1

General setting for PDE's

in n-variables $x_1, \dots, x_n$

$$F\left(\frac{\partial^\alpha}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} u, \ \underline{x}\right) = 0$$

instead of constant initial conditions like ODEs solutions depend on functions of $n-1$ variables

(values of u on hypersurfaces)

e.g. <u>initial value</u> $u\Big|_{t=t_0} = u_0(\underline{x})$

when $x_0 = t$ $(x_1, \dots, x_{n-1})$ space variables

<u>boundary value</u>

$\Omega \subset \mathbb{R}^n$ domain for u & equation

$\partial\Omega$ boundary e.g. $u\Big|_{\partial\Omega} = 0$

or $u\big|_{\partial\Omega} = \varphi$ some function

<u>mixed problem</u>

some initial &
some boundary conditions

Well posedness

- solutions exist (for given F form of the equation)
- solutions are unique w/ specified boundary/initial data
- dependence on data of eq. & b/in cond is continuous

$\boxed{2}$ Same PDE form F can have well posed & ill posed problems depending on data a/in

- Linear PDE's

$$\mathcal{L}\, u = f \qquad f = f(\underline{x})$$

assigned function

$\mathcal{L}$: linear differential operators (coeff's are functions of $\underline{x}$)

$$\mathcal{L} = \sum_{\alpha} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

$\alpha = (\alpha_1, \dots, \alpha_n)$

$|\alpha| = \alpha_1 + \dots + \alpha_n$

(several α_i can be 0)

[in general u vector valued $\underline{u} = (u_1, \dots u_M)$

$u_i(\underline{x})$

(so these coeff's are matrix valued)]

- Elliptic / hyperbolic / parabolic

coefficients of highest order term

$\Rightarrow$ Matrix of principal coefficients A

e.g. second order equation

$$\mathcal{L} = a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \text{lower order terms}$$

$$A = (a_{ij})$$

w/ $a_{ij} = a_{ji}$

3

a change of variables $\underline{x} = \underline{x}(\underline{x}')$
transforms A by conjugating w/
the Jacobian matrix of the change of
coord's

$\Longrightarrow$ if A constant entries $\Rightarrow$ diagonalizable ($a_{ij} = a_{ji}$)
$\exists$ linear change of coordinates
$\rightsquigarrow$ diag form

can absorbe then the λ_j diagonal entries
in a rescaling of the variables so
diag. entries $\in \{\pm 1, 0\}$

<u>case of order 2</u>

$u_{xx} + u_{yy}$ — elliptic
$u_{xx} - u_{yy}$ — hyperbolic
u_{xx} — parabolic

in terms of the quadratic form

Second order equations
are usually most interesting
due to relation to physical problems

Classification for higher dim

<u>Differential operators</u> $\longrightarrow$ elliptic if matrix of principal coeff is invertible

$$\mathbb{R}^n \ni \xi \longmapsto \sum_{|\alpha| = m} a_\alpha \xi^\alpha \neq 0 \quad \forall \xi \neq 0 \qquad \xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

$m \leftarrow$ order of diff op

4

sometimes a <u>uniform</u> ellipticity condition also considered

order $m = 2K$

$$(-1)^K \sum_{|\alpha|=2K} a_\alpha(\underline{x})\, \xi^\alpha > c\, \|\xi\|^2$$

stronger than just invertibility

[<u>Note</u>: nonlinear case elliptic if linearization (at a sol'n) is elliptic]

$$\xi \longmapsto \sum_{|\alpha|=m} a_\alpha \xi^\alpha$$

principal symbol

associated to a lin. differential operator

$\sigma(L)$

$$L = \sum_{|\alpha|\le m} a_\alpha \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_n^{\alpha_n}}$$

ellipticity = invertibility of principal symbol

<u>Note</u>: can also define a "total symbol"

$$\sum_{|\alpha|\le m} a_\alpha \xi^\alpha = \sigma_{tot}(L)$$

but principal symbol has better properties

transforms "well" under changes of coordinates as a section of $\underbrace{\text{cotangent}}_{\text{symm. power of}}$ bundle $\mathrm{Sym}^m(T^*X)$

$\boxed{5}$

For hyperbolic / parabolic cases
the principal symbol has a kernel

$$\{ \xi \mid \sum_{\alpha = |m|} a_\alpha \xi^\alpha = 0 \} \supsetneq \{0\}$$

these loci are the "characteristics"
of the corresponding PDE

What is a general form of the hyperbolic condition ?

- for 2D $u_{xx} - u_{yy} = 0$ case

signature $(+1, -1)$ in quadratic form

$$\sigma(\mathcal{L}) : \xi = (\xi_1, \xi_2) \longmapsto \xi_1^2 - \xi_2^2$$

Wave equation

2Dim Minkowski metric

$$= -g(\xi, \xi) \text{ for } ds^2 = -dx^2 + dy^2 = g_{\mu\nu}$$

- So wave equation is closely related (see more below) to Minkowski metric

still order 2 but dimension M

- generalize: normally hyperbolic diff op's

$$\mathcal{L} \quad \text{s.t. } \sigma(\mathcal{L})(\xi) = -g(\xi, \xi)$$

for a Lorentzian metric g signature $(-1, 1, 1, \cdots, 1)$

6

<u>Note</u>: wave equation in n Dim is

$$u_{tt} - \sum_i u_{x_i x_i} = 0$$ so again Minkowski metric

$$\| \quad u_{tt} - \Delta u$$

$$-g(\xi,\xi) = \xi_0^2 - \sum_{i=1}^{n-1} \xi_i^2 = \xi_0^2 - \|\hat{\xi}\|^2 \qquad \hat{\xi} = (\xi_1, \dots, \xi_{n-1})$$

order 2 dimension n

So
$$\mathcal{L} = -\sum_{i,j} g^{ij}(\underline{x})\, \partial_i \partial_j + \sum_i a^i(\underline{x})\, \partial_i + b(\underline{x})$$

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general idea: hyperbolic equations
<u>one variable is time</u> and describe wave propag.

