 1 \\ (a_j + a_{j-1}) - (a_{j+2} + a_{j+1}) & \text{if } j \geqslant 1, k=0 \\ a_0 - a_2 & \text{if } j=k=0 \end{cases} \\ &= \begin{cases} a_{j-k}^- - a_{j+k+1}^- & \text{if } j,k \geqslant 1 \\ a_{-k}^- - a_{k+1}^- & \text{if } j=0, k \geqslant 1 \\ a_j^- - a_{j+1}^- & \text{if } j \geqslant 1, k=0 \\ a_0^- - a_1^- & \text{if } j=k=0 \end{cases} = [A^-]_{j,k}.\end{aligned}$$

Here we used $a_k + a_{k-1} - a_{k+2} - a_{k+1} = (2a_k + a_{k-1} + a_{k+1}) - (2a_{k+1} + a_k + a_{k+2})$, a similar identity for j, and $a_0 - a_2 = (2a_0 + 2a_1) - (2a_1 + a_0 + a_2)$, and again $a_n = a_{-n}$.

As for statement (3) we consider

$$[D_-A^+D_-^T]_{j,k} = \begin{cases} \left(2a^+_{j-k} + a^+_{j-k-1} + a^+_{j-k+1}\right) + \left(a^+_{j+k+1} + 2a^+_{j+k} + a^+_{j+k-1}\right) & \text{if } j,k \geqslant 1 \\ \left(a^+_{-k} + a^+_{-k+1}\right) + \left(a^+_{k+1} + a^+_k\right) & \text{if } j = 0, k \geqslant 1 \\ \left(a^+_j + a^+_{j-1}\right) + \left(a^+_{j+1} + a^+_j\right) & \text{if } j \geqslant 1, k = 0 \\ a^+_0 + a^+_1 & \text{if } j = k = 0 \end{cases}$$

$$= \begin{cases} a^\#_{j-k} + a^\#_{j+k} & \text{if } j,k \geqslant 1 \\ \frac{1}{2}(a^\#_{-k} + a^\#_k) & \text{if } j = 0, k \geqslant 1 \\ \frac{1}{2}\left(a^\#_j + a^\#_j\right) & \text{if } j \geqslant 1, k = 0 \\ \frac{1}{4}\left(a^\#_0 + a^\#_0\right) & \text{if } j = k = 0 \end{cases} = [RA^\#R]_{j,k}.$$

Again, we used that $a^+_n = a^+_{-n}$ and $a^\#_n = a^\#_{-n}$.

Statement (4) can be proven in the same way as statement (3). In fact, if we assume all the hypotheses in (1)–(4), then (4) follows with a little algebra from the previous three statements (and from the fact that D_+ and D_- commute).

In order to prove formula (6.7) first observe that

$$S_\# = (\xi(i,j))_{i,j=0}^\infty, \qquad \xi(i,j) = \begin{cases} \binom{i}{\frac{i-j}{2}} & \text{if} \quad i \geqslant j \quad \text{and} \quad i-j \text{ even} \\ 0 & \text{otherwise.} \end{cases}$$

Put $r_0 = 1/2$ and $r_n = 1$ for $n \geqslant 1$. Then the identity $B = S_\# RA^\# RS_\#^T$ can be rephrased as

$$\frac{1}{2}\sum_{m=0}^{i+l} a^\#_{i+l-2m}\binom{i+l}{m} = \sum_{j=0}^{i}\sum_{k=0}^{l} \xi(i,j)\xi(l,k) r_j r_k \left(a^\#_{j-k} + a^\#_{j+k}\right) \tag{6.8}$$

to hold true for each $i, l \geqslant 0$. These identities are valid if for each integer $s \geqslant 0$, the coefficients for $a^\#_s = a^\#_{-s}$ are the same on the left and right-hand side.

