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Painlevé equations—nonlinear special functions

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Abstract

The six Painlevé equations (P_I–P_{VI}) were first discovered about a hundred years ago by Painlevé and his colleagues in an investigation of nonlinear second-order ordinary differential equations. Recently, there has been considerable interest in the Painlevé equations primarily due to the fact that they arise as reductions of the soliton equations which are solvable by inverse scattering. Consequently, the Painlevé equations can be regarded as completely integrable equations and possess solutions which can be expressed in terms of solutions of linear integral equations, despite being nonlinear equations. Although first discovered from strictly mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications including statistical mechanics, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics.

The Painlevé equations may be thought of a nonlinear analogues of the classical special functions. They possess hierarchies of rational solutions and one-parameter families of solutions expressible in terms of the classical special functions, for special values of the parameters. Further the Painlevé equations admit symmetries under affine Weyl groups which are related to the associated Bäcklund transformations.

In this paper, I discuss some of the remarkable properties which the Painlevé equations possess including connection formulae, Bäcklund transformations associated discrete equations, and hierarchies of exact solutions. In particular, the second Painlevé equation P_{II} is used to illustrate these properties and some of the applications of P_{II} are also discussed.

Keywords: Painlevé equations; Bäcklund transformations; Connection formulae; Exact solutions

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1. Introduction

In this paper, our interest is in the six Painlevé equations ($P_{\rm I}$–$P_{\rm VI}$)

$$w'' = 6w^2 + z, \tag{1.1}$$

$$w'' = 2w^3 + zw + \alpha, \tag{1.2}$$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}, \tag{1.3}$$

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}, \tag{1.4}$$

$$w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}, \tag{1.5}$$

$$\begin{aligned} w'' = {} & \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w' \\ & + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left\{\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right\}, \end{aligned} \tag{1.6}$$

where $' \equiv \mathrm{d}/\mathrm{d}z$ and α, β, γ and δ are arbitrary constants. The solutions of $P_{\rm I}$–$P_{\rm VI}$ are called the *Painlevé transcendents*. The Painlevé equations $P_{\rm I}$–$P_{\rm VI}$ were discovered about a hundred years ago by Painlevé and his colleagues whilst studying a problem posed by Picard [71]. Picard asked which second-order ordinary differential equations of the form

$$w'' = F(z; w, w'), \tag{1.7}$$

where F is rational in w' and w and analytic in z, have the property that the solutions have no movable branch points, i.e., the locations of multi-valued singularities of any of the solutions are independent of the particular solution chosen and so are dependent only on the equation; this is now known as the *Painlevé property*. Painlevé et al. showed that there were 50 canonical equations of form (1.7) with this property, up to a Möbius (bilinear rational) transformation

$$W(\zeta) = \frac{a(z)w + b(z)}{c(z)w + d(z)}, \quad \zeta = \phi(z), \tag{1.8}$$

where $a(z)$, $b(z)$, $c(z)$, $d(z)$ and $\phi(z)$ are locally analytic functions. Further, they showed that of these 50 equations, 44 are either integrable in terms of previously known functions (such as elliptic functions or are equivalent to linear equations) or reducible to one of six new nonlinear ordinary differential equations, which define new transcendental functions (cf. [37]). Although first discovered from strictly mathematical considerations, the Painlevé equations have arisen in a variety of important physical applications including statistical mechanics, plasma physics, nonlinear waves, quantum gravity, quantum field theory, general relativity, nonlinear optics and fibre optics. Further, the Painlevé equations have attracted much interest since they arise in many physical situations and as reductions of the soliton equations which are solvable by inverse scattering (cf. [1,4] for further details and references).

The Painlevé equations can be thought of as nonlinear analogues of the classical special functions. Their general solutions are transcendental, i.e., irreducible in the sense that they cannot be expressed in terms of previously known functions, such as rational functions or the special functions. However, they possess many rational solutions and solutions expressible in terms of special functions for certain values of the parameters (these special solutions are called "classical solutions" [82]), and they possess Bäcklund transformations which relate one solution to another solution either of the same equation, with different values of the parameters, or another equation (further details and references are given in Section 2). The isomonodromy method has been developed for the study of the Painlevé equations (cf. [14,15,17,38,40,42–44,65]) and in this sense they are said to be integrable. The Painlevé equations have a plethora of other fascinating properties. For example, they can be written in Hirota bilinear form [36] and have the following coalescence cascade (see, for example, [37,41] for details)

$$\begin{array}{ccccc} \mathrm{P_{VI}} & \to & \mathrm{P_V} & \to & \mathrm{P_{IV}} & & \\ & & \downarrow & & \downarrow & & \\ & & \mathrm{P_{III}} & \to & \mathrm{P_{II}} & \to & \mathrm{P_I}. \end{array}$$