⇝ will describe good geometric definition of hyperbolicity
Atiyah, Bott, Gårding (1970)

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for parabolic case one considers as model the heat equation

$$\frac{\partial u}{\partial t} = \alpha \Delta u$$

so generally one assumes a similar form
one time variable t and $\underline{x}$ space variables

and $\frac{\partial}{\partial t} = -\mathcal{L}$ with $\mathcal{L}$ elliptic

$+\mathcal{L}$ (backward parabolic PDE)

backward probl. ill posed in gen.

$\boxed{7}$ 1-dim Wave equation

consider a family of PDE's of form

$l = a\frac{\partial}{\partial t} + b\frac{\partial}{\partial x}$

$a u_t + b u_x = 0$ $\quad l = (a,b)$ vector field $l = l(x,t)$ in the (x,t)-plane

$l(u) = 0$

(can assume l is normalized $\|l\| = 1$)

$\frac{dt}{a} = \frac{dx}{b}$ integral curves of the vector field l

in general $a = a(x,t)$ $\quad b = b(x,t)$

if constant $\rightsquigarrow$ integral curves straight lines

$\frac{t}{a} - \frac{x}{b} = c$

so $\quad u(x,t) = \phi\left(\frac{t}{a} - \frac{x}{b}\right)$

is a <u>general solution</u> of the equation

initial value condition

$u\big|_{t=0} = f(x) \quad \Rightarrow \quad \phi\left(\frac{-x}{b}\right) = f(x)$

$\phi(x) = f(-bx)$

$\Rightarrow u(x,t) = f(x - ct)$ for $c = b/a$

$$\begin{cases} a u_t + b u_x = 0 \\ u(x,0) = f(x) \end{cases}$$ Solution of IVP

8 Nonconstant coeff's case:

e.g. $u_t + t u_x = 0$

integral curves $\quad dt = \frac{dx}{t} \quad t\,dt - dx = 0$

$\Rightarrow \quad x - \frac{1}{2}t^2 = C$

$u(x,t) = \phi\left(x - \frac{1}{2}t^2\right)$ general solution

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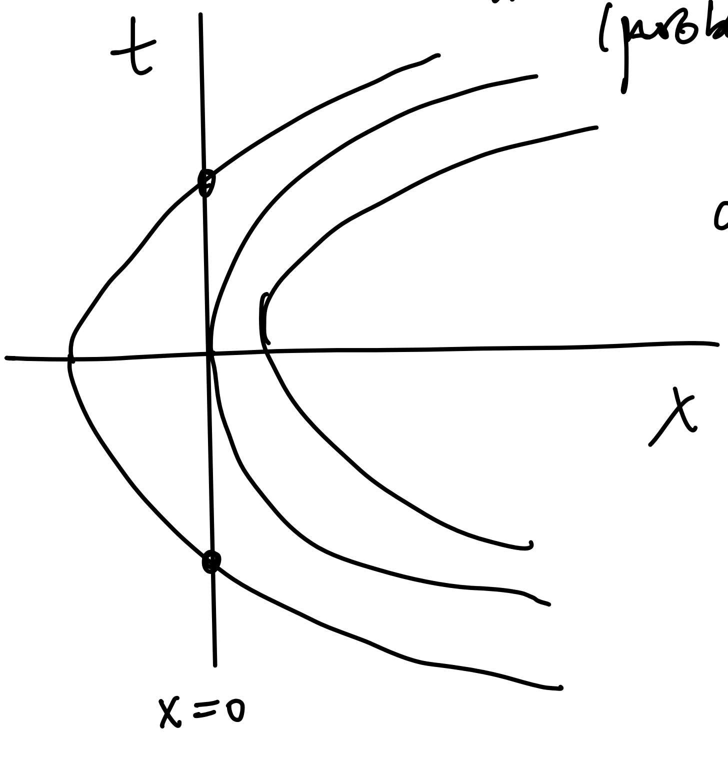
IVP: $u(0,t) = g(t)$ <u>not</u> a good problem:

- Some curves intersect $x=0$ line more than once
  if in different inters. pts different initial values incompatible (problem not solvable)
  <u>existence fails</u>



also even if g has same value at two pts (e.g. g even function)

$u(x,t)$ not uniquely determined curves that do not intersect $x=0$ line
So <u>uniqueness fails</u>