First assume $s > 0$. On the right-hand side, the coefficient for $a^\#_s = a^\#_{-s}$ is equal to the sum $N_1 + N_2 + N_3$, where

$$N_1 = \sum_{\substack{0 \leqslant j \leqslant i \\ 0 \leqslant k \leqslant l \\ s = j-k}} \xi(i,j)\xi(l,k) r_j r_k = \sum_{\substack{0 \leqslant u \leqslant i/2 \\ 0 \leqslant v \leqslant l/2 \\ s = i-2u-l+2v}} \binom{i}{u}\binom{l}{v} r_{i-2u} r_{l-2v},$$

$$N_2 = \sum_{\substack{0 \leqslant j \leqslant i \\ 0 \leqslant k \leqslant l \\ s = k-j}} \xi(i,j)\xi(l,k) r_j r_k = \sum_{\substack{0 \leqslant u \leqslant i/2 \\ 0 \leqslant v \leqslant l/2 \\ s = -i+2u+l-2v}} \binom{i}{u}\binom{l}{v} r_{i-2u} r_{l-2v},$$

$$N_3 = \sum_{\substack{0 \leqslant j \leqslant i \\ 0 \leqslant k \leqslant l \\ s = j+k}} \xi(i,j)\xi(l,k) r_j r_k = \sum_{\substack{0 \leqslant u \leqslant i/2 \\ 0 \leqslant v \leqslant l/2 \\ s = i-2u+l-2v}} \binom{i}{u}\binom{l}{v} r_{i-2u} r_{l-2v}.$$

Therein, we made a change of variables $j \mapsto u = (i-j)/2$ and $k \mapsto v = (l-k)/2$. The summation is over integer pairs (u,v). In the above expressions for N_1 and N_2 we make another change of variables $v \mapsto l-v$ and $u \mapsto i-u$ to get the expressions

$$\sum_{\substack{0 \leqslant u \leqslant i/2 \\ l/2 \leqslant v \leqslant l \\ s = i-2u+l-2v}} \binom{i}{u}\binom{l}{v} r_{i-2u} r_{2v-l} \qquad \text{and} \qquad \sum_{\substack{i/2 \leqslant u \leqslant i \\ 0 \leqslant v \leqslant l/2 \\ s = i-2u+l-2v}} \binom{i}{u}\binom{l}{v} r_{2u-i} r_{l-2v}.$$

Since there are no indices (u, v) satisfying $i/2 \leqslant u \leqslant i$, $l/2 \leqslant v \leqslant l$ and $s = i - 2u + l - 2v$, we obtain that $N_1 + N_2 + N_3$ equals

$$\sum_{\substack{0 \leqslant u \leqslant i \\ 0 \leqslant v \leqslant l \\ s = i-2u+l-2v}} \binom{i}{u}\binom{l}{v} = \xi(i+l, s) = \begin{cases} \binom{i+l}{\frac{i+l-s}{2}} & \text{if} \quad s \leqslant i+l \quad \text{and} \quad i+l-s \text{ even} \\ 0 & \text{otherwise.} \end{cases} \tag{6.9}$$

This is the desired result since the coefficient for $a^\#_s = a^\#_{-s}$ on the left-hand side of (6.8) is zero if $i + l - s$ is odd and

$$\frac{1}{2}\left(\binom{i+l}{\frac{i+l-s}{2}} + \binom{i+l}{\frac{i+l+s}{2}}\right) = \binom{i+l}{\frac{i+l-s}{2}}$$

otherwise.

In the case $s = 0$, the coefficient for the term $a^\#_0$ on the right-hand side of (6.8) equals $N := N_1 + N_3 = N_2 + N_3$, while it equals $\frac{1}{2}\xi(i + l, 0)$ on the left-hand side. The manipulation of the expressions N_k can be done in the same way, with the only difference that in the end there are indices (u, v) satisfying $i/2 \leqslant u \leqslant i$, $l/2 \leqslant v \leqslant l$, $i+l = 2(u+v)$. This corresponds to a term N_4, which happen to be equal to N_3. Thus, $N = \frac{1}{2}(N_1 + \cdots + N_4)$ with $N_1 + \cdots + N_4$ equaling (6.9). This settles the case $s = 0$.