2. Mathematical properties of the Painlevé equations

2.1. Asymptotic expansions and connection formulae

Consider the special case of $\mathrm{P_{II}}$ (1.2) with $\alpha = 0$, i.e.,

$$w'' = 2w^3 + zw \tag{2.1}$$

with boundary condition

$$w(z) \to 0 \quad \text{as } z \to \infty. \tag{2.2}$$

The "classic problem" for (2.1) and (2.2) is given in the following theorem, proved in [34].

Theorem 2.1. *Any solution of* (2.1), *satisfying* (2.2) *is asymptotic to* $k\,\mathrm{Ai}(x)$, *for some* k, *with* $\mathrm{Ai}(z)$ *the Airy function. Conversely, for any* k, *there is a unique solution* $w_k(z)$ *of* (2.1) *which is asymptotic to* $k\,\mathrm{Ai}(z)$ *as* $z \to +\infty$, *for some* k. *If* $|k| < 1$, *then this solution exists for all real* z *as* $z \to -\infty$, *and as* $z \to -\infty$

$$w(z) = d|z|^{-1/4} \sin\{\tfrac{2}{3}|z|^{3/2} - \tfrac{3}{4}d^2 \log|z| - \theta_0\} + \mathrm{o}(|z|^{-1/4}) \tag{2.3}$$

for some constants d *and* θ_0 *which depend on* k.

If $|k| = 1$ *then*

$$w_k(z) \sim \mathrm{sgn}(k)\sqrt{-\tfrac{1}{2}z} \quad \textit{as } z \to -\infty. \tag{2.4}$$

If $|k| > 1$ *then* $w_k(z)$ *has a pole at a finite* z_0, *dependent on* k,

$$w_k(z) \sim \mathrm{sgn}(k)(z - z_0)^{-1} \quad \textit{as } z \downarrow z_0. \tag{2.5}$$

The specific dependent of the constants d and θ_0 in (2.3) on the parameter k is given as follows.

Theorem 2.2. *The connection formulae d and θ_0 in the asymptotic expansion (2.3) are given by*

$$d^2(k) = -\pi^{-1} \ln(1-k^2), \tag{2.6}$$

$$\theta_0(k) = \tfrac{3}{2} d^2 \ln 2 + \arg\{\Gamma(1-\tfrac{1}{2}\mathrm{i}d^2)\} - \tfrac{1}{4}\pi. \tag{2.7}$$

with $\Gamma(z)$ the Gamma function.

The amplitude connection formula (2.6) and the phase connection formula (2.7) were first conjectured, derived heuristically and subsequently verified numerically in [3,75]. Some years later Clarkson and McLeod [11] gave a rigorous proof of (2.6), using the Gel'fand–Levitan–Marchenko integral equation (2.9). Suleĭmanov [77] derived (2.6) and (2.7) using the isomonodromy problem (2.13)—see also [38,39]. Subsequently, Deift and Zhou [14,15] rigorously proved these connection formulae using a nonlinear version of the classical steepest descent method for oscillating Riemann–Hilbert problems. Recently, Bassom et al. [8] have developed a uniform approximation method, which is rigorous, removes the need to match solutions and can leads to a simpler solution of this connection problem for the special case of $\mathrm{P_{II}}$ given by (2.1). Numerical studies of this boundary value problem are discussed in [58,59,74], which also arises in a number of mathematical and physical problems, as discussed in Section 3.2.

2.2. Integral equations

The Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ arise as similarity reductions of partial differential equations solvable by inverse scattering (cf. [1–4]). For example, if we make the scaling reduction $u(x,t)=(3t)^{-1/3}w(z)$, with $z = x/(3t)^{1/3}$, in the modified Korteweg–de Vries (mKdV) equation

$$u_t - 6u^2 u_x + u_{xxx} = 0, \tag{2.8}$$

then after integrating once, $w(z)$ satisfies $\mathrm{P_{II}}$ (1.2) with α the arbitrary constant of integration [2,3].