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more general problem

$a u_t + b u_x = f(x,t,u)$

again consider curves
$\frac{dt}{a} - \frac{dx}{b} = 0$

$$du = u_t\,dt + u_x\,dx = (a u_t + b u_x)\frac{dt}{a} = f\frac{dt}{a}$$

using

$$\frac{dx}{b} = \frac{dt}{a}$$

so get

$$\frac{dt}{a} = \frac{dx}{b} = \frac{du}{f}$$

Example: $u_t + u_x = x$ $\qquad dx = dt = \dfrac{du}{x}$

$$\begin{cases} x - t = C \\ u - \frac{1}{2}x^2 = D \end{cases}$$

constant along integral curves

$$u - \tfrac{1}{2}x^2 = \phi(x-t)$$

$$u = \tfrac{1}{2}x^2 + \phi(x-t) \qquad \text{general solution}$$

initial condition $u\big|_{t=0} = 0$ for instance

gives $\phi(x) = -\frac{1}{2}x^2$ hence

$$u(x,t) = \frac{1}{2}x^2 - \frac{1}{2}(x-t)^2 = xt - \frac{1}{2}t^2$$

Example w/ a nonlinearity

$$u_t + x u_x = -u^2 \qquad dt = \frac{dx}{x} = -\frac{du}{u^2}$$

$$\Rightarrow\ -\frac{du}{u^2} = dt \Rightarrow \begin{cases} u^{-1} = t + D \\ xe^{-t} = C \end{cases} \qquad u(x,t) = \left(t + \phi(xe^{-t})\right)^{-1}$$

and $x = Ce^{t}$

other nonlinearities (a, b depending on u)

$$\frac{dt}{a} = \frac{dx}{b} = \frac{du}{f}$$

example: Burgers equation (gas dynamics) $u_t + u u_x = 0$

$$\frac{dt}{1} = \frac{dx}{u} = \frac{du}{0}$$

$du = 0$ on int curves

∞

$\Rightarrow$ $\underline{u = \text{constant}}$ $\Rightarrow$ $\underline{x - ut = C}$ (w/ u = constant)

IBVP: $\begin{cases} u_t + c\, u_x = 0 & x > 0, t > 0 \\ u|_{t=0} = f(x) & x > 0 \end{cases}$

general solution $\phi(x - ct) = u$

$\rightarrow$ initial data: $\phi(x) = f(x)$ $x > 0$

So $u(x,t) = \underbrace{f(x-ct)}$

f defined for $x > 0$ $\Rightarrow$ u defined for $x - ct > 0$, $x > ct$

[if $c \leq 0$ fills quadrant (waves run to the left)

when $c > 0$ (waves to the right)

u not defined (as $x - ct < 0$)

need to define via a ∂-condition:

11

IBVP: $\begin{cases} u_t + c u_x = 0 & x>0, t>0 \\ u|_{t=0} = f(x) & x>0 \quad (IV) \\ u|_{x=0} = g(t) & t>0 \quad (BV) \end{cases}$

$\Rightarrow \quad \phi(-ct) = g(t) \quad t>0$

$\Rightarrow \quad \phi(x) = g(-\frac{1}{c}x) \quad x<0$

$\Rightarrow \quad u(x,t) = g(-\frac{1}{c}(x-ct)) \quad x<ct$

so get overall solution

$$u = \begin{cases} f(x-ct) & x>ct \\ g(t-\frac{1}{c}x) & x<ct \end{cases}$$

discontinuous at $x=ct$
unless $f(0)=g(0)$

not a classical solution (derivatives of u not def. at the discontinuity)
but weak solution (distributional sense)

A higher dimensional example

$$a u_t + \sum_{j=1}^{n} b_j u_{x_j} = f(x_1, \dots, x_n, t, u)$$

possible nonlinearity

arb. function

e.g. for $a=1$, $b_j = \text{const}$, $f=0 \Rightarrow u = \phi(x_1 - b_1 t, \dots, x_n - b_n t)$ sol'n

<u>Example:</u> $u_t + 2t u_x - x u_y = 0$

$$\underbrace{\frac{dt}{1} = \frac{dx}{2t}} = \frac{dy}{x} = \frac{du}{0}$$

$x - t^2 = C$ $\quad$ $dy = x\,dt = (C + t^2)\,dt$

$\Downarrow$

$$y - Ct - \frac{t^3}{3} = D$$

$$D = y - (x - t^2)t - \frac{t^3}{3} = y - xt + \frac{2t^3}{3}$$

So get general solution

$$u(x,y,t) = \phi\left(x - t^2,\ y - xt + \frac{2t^3}{3}\right)$$

<u>higher dim</u> examples w/ <u>nonlinearities</u>

$$F(x, u, \nabla u) = 0$$

$x = (x_1, \dots, x_n)$ $\quad$ $\nabla u = (u_{x_1}, \dots, u_{x_n})$

$\left(u_{x_i} = \frac{\partial}{\partial x_i} u\right)$ $\quad$ $\begin{cases} \text{initial cond. } u\big|_\Sigma = g \\ \Sigma \text{ hypersurface} \end{cases}$

char curves transversal to Σ

assume $\boxed{\sum_j F_{p_j}(x,u,p)\Big|_{p = \nabla u} \cdot \nu_j \neq 0}$

for $\nu = (\nu_1, \dots, \nu_n)$ normal vector to Σ at pt. x

13

$p = \nabla u$: characteristic curve in x-space (n-dim)

$$\frac{dx_j}{dt} = F_{p_j}$$

chain rule:

$$\frac{dp_j}{dt} = \sum_k p_{j,x_k} \frac{dx_k}{dt} = \sum_k u_{x_j x_k} F_{p_k}$$

and $\frac{du}{dt} = \sum_k u_{x_k} \frac{dx_k}{dt} = \sum_k p_k F_{p_k}$

also from differentiating the equation get:

$$0 = \partial_{x_j}(F(x, u, \nabla u)) = F_{x_j} + F_u u_{x_j} + \sum_k F_{p_k} p_{k,x_j} =$$

$$= F_{x_j} + F_u u_{x_j} + \sum_k F_{p_k} u_{x_k x_j}$$

So this term is $-F_{x_j} - F_u u_{x_j}$

$\Rightarrow$ obtain system

char. trajectory equations

$$\begin{cases} \frac{dx_j}{dt} = F_{p_j} \\ \frac{dp_j}{dt} = -F_{x_j} - F_u p_j \\ \frac{du}{dt} = \sum_j F_{p_j} p_j \end{cases}$$