Hence, we have shown that identity (6.8) holds, and this implies formula (6.7). □

In regard to the first part of the theorem we remark that the hypotheses in (1)–(4) are compatible to each other in the sense that the hypotheses in (1) and (3), as well as those in (2) and (4) imply that

$$a^\#_k = 2a_k - a_{k-2} - a_{k+2}.$$

Correspondingly, we have

$$DAD^T = RA^\# R \quad \text{with} \quad D = D_+ D_- = D_- D_+.$$

Elaborating on formula (6.7) we remark that assuming the hypotheses in (1)–(4), one can express the coefficients b_n in terms of $a^\pm_k$ and a_k as well. We record the corresponding results for completeness sake:

$$b_n = \sum_{k=0}^{n} \binom{n}{k}\left(a^+_{n-2k} + a^+_{2n+1-k}\right), \qquad B = S_+ A^+ S_+^T, \quad S_+ = S_\# D_- \tag{6.10}$$

$$b_n = \sum_{k=0}^{n} \binom{n}{k}\left(a^-_{n-2k} - a^-_{2n+1-k}\right), \qquad B = S_- A^- S_-^T, \quad S_- = S_\# D_+ \tag{6.11}$$

$$b_n = \sum_{k=0}^{n} \binom{n}{k}\left(a_{n-2k} - a_{2n+2-k}\right), \qquad B = SAS^T, \quad S = S_\# D \tag{6.12}$$

The matrices $S_\pm$ and S are evaluated as follows:

$$S_+ = \begin{pmatrix} \binom{0}{0} & & & & & \\ \binom{1}{0} & \binom{1}{0} & & & & \\ \binom{2}{1} & \binom{2}{0} & \binom{2}{0} & & & \\ \binom{3}{1} & \binom{3}{1} & \binom{3}{0} & \binom{3}{0} & & \\ \binom{4}{2} & \binom{4}{1} & \binom{4}{1} & \binom{4}{0} & \binom{4}{0} & \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$S_- = \begin{pmatrix} \binom{0}{0} \\ -\binom{1}{0} & \binom{1}{0} \\ \binom{2}{1} & -\binom{2}{0} & \binom{2}{0} \\ -\binom{3}{1} & \binom{3}{1} & -\binom{3}{0} & \binom{3}{0} \\ \binom{4}{2} & -\binom{4}{1} & \binom{4}{1} & -\binom{4}{0} & \binom{4}{0} \\ \vdots & & & & & \ddots \end{pmatrix},$$

$$S = \begin{pmatrix} \binom{0}{0} \\ 0 & \binom{1}{0} \\ \binom{2}{1}-\binom{2}{0} & 0 & \binom{2}{0} \\ 0 & \binom{3}{1}-\binom{3}{0} & 0 & \binom{3}{0} \\ \binom{4}{2}-\binom{4}{1} & 0 & \binom{4}{1}-\binom{4}{0} & 0 & \binom{4}{0} \\ \vdots & & & & & \ddots \end{pmatrix}.$$

Finally we remark that the recurrence relation (6.6) allows us to express the coefficients $a_n^\#$ in terms of b_n,

$$a_0^\# = 2b_0, \qquad a_n^\# = a_{-n}^\# = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k b_{n-2k} \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right), \qquad n \geqslant 1.$$

Let us now proceed with establishing the identities for the determinants of the finite matrices. We restrict to the cases where the symbols are L^1-functions because this is what is of interest to us.

Theorem 6. *Let $a, a^+, a^-, a^\# \in L^1(\mathbb{T})$ be even, and let $b \in L^1[-1, 1]$. Assume that*
(1) $a(e^{i\theta}) = b(\cos\theta)(2 + 2\cos\theta)^{-1/2}(2 - 2\cos\theta)^{-1/2}$,
(2) $a^+(e^{i\theta}) = b(\cos\theta)(2 + 2\cos\theta)^{-1/2}(2 - 2\cos\theta)^{1/2}$,
(3) $a^-(e^{i\theta}) = b(\cos\theta)(2 + 2\cos\theta)^{1/2}(2 - 2\cos\theta)^{-1/2}$,
(4) $a^\#(e^{i\theta}) = b(\cos\theta)(2 + 2\cos\theta)^{1/2}(2 - 2\cos\theta)^{1/2}$.