Consequently, certain solutions of Painlevé equations can be expressed in terms of solutions of linear integral equations. Consider the integral equation

$$K(z,\xi) = k\,\mathrm{Ai}\left(\frac{z+\xi}{2}\right) + \frac{1}{4}k^2 \int_z^\infty \int_z^\infty K(z,s)\,\mathrm{Ai}\left(\frac{s+t}{2}\right)\mathrm{Ai}\left(\frac{t+\xi}{2}\right) \mathrm{d}s\,\mathrm{d}t \tag{2.9}$$

with $\mathrm{Ai}(z)$ the Airy function, then it can be shown that $w_k(z)=K(z,z)$, satisfies (2.1), i.e., $\mathrm{P_{II}}$ with $\alpha = 0$, with the boundary condition

$$w_k(z) \sim k\,\mathrm{Ai}(z) \quad \text{as } z \to \infty. \tag{2.10}$$

Integral equation (2.9) is derived by making a scaling reduction of the Gel'fand–Levitan–Marchenko integral equation for solving the mKdV equation (2.8) by inverse scattering (see [1–3] for further details). The construction of the one-parameter transcendental solution of $\mathrm{P_{II}}$ with $\alpha = 0$ satisfying the boundary condition (2.10) in [3] through the linear integral equation (2.9) is the first such construction for a Painlevé equation.

2.3. Isomonodromy problems

Each of the Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be expressed as the compatibility condition of the linear system

$$\boldsymbol{\Psi}_\lambda = \boldsymbol{A}(z;\lambda)\boldsymbol{\Psi}, \quad \boldsymbol{\Psi}_z = \boldsymbol{B}(z;\lambda)\boldsymbol{\Psi}, \tag{2.11}$$

where $\boldsymbol{A}$ and $\boldsymbol{B}$ are matrices. The equation $\boldsymbol{\Psi}_{z\lambda} = \boldsymbol{\Psi}_{\lambda z}$ is satisfied provided that

$$\boldsymbol{A}_z - \boldsymbol{B}_\lambda + \boldsymbol{AB} - \boldsymbol{BA} = 0, \tag{2.12}$$

which is the compatibility condition of (2.11). For example, $\mathrm{P_{II}}$ (1.2) arises for the matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ given in [17] (see also [19])

$$\boldsymbol{A}(z;\lambda) = \begin{pmatrix} -\mathrm{i}(4\lambda^2+2w^2+z) & 4\lambda w + 2\mathrm{i}w' + \alpha/w \\ 4\lambda w - 2\mathrm{i}w' + \alpha/w & \mathrm{i}(4\lambda^2+2w^2+z) \end{pmatrix}, \quad \boldsymbol{B}(z;\lambda) = \begin{pmatrix} -\mathrm{i}\lambda & w \\ w & \mathrm{i}\lambda \end{pmatrix}. \tag{2.13}$$

These are derived through a scaling reduction of the Lax pair of the mKdV equation (2.8) [17].

Matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ for $\mathrm{P_I}$–$\mathrm{P_{VI}}$ satisfying (2.13) are given by in [43], though these are not unique.

2.4. Hamiltonian structure

The Hamiltonian structure associated with the Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ is $\mathscr{H}_\mathrm{J} = (q, p, \mathrm{H_J}, z)$, where $\mathrm{H_J}$, the Hamiltonian function associated with $\mathrm{H_J}$ is a polynomial in q, p and z. Each of the Painlevé equations $\mathrm{P_I}$–$\mathrm{P_{VI}}$ can be written as a Hamiltonian system

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathrm{H_J}}{\partial p}, \quad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathrm{H_J}}{\partial q}, \tag{2.14}$$

for a suitable Hamiltonian function $\mathrm{H_J}(q,p,z)$ (cf. [70]). Further the function $\sigma_\mathrm{J}(z) \equiv \mathrm{H_J}(q,p,z)$ satisfies a second-order, second-degree ordinary differential equation, whose solution is expressible in terms of the solution of the associated Painlevé equation.

For example, the second Painlevé equation $\mathrm{P_{II}}$ (1.2) can be written as the Hamiltonian system [66]

$$\frac{\mathrm{d}q}{\mathrm{d}z} = \frac{\partial \mathrm{H_{II}}}{\partial p} = p - q^2 - \frac{1}{2}z, \qquad \frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{\partial \mathrm{H_{II}}}{\partial q} = 2qp + \alpha + \frac{1}{2}, \tag{2.15}$$

where the (nonautonomous) Hamiltonian $\mathrm{H_{II}}(q,p,z;\alpha)$ is given by

$$\mathrm{H_{II}}(q,p,z;\alpha) = \tfrac{1}{2}p^2 - (q^2 + \tfrac{1}{2}z)p - (\alpha + \tfrac{1}{2})q. \tag{2.16}$$