So to solve $\begin{cases} F(x,u,\nabla u)=0 \\ u|_\Sigma = g \end{cases}$ w/ transversality assumption

- $\nabla_\Sigma u = \nabla_\Sigma g$ gradient of u along Σ

  (e.g. $\Sigma = \{x_1 = 0\} \Rightarrow \nabla_\Sigma u = (u_{x_2}, \dots, u_{x_n})$)

- normal components of ∇u at Σ from equation

$\rightsquigarrow \underbrace{\Sigma^* = \{(x,u,\nabla u) : x\in\Sigma\}}_{(n-1)\text{-dim}} \subset (2n+1)\text{-dim space}$

from each pt of Σ^* characteristic trajectory eq w/ this pt as initial condition

$\hookrightarrow \underbrace{\Lambda}_{n\text{-dim}} \subset \mathbb{R}^{2n+1}$ (evolution of $(n-1)$-dim Σ^*)

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special case: F indep of u (only dep on $x, p=\nabla u$)

$$\begin{cases} \dfrac{dx_j}{dt} = F_{p_j} \\[2mm] \dfrac{dp_j}{dt} = -F_{x_j} \\[2mm] \dfrac{du}{dt} = \sum_j p_j F_{p_j} \quad (*) \end{cases}$$

Hamiltonian system w/ Hamiltonian: $F(x,p)$

u can be dropped and solve Hamilt. dyn for (x,p) then u from $(*)$ alone

$\boxed{15}$

Homogeneous 1D wave equation

$$u_{tt} - c^2 u_{xx} = 0$$

$$\underbrace{(\partial_t^2 - c^2 \partial_x^2)}_{\mathcal{L}} u = 0 \qquad \mathcal{L} = \mathcal{L}_- \cdot \mathcal{L}_+$$

$$\mathcal{L}_- = \partial_t - c\,\partial_x \qquad \mathcal{L}_+ = \partial_t + c\,\partial_x$$

$$v = \mathcal{L}_+ u = u_t + c u_x \qquad w = \mathcal{L}_- u = u_t - c u_x$$

$$\begin{cases} v_t - c v_x = 0 \\ w_t + c w_x = 0 \end{cases} \qquad \begin{aligned} v &= 2c\,\tilde{\phi}(x+ct) \\ w &= -2c\,\tilde{\psi}(x-ct) \end{aligned}$$

$\tilde{\phi}, \tilde{\psi}$ arb. functs

$$\Rightarrow \begin{cases} u_t + c u_x = 2c\,\tilde{\phi}(x+ct) = 2c\,\phi'(x+ct) \\ u_t - c u_x = -2c\,\tilde{\psi}(x-ct) = -2c\,\psi'(x-ct) \end{cases}$$

$$\Rightarrow (\partial_t + c\,\partial_x)(\phi(x+ct))$$

So $(\partial_t + c\,\partial_x)\underbrace{(u - \phi(x+ct))}_{} = 0 \longrightarrow$ funct of $x-ct$

$\Rightarrow \quad u - \phi(x+ct) = \chi(x-ct)$

then using

get $\chi = \psi$ (up to const) but IC in ϕ, ψ so same

$\Rightarrow \quad \underline{u = \phi(x+ct) + \psi(x-ct)}$ general solution

superposition of two waves traveling in opposite direction left $\rightleftarrows$ right w/ same speed

<u>then IVP</u>: $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u\big|_{t=0} = g(x) \\ u_t\big|_{t=0} = h(x) \end{cases}$ Cauchy problem

$u = \phi(x+ct) + \psi(x-ct)$

$\Rightarrow \begin{cases} \phi(x) + \psi(x) = g(x) \\ c\phi'(x) - c\psi'(x) = h(x) \end{cases} \Rightarrow \phi(x) - \psi(x) = \frac{1}{c}\int h(y)\,dy$

$\Rightarrow \begin{cases} \phi(x) = \frac{1}{2} g(x) + \frac{1}{2c}\int^x h(y)\,dy \\ \psi(x) = \frac{1}{2} g(x) - \frac{1}{2c}\int^x h(y)\,dy \end{cases}$

17

then get:

$$u(x,t) = \frac{1}{2} g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy$$

$$+ \frac{1}{2} g(x-ct) - \frac{1}{2c}\int^{x-ct} h(y)\,dy$$

$\Rightarrow$ D'Alembert:

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy \quad (*)$$

1D wave equation: characteristic coord's

$x+ct = \text{const}$ $\qquad$ $x-ct = \text{const}$

characteristic lines (or "characteristics")

char coordinates: $\begin{cases} \xi = x+ct \\ \eta = x-ct \end{cases}$

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta}$$

since

$x = \frac{1}{2}(\xi+\eta) \qquad t = \frac{1}{2c}(\xi-\eta)$

$\boxed{\text{so eq. } u_{\xi\eta} = 0}$

$$u_{\xi\eta} = 0 \Rightarrow u_\xi = \phi'(\xi)$$

i.e. u_ξ does not depend on η

$$\Rightarrow (u - \phi(\xi))_\xi = 0$$

primitive of ϕ'

$$\Rightarrow u - \phi(\xi) = \psi(\eta) \qquad \Rightarrow u = \phi(\xi) + \psi(\eta)$$

same 2 traveling waves

D'Alambert formula

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t) \\ u|_{t=0} = g(x) \\ u_t|_{t=0} = h(x) \end{cases}$$

first for $g = h = 0$

$$u_{\xi\eta} = -\frac{1}{4c^2}\tilde{f}(\xi,\eta)$$

after change of coords

$$u_\xi = -\frac{1}{4c^2}\int^{\eta}\tilde{f}(\xi,\eta')\,d\eta' = -\frac{1}{4c^2}\int_{\xi}^{\eta}\tilde{f}(\xi,\eta')\,d\eta' + \phi'(\xi)$$

at $t=0$ $u_t, u_x = 0$ so $\phi'(\xi) = 0$

$$u_\xi = \frac{1}{4c^2}\int_{\eta}^{\xi}\tilde{f}(\xi,\eta')\,d\eta'$$

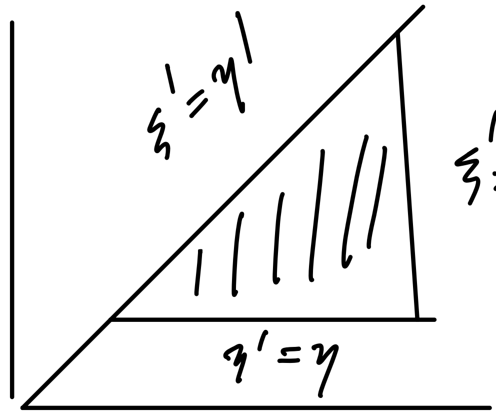
integrate in ξ

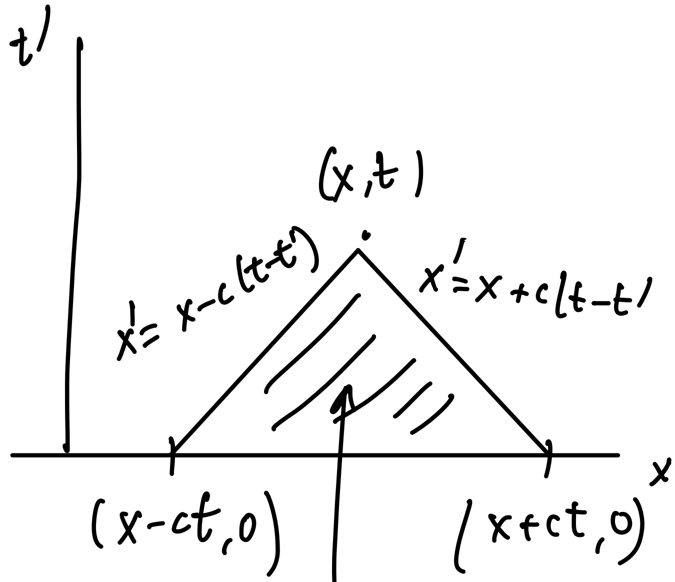
$$u = \frac{1}{4c^2}\int_{\eta}^{\xi}\int_{\eta}^{\xi'}\tilde{f}(\xi',\eta')\,d\eta'\,d\xi' + \cancel{\psi(\eta)}$$

for $u=0$ at $t=0$ $(\xi = \eta)$

[19]

$$\Rightarrow u(\xi,\eta) = \frac{1}{4c^2}\int_\eta^\xi \int_\eta^{\xi'} \tilde{f}(\xi',\eta')\,d\eta'\,d\xi'$$





$$\boxed{u(x,t) = \frac{1}{2c}\int_{\Delta(x,t)} f(x',t')\,dx'\,dt'}$$

factor changed by Jacobian of change of coord's

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when f, g, h nonzero

Linear eq $\rightsquigarrow$ superposition $\rightsquigarrow$ $u = u_1 + u_2$

u_1 sol'n of probl. w/ $f \equiv 0$ g, h nonzero

u_2 sol'n of probl. with $f \neq 0$ $g, h \equiv 0$

u_1 from (*) and u_2 from

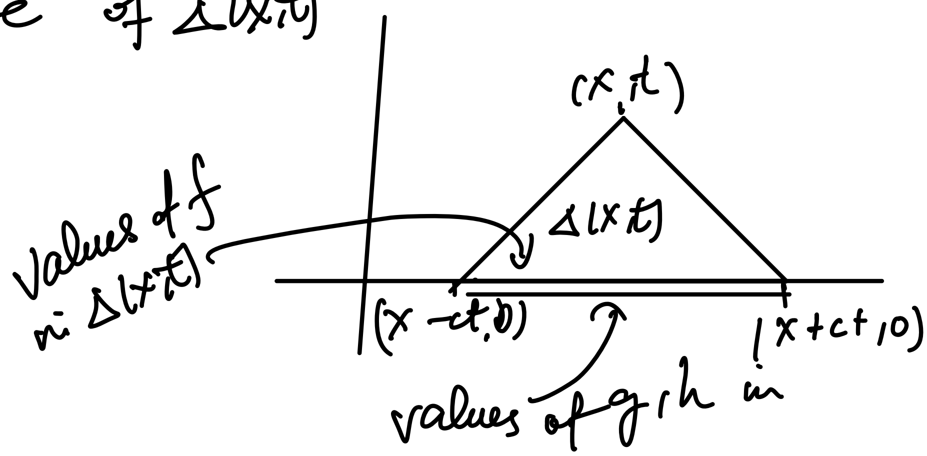
[D'Alambert formula]