Then, for each $n \geqslant 1$,

$$\begin{aligned} \det H_n[b] &= \det(T_n(a) - H_n(z^{-1}a)) = \det(T_n(a^+) + H_n(a^+)) \\ &= \det(T_n(a^-) - H_n(a^-)) = \tfrac{1}{4} \det(T_n(a^\#) + H(za^\#)). \end{aligned} \tag{6.13}$$

Proof. We first note that the hypotheses on the coefficients stated in (1)–(4) of theorem 5 can be rephrased in terms of the corresponding generating functions (see (6.1) and (6.3)) as follows:

$$\begin{aligned} a^+(z) &= a(z)(1-z)(1-z^{-1}), & a^-(z) &= a(z)(1+z)(1+z^{-1}), \\ a^\#(z) &= a^+(z)(1+z)(1+z^{-1}), & a^\#(z) &= a^-(z)(1-z)(1-z^{-1}). \end{aligned} \tag{6.14}$$

Here $z = e^{i\theta} \in \mathbb{T}$. Incidentally, the relations between a, a^+, a^-, and $a^\#$ implied by the assumption 1–4 above are precisely those in (6.14).

Now assume 4 and compute

$$\begin{aligned} b_n &= \frac{1}{\pi} \int_1^{-1} b(x)(2x)^n \, dx = \frac{1}{\pi} \int_0^\pi b(\cos\theta)(2\cos\theta)^n \sin(\theta) \, d\theta \\ &= \frac{1}{4\pi} \int_0^{2\pi} a^\#(e^{i\theta})(e^{i\theta} + e^{-i\theta})^n \, d\theta = \frac{1}{2} \sum_{k=0}^n a_{n-2k}^\# \binom{n}{k}, \end{aligned}$$

which is precisely the condition (6.6).

In order to use the results of theorem 5 we take the finite sections of the various identities (i.e. we consider the $n \times n$ upper-left corners of the infinite matrices),

$$D_+AD_+^T = A^+, \qquad D_-AD_-^T = A^-,$$

$$D_-A^+D_-^T = D_+A^-D_+^T = RA^\# R, \qquad B = S_\# RA^\# RS_\#^T,$$

and then take the determinants. The crucial point is that D_+, D_- and $S_\#$ are lower triangular and have ones on their diagonals. The diagonal matrices R give a factor $\frac{1}{4}$ in the determinants. Now it just remains to check that the finite sections of the infinite matrices (6.5) and (6.6) are indeed the matrices occurring in (6.13). But this follows from the definitions (6.2) and (6.4). □

It is apparent from the proof that if we are only interested in an identity between two types of determinants featuring (6.13), then it is enough to assume that only the corresponding symbols are L^1-functions and that the appropriate relationships between these symbols hold (see also (6.14)). For instance, if we assume $a^+, a^- \in L^1(\mathbb{T})$ and

$$a^+(z)(1+z)(1+z^{-1}) = a^-(z)(1-z)(1-z^{-1}),$$

then we can conclude that

$$\det(T_n(a^+) + H_n(a^+)) = \det(T_n(a^-) - H_n(a^-)). \tag{6.15}$$

By the way, this relationship between these two types of determinants is not the trivial one featuring the 'equivalence' between cases 2 and 3, which has been pointed out earlier.

7. Results from the Toeplitz theory

The idea for this section is that if the α and β are any combination of $\pm\frac{1}{2}$, then we may choose an operator of the form $T_n(a) + H_n(b)$ from our list of identities 1–4 that has, in a certain sense, a nice symbol and find an explicit formula for the determinants of the associated Hankel matrices, $H_n[b]$. This is because for these values of the parameters and the right choice of operator, we lose the square root singularities. Fortunately in [6] exact formulas for the types of Toeplitz plus Hankel determinants that appear in the previous theorem were found. If we specialize the results to the cases at hand we can state the exact formula of the determinants of the matrices $H_n[b]$. The four different determinants all have the form

$$G[a]^n F[a] \det(I + Q_n K Q_n), \tag{7.1}$$

where $F[a]$ and $G[a]$ are certain constants that depend on our choice of parameters for α and β. The last operator determinant involves orthogonal projections $Q_n = I - P_n$, where the projections P_n acting on $\ell^2(\mathbb{Z}_+)$, $\mathbb{Z}_+ = \{0, 1, \ldots, \}$, are defined by

$$P_n(a_0, a_1, \ldots) = (a_0, a_1, \ldots, a_{n-1}, 0, 0, \ldots).$$

The operator K, acting on $\ell^2(\mathbb{Z})$, is a certain (trace class) semi-infinite Hankel operator.