Eliminating p in (2.15) then $q = w$ satisfies $\mathrm{P_{II}}$ (1.2) whilst eliminating q yields

$$pp'' = \tfrac{1}{2}(p')^2 + 2p^3 - zp^2 - \tfrac{1}{2}(\alpha + \tfrac{1}{2})^2, \tag{2.17}$$

which is known as $\mathrm{P_{34}}$, since it is equivalent to [37, Eq. (XXXIV) Chapter 14]. Further, if q satisfies $\mathrm{P_{II}}$ (1.2) then $p = q' + q^2 + \frac{1}{2}z$ satisfies (2.17) and conversely, if p satisfies (2.17) then $q = (p' - \alpha - \frac{1}{2})/(2p)$ satisfies $\mathrm{P_{II}}$ (1.2). Thus, there is a one-to-one correspondence between solutions of $\mathrm{P_{II}}$ (1.2) and those of $\mathrm{P_{34}}$ (2.17). The function $\sigma(z;\alpha) = \mathrm{H_{II}}(q,p,z;\alpha)$ defined by (2.16) satisfies the second-order, second-degree equation

$$(\sigma'')^2 + 4(\sigma')^3 + 2\sigma'(z\sigma' - \sigma) = \tfrac{1}{4}(\alpha + \tfrac{1}{2})^2. \tag{2.18}$$

Conversely, if $\sigma(z;\alpha)$ is a solution of (2.18), then

$$q(z;\alpha) = (\tfrac{1}{2}\sigma''(z;\alpha) + \tfrac{1}{4}\alpha + \tfrac{1}{8})/\sigma'(z;\alpha), \qquad p(z;\alpha) = -2\sigma'(z;\alpha) \tag{2.19}$$

are solutions of (2.15).

We remark that Eq. (2.18) is equation SD-I.d in the classification of second-order, second-degree equations which have the Painlevé property in [12], an equation first derived in [9].

2.5. Bäcklund transformations

The Painlevé equations P_{II}–P_{VI} possess Bäcklund transformations which relate one solution to another solution either of the same equation, with different values of the parameters, or another equation (cf. [5,7,18,23,31,40,48,60–62,66–69] and the references therein).

For example, if $w \equiv w(z;\alpha)$ is a solution of P_{II} (1.2) then

$$\mathscr{S}: \quad w(z;-\alpha) = -w, \tag{2.20}$$

$$\mathscr{T}_{\pm}: \quad w(z;\alpha \pm 1) = -w - \frac{2\alpha \pm 1}{2w^2 \pm 2w' + z} \tag{2.21}$$

are also solutions of P_{II} (1.2), provided that $\alpha \neq \mp\frac{1}{2}$ [26,52]. Umemura [83] discusses geometrical aspects of the Bäcklund transformations of P_{II} (1.2). Gambier [26] also discovered the following special transformation of P_{II} (1.2):

$$\begin{gathered} W\left(\zeta; \frac{1}{2}\varepsilon\right) = \frac{2^{-1/3}\varepsilon}{w(z;0)}\,\frac{\mathrm{d}w}{\mathrm{d}z}(z;0), \\ w^2(z;0) = 2^{-1/3}\left\{W^2\left(\zeta;\frac{1}{2}\varepsilon\right) - \varepsilon\,\frac{\mathrm{d}W}{\mathrm{d}\zeta}\left(\zeta;\frac{1}{2}\varepsilon\right) + \frac{1}{2}\zeta\right\}, \end{gathered} \tag{2.22}$$

where $\zeta = -2^{1/3}z$ and $\varepsilon = \pm 1$ (see also [10]). Combined with the Bäcklund transformation (2.21), transformation (2.22) provides a relation between two P_{II} equations whose parameters α are either integers or half odd-integers. Hence, this yields a mapping between the rational solutions of P_{II}, which arise when $\alpha = n$ for $n \in \mathbb{Z}$ and the one-parameter Airy function solutions, which arise when $\alpha = n + \frac{1}{2}$ for $n \in \mathbb{Z}$.

The solutions $w_\alpha = w(z;\alpha)$, $w_{\alpha\pm 1} = w(z;\alpha \pm 1)$ also satisfy the nonlinear three-point recurrence relation

$$\frac{2\alpha+1}{w_{\alpha+1} + w_\alpha} + \frac{2\alpha-1}{w_\alpha + w_{\alpha-1}} + 4w_\alpha^2 + 2z = 0, \tag{2.23}$$

a difference equation which is known as an alternative form of discrete P_I [20]. This is analogous to the situation for classical special functions such as Bessel function $J_\nu(z)$ which satisfies both a differential equation and a difference equation. We remark that for P_{II} (1.2), the independent variable z varies and the parameter α is fixed, whilst for the discrete equation (2.23), z is a fixed parameter and α varies.