[20]

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(x')\,dx'$$
$$+ \frac{1}{2c}\iint_{\Delta(x,t)} f(x',t')\,dx'dt'$$

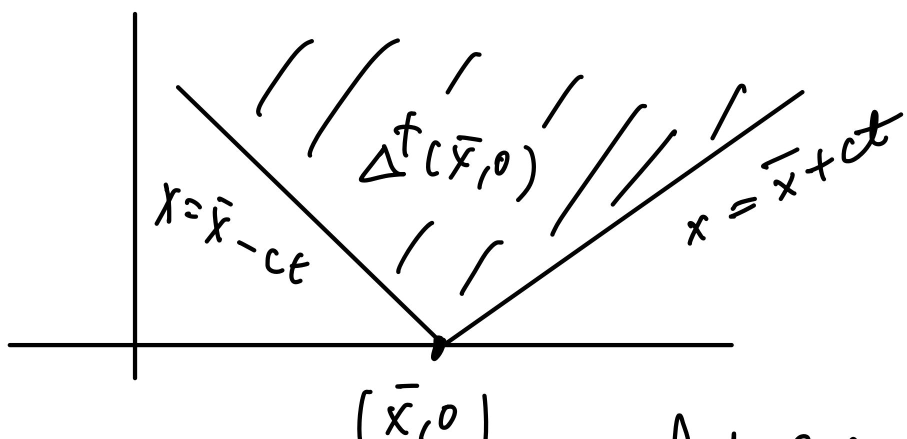
value $u(x,t)$ at pt (x,t)
depends only on f through values in region $\Delta(x,t)$
and on initial data g, h values on
bottom base of $\Delta(x,t)$

region of
dependence



- if change datum
  g, h at some pt $(\bar{x}, 0)$

  $\rightsquigarrow$ $u(x,t)$ can change only in region $\Delta^{+}(\bar{x},0)$



future cone of $(\bar{x}, 0)$
region of influence

$\boxed{21}$ for 3D wave eq'n

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = f$$

backward / forward light cones in Minkowski metric

(if c not constant but variable ⇝ "conoids" future/past)

solutions propagate at speed not exceeding c (as stays inside the cone of influence)

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higher dim wave eq.

$$u_{tt} - c^2 \Delta u = 0$$

$$\left(\Delta u = \sum_i u_{x_i x_i} = \nabla \cdot \nabla u \right)$$

mult. by u_t & get

$$u_t u_{tt} - c^2 u_t \nabla^2 u =$$

$$\partial_t \left(\tfrac{1}{2} u_t^2\right) + \nabla \cdot \left(-c^2 u_t \nabla u\right)$$

$$+ c^2 \nabla u_t \cdot \nabla u$$

$$= \partial_t \left(\tfrac{1}{2} u_t^2 + \tfrac{1}{2} c^2 |\nabla u|^2 \right) + \nabla \cdot \left(-c^2 u_t \nabla u\right)$$

$\boxed{22}$

$$\partial_t\left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2|\nabla u|^2\right) + \nabla\cdot\left(-c^2 u_t \nabla u\right) = 0$$

Energy conservation law

$$e = \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2|\nabla u|^2\right)$$

energy density

$$S = -c^2 u_t \nabla u \quad \text{energy flow}$$

fix a region of fin. volume V (n-dim)

$$E_V(t) = \int_V \left(\tfrac{1}{2}u_t^2 + \tfrac{1}{2}c^2|\nabla u|^2\right) dV$$

$$E_V(t_2) - E_V(t_1) + \int_{t_1}^{t_2} dt \int_\Sigma S\cdot n \, d\sigma = 0$$

$\Sigma = \partial V$ $(n-1)$-boundary

n = normal vector to Σ (outward pointing)

[23] Heat equation in 1D :

⊛ $u_t = K u_{xx}$ diffusion equation

(IVP) $t > 0 \quad -\infty < x < \infty$

$u_{\alpha,\beta,\gamma}(x,t) := \gamma\, u(\alpha x, \beta t)$ Scaling properties (self-similar solutions)

if u satisfies ⊛
then $u_{\alpha,\beta,\gamma}$ also satisfies ⊛ whenever $\beta = \alpha^2$

because order 1 in t but order 2 in x

assume:
finite energy condition

$I(t) := \int_{-\infty}^{+\infty} u(x,t)\,dx$ total energy of the solution

- $I(t) < \infty$ and
- independent of t (energy conservation)

then note that
$$\int_{-\infty}^{+\infty} u_{\alpha,\beta,\gamma}(x,t)\,dx = \gamma |\alpha|^{-1} \int_{-\infty}^{+\infty} u\,dx$$

so same I if $\gamma = |\alpha|$ (in fact assume $\alpha > 0$ so just α)