The precise reference for result (7.1) is proposition 4.1 and the remark afterward in [6]. Propositions 3.1 and 3.3 in [6] also have to be considered. For the sake of clarification we remark that our cases 1–4 correspond to cases I–IV in [6] as follows: 1 = III, 2 = I, 3 = II, 4 = IV, where in case 4, the operators differ by a constant.

In our case the symbol is (up to a constant) $a(e^{i\theta}) = e^{-t\cos\theta}$ whence $\psi = a_+^{-1}(e^{i\theta})\widetilde{a}_+(e^{i\theta}) = e^{it\sin\theta}$, which occurs in the definition of the operator K. The Fourier coefficients ψ_k ($k \geqslant 0$) are precisely equal to the value of the Bessel function $J_k(t)$ of order j

with the argument t. The precise description of K is as follows

(1) Let $\alpha = \frac{1}{2},\ \beta = \frac{1}{2}$, then $K = K_1$, where K_1 has (j,k)-entry $-J_{j+k+2}(t)$.
(2) Let $\alpha = -\frac{1}{2},\ \beta = \frac{1}{2}$, then $K = K_2$, where K_2 has (j,k)-entry $J_{j+k+1}(t)$.
(3) Let $\alpha = \frac{1}{2},\ \beta = -\frac{1}{2}$, then $K = K_3$, where K_3 has (j,k)-entry $-J_{j+k+1}(t)$.
(4) Let $\alpha = -\frac{1}{2},\ \beta = -\frac{1}{2}$, then $K = K_4$, where K_4 has (j,k)-entry $J_{j+k}(t)$.

Here $j,k \geqslant 0$. It is known that the operator K is trace class. This is not hard to see since for fixed t the entries in the Hankel matrix tend to zero very rapidly. We state the four cases below. In all cases our function a in the previous identities is $\mathrm{e}^{-t\cos\theta}$ times a factor of a power of 2.

Theorem 7. *Let* $b(x) = (1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{e}^{-tx}$.

(1) Let $\alpha = \frac{1}{2},\ \beta = \frac{1}{2}$, *then*

$$\det H_n[b] = 2^{-n}\,\mathrm{e}^{t^2/8}\det(I + Q_n K_1 Q_n).$$

(2) Let $\alpha = -\frac{1}{2},\ \beta = \frac{1}{2}$, *then*

$$\det H_n[b] = \mathrm{e}^{t^2/8-t/2}\det(I + Q_n K_2 Q_n).$$

(3) Let $\alpha = \frac{1}{2},\ \beta = -\frac{1}{2}$, *then*

$$\det H_n[b] = \mathrm{e}^{t^2/8+t/2}\det(I + Q_n K_3 Q_n).$$

(4) Let $\alpha = -\frac{1}{2}, \beta = -\frac{1}{2}$, *then*

$$\det H_n[b] = 2^{n-1}\,\mathrm{e}^{t^2/8}\det(I + Q_n K_4 Q_n).$$

This does not quite give us the identity of the original $D_n(t)$ since the above Hankel was defined with some extra constants of π and 2. So first we adjust for these to yield the following.

Theorem 8. *Let* $b(x) = (1-x)^{\alpha}(1+x)^{\beta}\mathrm{e}^{-tx}$.

(1) Let $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$, *then*

$$D_n(t) = 2^{-n(n+1)}(2\pi)^n\,\mathrm{e}^{t^2/8}\det(I + Q_n K_1 Q_n).$$

(2) Let $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$, *then*

$$D_n(t) = 2^{-n^2}(2\pi)^n\,\mathrm{e}^{t^2/8-t/2}\det(I + Q_n K_2 Q_n).$$

(3) Let $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, *then*

$$D_n(t) = 2^{-n^2}(2\pi)^n\,\mathrm{e}^{t^2/8+t/2}\det(I + Q_n K_3 Q_n).$$

(4) Let $\alpha = -\frac{1}{2}, \beta = -\frac{1}{2}$, *then*

$$D_n(t) = 2^{-n(n-1)-1}(2\pi)^n\,\mathrm{e}^{t^2/8}\det(I + Q_n K_4 Q_n).$$

Since Q_n tends to zero strongly and the operator K is trace class, the term $\det(I + Q_n K Q_n)$ tends to one and the asymptotics are given by the previous factors in each case of the above result.