2.6. Affine Weyl groups

The parameter space of P_{II}–P_{VI} can be identified with the Cartan subalgebra of a simple Lie algebra and the corresponding affine Weyl groups $\tilde{A}_1$, $\tilde{C}_2$, $\tilde{A}_2$, $\tilde{A}_3$, $\tilde{D}_4$, act on P_{II}–P_{VI}, respectively, as a group of Bäcklund transformations (cf. [63,64,66–69,83]). An affine Weyl group is essentially a group of translations and reflections on a lattice. For the Painlevé equations, this lattice is in the parameter space.

The Bäcklund transformations $\mathscr{S}$ (2.20) and $\mathscr{T}_\pm$ (2.21) are affine transformations $\mathscr{S}(\alpha)=-\alpha$ and $\mathscr{T}_\pm(\alpha)=\alpha\pm 1$, for $\alpha\in\mathbb{C}$. Consider the subgroup $\mathscr{G}$ of the affine transformation group on $\mathbb{C}$ generated by $\langle\mathscr{S},\mathscr{T}_+,\mathscr{T}_-\rangle$. Then $\mathscr{S}^2=\mathscr{I}$, $\mathscr{T}_+\mathscr{T}_-=\mathscr{T}_-\mathscr{T}_+=\mathscr{I}$ and $\mathscr{T}_+=\mathscr{S}\mathscr{T}_-\mathscr{S}$, with $\mathscr{I}$ the identity transformation, and so $\langle\mathscr{S}\rangle\cong\mathbb{Z}/2\mathbb{Z}$, the Weyl group of the root system of type A_1, and $\langle\mathscr{T}_+,\mathscr{T}_-\rangle\cong\mathbb{Z}$. Therefore, $\mathscr{G}\cong\mathbb{Z}/2\mathbb{Z}\ltimes\mathbb{Z}$, the Weyl group of the affine root system of type $\tilde{A}_1$.

2.7. Exact solutions

The generic solutions of the Painlevé equations are transcendental in the sense that they cannot be expressed in terms of known functions. However, for special values of the parameters, P_{II}–P_{VI} possess rational solutions, algebraic solutions and solutions expressible in terms of special functions (cf. [1,5,16,23,25,29,31,40,48,56,57,60–62,66–69] and the references therein). These special solutions are called "classical solutions" [82].

2.7.1. Rational and algebraic solutions

The Painlevé equations P_{II}–P_{VI} possess hierarchies of rational solutions and P_{III}, P_V and P_{VI} also possess algebraic solutions for special values of the parameters. These hierarchies are generated from "seed solutions" using the Bäcklund transformations and often are expressed in the form of determinants. This is illustrated for P_{II}.

Theorem 2.3 (Vorob'ev [86], Yablonski [88]—see also Fukutani et al. [25], Taneda [78]). *Rational solutions of* P_{II} (1.2) *exist for* $\alpha=n\in\mathbb{Z}$ *and have the form*

$$w(z;n)=\frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{Q_{n-1}(z)}{Q_n(z)}\right]\right\},\tag{2.24}$$

where the polynomials $Q_n(z)$, *for* $n\geqslant 1$, *satisfy*

$$Q_{n+1}(z)Q_{n-1}(z)=zQ_n^2(z)+4[Q_n'(z)]^2-4Q_n(z)Q_n''(z)\tag{2.25}$$

with $Q_0(z)=1$ *and* $Q_1(z)=z$.

The polynomials $Q_n(z)$ are monic polynomials of degree $\frac{1}{2}n(n+1)$ and referred to as the *Yablonski–Vorob'ev polynomials*. The first few of these polynomials and the associated rational solutions $w(z;n)$ of P_{II} (1.2) are given below. Fukutani et al. [25], see also [78], prove that the polynomials $Q_n(z)$

have simple roots and that for a positive integer n, $Q_n(z)$ and $Q_{n+1}(z)$ do not have a common root.

n	$Q_n(z)$	$w(z;n)$
1	z	$-1/z$
2	z^3+4	$\dfrac{1}{z}-\dfrac{3z^2}{z^3+4}$
3	z^6+20x^3-80	$\dfrac{3z^2}{z^3+4}-\dfrac{6z^2(z^3+10)}{z^6+20z^3-80}$
4	$z^{10}+60z^7+11200z$	$-\dfrac{1}{z}+\dfrac{6z^2(z^3+10)}{z^6+20z^3-80}-\dfrac{9z^5(z^3+40)}{z^9+60z^6+11200}$

Kajiwara and Ohta [47] have derived a determinantal representation of rational solutions of $\mathrm{P_{II}}$.