$\Rightarrow$ so get scaling just by

$$u_\alpha(x,t) = \alpha u(\alpha x, \alpha^2 t)$$ $\leftarrow$ similarity transformation

$\Rightarrow$ self-similar solution if

$$u(x,t) = \alpha u(\alpha x, \alpha^2 t) \quad \forall \alpha > 0 \;\; \forall t > 0 \;\; \forall x \in \mathbb{R}$$

$\rightsquigarrow$ suggests relation to consider between the variables x, t

take $\alpha = t^{-1/2}$

$$u(x,t) = t^{-1/2} u(t^{-1/2} x, 1) = t^{-1/2} \phi(t^{-1/2} x)$$

$\phi(x) := u(x,1)$

substituting back into eq.

$$u_t = -\frac{1}{2} t^{-3/2} \phi(t^{-1/2} x) + t^{-1/2} \phi'(t^{-1/2} x) \cdot \left(-\frac{1}{2} t^{-3/2} x\right)$$
$$= -\frac{1}{2} t^{-3/2} \left(\phi(t^{-1/2} x) + t^{-1/2} x \phi'(t^{-1/2} x)\right)$$

$$u_x = t^{-1} \phi'(t^{-1/2} x)$$

$$u_{xx} = t^{-3/2} \phi''(t^{-1/2} x)$$

taking $u_t = k u_{xx}$ (mult by $t^{3/2}$ and take $\xi = t^{-1/2} x$)

25

$$-\frac{1}{2}(\phi(\xi)+\xi\phi'(\xi)) = k\phi''(\xi) \quad \text{ODE}$$

and

$\phi(\xi)+\xi\phi'(\xi) = (\xi\phi(\xi))'$ So integrating get

$$-\frac{1}{2}\xi\phi(\xi) = k\phi'(\xi)$$

(take integration constant $C=0$ to have decay at ∞)

separate variables

$$\frac{d\phi}{\phi} = -\frac{1}{2k}\xi\, d\xi \Rightarrow \log\phi = -\frac{1}{4k}\xi^2 + \log(c)$$

$$\phi(\xi) = c e^{-\frac{1}{4k}\xi^2}$$

$$\Rightarrow \quad u(x,t) = \frac{1}{2\sqrt{\pi k t}} e^{-\frac{x^2}{4kt}}$$

normalization factor

$c = \frac{1}{2\sqrt{\pi k t}}$ so that $I(t)=1$

normalization of total energy

<u>What is the initial condition?</u>

$u(x,t) \to 0$ as $t\to 0^+$ with $x\neq 0$

$u(x,t) \to 1$ as $t\to 0^+$ with $x=0$

but with $\int_{-\infty}^{+\infty} u(x,t) = 1$ (!)

(26) in fact correct way to interpret this initial condition is as δ-function i.e. as distribution rather than function

$\rightsquigarrow$ solutions by distributions

take $U(x,t) = \int_{-\infty}^{x} u(x,t)\,dt$

$$\begin{cases} U(x,t) \text{ also satisfies } (*) \text{ and} \\ U(x,0^+) = \lim_{t\to 0^+} U(x,t) = \Theta(x) = \begin{cases} 0 & x<0 \\ 1 & x>0 \end{cases} \end{cases}$$

Heaviside function

in fact:

$u = U_x$ in eq. gives $(U_t - kU_{xx})_x = 0$

$\Rightarrow \quad U_t - kU_{xx} = \phi(t)$

at $x \to -\infty$ $\quad U$ is decaying to zero w/ all derivatives

$\Rightarrow \phi(t) \equiv 0$ so U sol'n of $(*)$

$$U(x,t) = \frac{1}{2} + \frac{1}{2}\,\mathrm{erf}\left(\frac{x}{\sqrt{4kt}}\right) = \frac{1}{\sqrt{4\pi}} \int_{-\infty}^{x/\sqrt{4kt}} e^{-z^2}\,dz$$

$\mathrm{erf}(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-x^2}dx$ and lim gives Heaviside

(27)

given $g(x)$ smooth with $g(-\infty)=0$

$$g(x)=\int_{-\infty}^{+\infty}\theta(x-y)\,g'(y)\,dy$$

as $=\int_{-\infty}^{x} g'(y)\,dy = g(x)-g(-\infty)$

note $U(x-y,0^+)=\theta(x-y)$ initial cond of $U(x-y,t)$ sol'n

So $u(x,t)=\int_{-\infty}^{+\infty}U(x-y,t)\,g'(y)\,dy$ solution w/ initial cond.

$$u(x,0^+)=g(x)$$

integrate by parts in y:

$$u(x,t)=\int_{-\infty}^{+\infty}U_x(x-y,t)\,g(y)\,dy$$

$$\boxed{\begin{aligned} u(x,t) &= \int_{-\infty}^{+\infty} G(x-y,t)\,g(y)\,dy \\ G(x-y,t) &= \frac{1}{2\sqrt{k\pi t}}\,e^{-\frac{(x-y)^2}{4kt}} \end{aligned}}$$

formula for solution of

$$\begin{cases} u_t = k u_{xx} & -\infty < x < +\infty \quad t > 0 \\ u\big|_{t=0} = g(x) \end{cases}$$