More precisely, we obtain the following result.

Theorem 9. *Let* $b(x) = (1-x)^{\alpha}(1+x)^{\beta}\,\mathrm{e}^{-tx}$.

(1) Let $\alpha = \frac{1}{2},\ \beta = \frac{1}{2}$, *then*

$$D_n(t) \sim 2^{-n(n+1)}(2\pi)^n\,\mathrm{e}^{t^2/8}.$$

(2) Let $\alpha = -\frac{1}{2}, \beta = \frac{1}{2}$, *then*

$$D_n(t) \sim 2^{-n^2}(2\pi)^n\,\mathrm{e}^{t^2/8-t/2}.$$

(3) Let $\alpha = \frac{1}{2}, \beta = -\frac{1}{2}$, *then*

$$D_n(t) \sim 2^{-n^2}(2\pi)^n\,\mathrm{e}^{t^2/8+t/2}.$$

(4) Let $\alpha = -\frac{1}{2},\ \beta = -\frac{1}{2}$, *then*

$$D_n(t) \sim 2^{-n(n-1)-1}(2\pi)^n\,\mathrm{e}^{t^2/8}.$$

If we expand $\det(I + Q_nK_1Q_n)$ using the fact that $\log\det(I + A) = \operatorname{tr}\log(I + A)$ using just the first couple of terms, it seems reasonable to conjecture that, for example,

$$D_n(t) \sim 2^{-n(n+1)}(2\pi)^n\,\mathrm{e}^{t^2/8}\,\mathrm{e}^{\frac{(t/2)^{2n+2}}{\Gamma(2n+3)}+\mathrm{O}\left(\frac{1}{\Gamma(2n+4)}\right)}.$$

Similar conjectures can be made in the other cases.

Before ending this section, we conjecture, with the aid of the linear statistics formula in [16] and [3] obtained through the heuristic Coulomb fluid approach [14], that for 'general' values α and β and for large n

$$\log\left(\frac{D_n(t)}{D_n(0)}\right) \sim \frac{t^2}{8} + (\alpha-\beta)t,$$

where

$$D_n(0) \sim 2^{-n(n+\alpha+\beta)}\,n^{(\alpha^2+\beta^2)/2-1/4}\,(2\pi)^n\,\frac{G\left(\frac{1+\alpha+\beta}{2}\right)G^2\left(\frac{2+\alpha+\beta}{2}\right)G\left(\frac{3+\alpha+\beta}{2}\right)}{G(1+\alpha+\beta)G(1+\alpha)G(1+\beta)}.$$

Here $G(z)$ is the Barnes G-function [2].

Finally, we have as a consequence of the previous sections the following remark: Let $\alpha = \frac{1}{2},\ \beta = \frac{1}{2}$, and let

$$\phi = t\frac{\mathrm{d}}{\mathrm{d}t}\log(I + Q_nK_1(t/2)Q_n).$$

Then the function

$$\phi(t) = \frac{t^2}{16} + \frac{t}{2}\mathsf{p}_1\left(n, \frac{t}{2}\right),$$

and thus also satisfies a related Painlevé equation. A similar expression can be obtained for the other three cases. This once again demonstrates that the most fundamental quantity in the theory is the coefficient $\mathsf{p}_1(n,t)$.

After the completion of our manuscript it was brought to our attention that a result similar to (1.1) was obtained in [27, theorem 2.2]. However, our main results, expressing the σ function in terms of $\mathsf{p}_1(n, t/2)$, the coefficient of z^{n-1} of our 'deformed' orthogonal polynomial, and consequently expressing $R_n(t)$, $r_n(t)$, $\alpha_n(t)$ and $\beta_n(t)$ all in terms of the same quantity, are distinct from the result in [27], where several τ- functions are involved.

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