Theorem 2.4. *Let* $p_k(z)$ *be the devisme polynomial defined by*

$$\sum_{k=0}^{\infty} p_k(z)\lambda^k = \exp\left(z\lambda - \frac{4}{3}\lambda^3\right) \tag{2.26}$$

with $p_k(z)=0$ *for* $k<0$, *and* $\tau_n(z)$ *be the* $n\times n$ *determinant*

$$\tau_n(z) = \begin{vmatrix} p_n(z) & p_{n+1}(z) & \cdots & p_{2n-1}(z) \\ p_{n-2}(z) & p_{n-1}(z) & \cdots & p_{2n-3}(z) \\ \vdots & \vdots & \ddots & \vdots \\ p_{-n+2}(z) & p_{-n+3}(z) & \cdots & p_1(z) \end{vmatrix}, \quad n \geqslant 1. \tag{2.27}$$

Then

$$w(z;n) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{\tau_{n-1}(z)}{\tau_n(z)}\right]\right\}, \quad n \geqslant 1 \tag{2.28}$$

satisfies $\mathrm{P_{II}}$ (1.2) *with* $\alpha = n$.

We remark that Flaschka and Newell [17], following the earlier work of Airault [5], expressed the rational solutions of $\mathrm{P_{II}}$ (1.2) as the logarithmic derivatives of determinants.

2.7.2. Special function solutions

The Painlevé equations $\mathrm{P_{II}}$–$\mathrm{P_{VI}}$ possess hierarchies of solutions expressible in terms of classical special functions, for special values of the parameters through an associated Riccati equation. These hierarchies of solutions, which are often referred to as “one-parameter solutions” since they have one arbitrary constant, are usually generated from “seed solutions” using the Bäcklund transformations and like the rational solutions, often are expressed in the form of determinants.

The second Painlevé equation $\mathrm{P_{II}}$ (1.2) can be written as

$$\varepsilon(\varepsilon w' + w^2 + \tfrac{1}{2}z)' = 2w(\varepsilon w' + w^2 + \tfrac{1}{2}z) + \alpha + \tfrac{1}{2}\varepsilon, \quad \varepsilon^2 = 1.$$

Hence if $\alpha = \frac{1}{2}\varepsilon$, then special solutions of $\mathrm{P_{II}}$ (1.2) can be obtained in terms of solutions of the Riccati equation

$$\varepsilon w' + w^2 + \tfrac{1}{2}z = 0. \tag{2.29}$$

Setting $w = \varepsilon\varphi'/\varphi$ in this equation yields

$$\varphi'' + \tfrac{1}{2}z\varphi = 0, \tag{2.30}$$

which is equivalent to the Airy equation and has general solution

$$\varphi(z) = C_1\,\mathrm{Ai}(\xi) + C_2\,\mathrm{Bi}(\xi), \quad \xi = -2^{-1/3}z, \tag{2.31}$$

where $\mathrm{Ai}(\xi)$ and $\mathrm{Bi}(\xi)$ are Airy functions and C_1, C_2 are arbitrary constants.

Theorem 2.5 (Airault [5], Flaschka and Newell [17], Okamoto [66]). *Let $\varphi(z)$ be the solution of (2.30) and $\tau_n(z)$ be the $n \times n$ determinant*

$$\tau_n(z) = \begin{vmatrix} \varphi(z) & \varphi'(z) & \cdots & \varphi^{(n-1)(z)} \\ \varphi'(z) & \varphi''(z) & \cdots & \varphi^{(n)(z)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)}(z) & \varphi^{(n)}(z) & \cdots & \varphi^{(2n-2)}(z) \end{vmatrix}, \quad n \geqslant 1, \tag{2.32}$$

where $\varphi^{(m)}(z) = \mathrm{d}^m\varphi/\mathrm{d}z^m$, then

$$w\left(z; n - \frac{1}{2}\right) = \frac{\mathrm{d}}{\mathrm{d}z}\left\{\ln\left[\frac{\tau_{n-1}(z)}{\tau_n(z)}\right]\right\}, \quad n \geqslant 1 \tag{2.33}$$

satisfies $\mathrm{P_{II}}$ (1.12) *with* $\alpha = n - \frac{1}{2}$.

If we set $\Phi(z) \equiv \varphi'(z)/\varphi(z)$, with $\varphi(z)$ given by (2.31), then the first few solutions in the Airy function solution hierarchy for $\mathrm{P_{II}}$ (1.2) are given below.

α	$w(z;\alpha)$
$\pm\frac{1}{2}$	$\mp\Phi$
$\pm\frac{3}{2}$	$\pm\Phi \mp \dfrac{1}{2\Phi^2 + z}$
$\pm\frac{5}{2}$	$\pm\dfrac{2z\Phi^2 + \Phi + z^2}{4\Phi^3 + 2z\Phi - 1} \pm \dfrac{1}{2\Phi^2 + z}$
$\pm\frac{7}{2}$	$\pm\dfrac{48\Phi^3 + 8z^2\Phi^2 + 28z\Phi + 4z^3 - 9}{z(8z\Phi^4 + 16\Phi^3 + 8z^2\Phi^2 + 8z\Phi + 2z^3 - 3)} \mp \dfrac{2z\Phi^2 + \Phi + z^2}{4\Phi^3 + 2z\Phi - 1} \mp \dfrac{3}{z}$

Special function families of solutions of P_{III} (1.3) are expressed in terms of Bessel functions $J_\nu(z)$ [50,55,60,62,69,85], of P_{IV} (1.4) in terms of Weber–Hermite (parabolic cylinder) functions $D_\nu(z)$ [7,30,49,61,66,84], of P_V (1.5) in terms of Whittaker functions $M_{\kappa,\mu}(z)$, or equivalently confluent hypergeometric functions ${}_1F_1(a;c;z)$ [51,28,68,87], and P_{VI} (1.6) in terms of hypergeometric functions ${}_2F_1(a,b;c;z)$ [23,53,67].

3. Applications of Painlevé equations

3.1. Combinatorics

Let S_N be the group of permutations $\boldsymbol{\pi}$ of the numbers $1,2,\ldots,N$. For $1 \leqslant i_1 < \cdots < i_k \leqslant N$, then $\boldsymbol{\pi}(i_1), \boldsymbol{\pi}(i_2), \ldots, \boldsymbol{\pi}(i_N)$ is an increasing subsequence of $\boldsymbol{\pi}$ of length k if $\boldsymbol{\pi}(i_1) < \boldsymbol{\pi}(i_2) < \cdots < \boldsymbol{\pi}(i_N)$. Let $\ell_N(\boldsymbol{\pi})$ be the length of the longest subsequence of $\boldsymbol{\pi}$ and define

$$q_N(n) \equiv \mathrm{Prob}(\ell_N(\boldsymbol{\pi}) \leqslant n). \tag{3.1}$$

The problem is to determine the asymptotics of $q_N(n)$ as $N \to \infty$, which Baik et al. [6] expressed in terms of solutions of P_{II} (see Theorem 3.1 below). Define the distribution function $F_2(s)$ by

$$F_2(s) = \exp\left\{-\int_s^\infty (z-s)w^2(z)\,\mathrm{d}z\right\}, \tag{3.2}$$

which is known as the Tracy–Widom distribution first introduced in [79], and $w(z)$ satisfies (2.1), the special case of P_{II} (1.2) with $\alpha = 0$, and the boundary conditions

$$w(z) \sim \begin{cases} \mathrm{Ai}(z) & \text{as } z \to \infty, \\ \sqrt{-\tfrac{1}{2}z} & \text{as } z \to -\infty \end{cases} \tag{3.3}$$

with $\mathrm{Ai}(z)$ the Airy function. Recall from Theorem 2.1 that Hastings and McLeod [34] proved there is a unique solution of (2.1) with boundary conditions (3.3). Baik et al. [6] proved the following theorem.

Theorem 3.1. *Let S_N be the group of all permutations of N numbers with uniform distribution and let $\ell_N(\boldsymbol{\pi})$ be the length of the longest increasing subsequence of $\boldsymbol{\pi} \in S_N$. Let χ be a random variable whose distribution function is the distribution function $F_2(t)$. Then, as $N \to \infty$,*

$$\chi_N := \frac{\ell_N(\boldsymbol{\pi}) - 2\sqrt{N}}{N^{1/6}} \to \chi$$

in distribution, i.e.,

$$\lim_{N\to\infty} \mathrm{Prob}\left(\frac{\ell_N - 2\sqrt{N}}{N^{1/6}} \leqslant s\right) = F_2(s),$$

for all $s \in \mathbb{R}$.

The Tracy–Widom distribution function $F_2(s)$ given by (3.2) arose in random matrix theory were it gives the limiting distribution for the normalised largest eigenvalue $\lambda_{\max}$ in the Gaussian unitary ensemble (GUE) of $N \times N$ Hermitian matrices [79]. Specifically, for the GUE

$$\lim_{N\to\infty} \mathrm{Prob}((\lambda_{\max} - \sqrt{2N})\sqrt{2}N^{1/6} \leqslant s) = F_2(s), \tag{3.4}$$

with $F_2(s)$ given by (3.2). See [13,24,80], and the references therein, for discussions of the application of Painlevé equations in combinatorics and random matrices.

We remark that the solution of (2.1) satisfying the boundary conditions (3.3) also arises several other applications including: (i) spherical electric probe in a continuum plasma [34]; (ii) Görtler vortices in boundary layers [32,33]; (iii) nonlinear optics [27]; (iv) Bose–Einstein condensation [81]; (v) superheating fields of superconductors [35]; (vi) universality of the edge scaling for non-Gaussian Wigner matrices [76]; (vii) shape fluctuations in polynuclear growth models [72,73]; (viii) distribution of eigenvalues for covariance matrices and Wishart distributions [45].

3.2. Orthogonal polynomials

Suppose $p_n(x)$, $n = 0, 1, \ldots, \infty$, is a set of orthonormal polynomials with respect to the weight function $w(x; z)$ on (α, β), with $-\infty \leqslant \alpha < \beta \leqslant \infty$,

$$\int_\alpha^\beta p_m(x) p_n(x) w(x; z)\, \mathrm{d}x = \delta_{nm}, \quad n, m = 0, 1, \ldots\ . \tag{3.5}$$

Then $p_n(x)$ satisfy the three-point recurrence relation (cf. [54])

$$a_{n+1}(z) p_{n+1}(x) = [x - b_n(z)] p_n(x) - a_n(z) p_{n-1}(x), \quad n = 1, 2, \ldots\ . \tag{3.6}$$

For example, consider the weight function $w(x; z) = \exp(-\frac{1}{4}x^4 - zx^2)$, so

$$\int_{-\infty}^{\infty} p_m(x) p_n(x) \exp\left(-\frac{1}{4}x^4 - zx^2\right) \mathrm{d}x = \delta_{nm}. \tag{3.7}$$

Here $\alpha = -\infty$, $\beta = \infty$, $b_n = 0$, since $w(-x) = w(x)$, and $u_n = a_n^2$ satisfies

$$\frac{\mathrm{d}u_n}{\mathrm{d}z} = u_n(u_{n-1} - u_{n+1}), \tag{3.8}$$

which is the Kac–van Morebeke equation [46], and

$$(u_{n+1} + u_n + u_{n-1})u_n = n - 2zu_n, \tag{3.9}$$

which is discrete $\mathrm{P_I}$ equation ($\mathrm{dP_I}$) [21,22]. From (3.8) and (3.9) we obtain

$$2u_{n+1} = \frac{n}{u_n} - \frac{1}{u_n}\frac{\mathrm{d}u_n}{\mathrm{d}z} - 2z - u_n, \tag{3.10a}$$

$$2u_{n-1} = \frac{n}{u_n} + \frac{1}{u_n}\frac{\mathrm{d}u_n}{\mathrm{d}z} - 2z - u_n. \tag{3.10b}$$

Letting $n \to n+1$ in (3.10b) and then eliminating u_{n+1} in (3.10a) yields $\mathrm{P_{IV}}$ (1.4) with $(\alpha, \beta) = (-\frac{1}{2}n, -\frac{1}{2}n^2)$. Further, Fokas et al. [21,22] demonstrated a relationship between solutions of $\mathrm{P_{IV}}$ (1.4) and $\mathrm{dP_I}$ (3.9) in the context of two-dimensional quantum gravity.

4. Discussion

This paper gives an introduction to some of the fascinating properties which the Painlevé equations possess including connection formulae, Bäcklund transformations, associated discrete equations, and hierarchies of exact solutions. I feel that these properties show that the Painlevé equations may be thought of as nonlinear analogues of the classical special functions.

Some important open problems relating to the Painlevé equations are: (i) asymptotics and connection formulae for the Painlevé equations using the isomonodromy method, (ii) Bäcklund transformations and exact solutions of Painlevé equations, and (iii) the relationship between affine Weyl groups, Painlevé equations, Bäcklund transformations and discrete equations. The ultimate objective is to provide a complete classification and unified structure for the exact solutions and Bäcklund transformations for the Painlevé equations and the discrete Painlevé equations—the presently known results are rather fragmentary and nonsystematic.

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