Inhomogeneous problem

$$(**) \begin{cases} u_t = k u_{xx} + f(x,t) & -\infty < x < +\infty \quad t > 0 \\ u\big|_{t=0} = g(x) \end{cases}$$

same argument as above shows, f contributes to sol'n through

$$\int_0^t \int_{-\infty}^{+\infty} G(x-y, t-\tau)\, f(y,\tau)\, dy\, d\tau$$

$\Rightarrow$ Solution of $(**)$:

$$u(x,t) = \int_0^t \int_{-\infty}^{+\infty} G(x-y, t-\tau)\, f(y,\tau)\, dy\, d\tau + \int_{-\infty}^{+\infty} G(x-y, t)\, g(y)\, dy$$

sol'n w/ f and $u\big|_{t=0} = 0$

sol'n without f and w/ $u_t\big|_{t=0} = g$

note that this ↑ does <u>not</u> mean solution is <u>unique</u> in fact it is not

∃ other solutions <u>not</u> decaying but rapidly oscillating at ∞

Other kinds of conditions

example problem on the half-line

$x > 0$
$t > 0$

$u_D \big|_{x=0} = 0$

Dirichlet ∂-cond

or $u_{N\,x} \big|_{x=0} = 0$

$\frac{\partial}{\partial x} u$ Neumann ∂-cond

$$u(x,t) = \int_{-\infty}^{+\infty} G(x,y,t)\, g(y)\, dt$$

$$G(x,y,t) = G_0(x-y,t) = \frac{1}{2\sqrt{k\pi t}} e^{-\frac{(x-y)^2}{4kt}}$$

continuation:
coeff's indep of x
equation only even order ∂ in x

continuation
even N
odd D

so obtain again sol'n of this form but w/ different G

$$G_D(x,y,t) = G_0(x-y,t) - G_0(x+y,t)$$

$$G_N(x,y,t) = G_0(x-y,t) + G_0(x+y,t)$$

$$G_D\big|_{x=0} = 0 \qquad \int_0^\infty G_D(x,y,t)\,dy \xrightarrow[t\to 0^+]{} 1$$

$$G_{N\,x}\big|_{x=0} = 0 \qquad \int_0^\infty G_N(x,y,t)\,dx = 1$$

$$\underset{N/D}{G}(x,y,t) = \underset{N/D}{G}(y,x,t)$$

---

instead of D/N vanishing ∂-condition
can take inhomogeneous ∂-cond

$$u_D\big|_{x=0} = p(t) \qquad \text{Dirichlet } \underline{\text{or}}$$

$$u_{N\,x}\big|_{x=0} = q(t) \qquad \text{Neumann}$$

take $\Pi = \Pi_\varepsilon = \{x>0,\ 0<\tau<t-\varepsilon\}$

have:

$$0 = \int_\Pi G(x,y,t-\tau)\left(-u_\tau(y,\tau) + k\,u_{yy}(y,\tau)\right)d\tau\,dy$$

then integrate by parts in τ in first term
& integrate twice in y in second term

$$0 = \int_\Pi \Big(-G_t(x,y,t-\tau) + k\,G_{yy}(x,y,t-\tau)\Big)\,u(y,\tau)\,d\tau\,dy$$

$$-\int_0^\infty G(x,y,\varepsilon)\,u(y,t-\varepsilon)\,dy + \int_0^\infty G(x,y,t)\,u(y,0)\,dy$$

$$+k\int_0^{t-\varepsilon}\Big(-G(x,y,t-\tau)\,u_y(y,\tau) + G_y(x,y,t-\tau)\,u(y,\tau)\Big)\Big|_{y=0}\,d\tau$$

the funct $G(x,y,t)$ satisfies $u_t = k u_{xx}$

in both (x,t) and (y,t) variables

(since $G(x,y,t) = G(y,x,t)$)

so $=0$

then when $\varepsilon \to 0^+$ second term $\longrightarrow -u(x,t)$

as $G(x,y,\varepsilon) \to \delta(x-y)$

$$\Rightarrow \quad u(x,t) = \int_0^\infty G(x,y,t)\,\underbrace{u(y,0)}_{=g(y)\text{ datum}}\,dy$$

$$+k\int_0^t \Big(-G(x,y,t-\tau)\,u_y(y,\tau) + G_y(x,y,t-\tau)\,u(y,\tau)\Big)\Big|_{y=0}\,d\tau$$

solution of IBVP with 0 boundary condition

for Dirichlet ∂-cond

$G(x,y,t)\big|_{y=0} \equiv 0$ so get

$$k\int_0^t G_y(x,0,t-\tau)\,\underset{\substack{\shortparallel \\ p(\tau)\text{ datum}}}{u(0,\tau)}\,d\tau$$

for Neumann ∂-cond

$$G_y(x,y,t)\big|_{y=0} \equiv 0 \quad \text{so get}$$

$$-k \int_0^t G(\tau, 0, t-\tau) \underbrace{u(0,\tau)}_{=q(\tau)} d\tau$$

$$u_D(x,t) = \int_0^\infty G_D(x,y,t)\, g(y)\, dy + k \int_0^t G_{D_y}(x,0,t-\tau)\, p(\tau)\, d\tau$$

$$u_N(x,t) = \int_0^\infty G_N(x,y,t)\, g(y)\, dy - k \int_0^t G_N(x,0,t-\tau)\, q(\tau)\, d\tau$$

$$u_t - k u_{xx} = \underline{f(x,t)}$$ inhomog. case

$$u(x,t) = \int_0^t \int_0^\infty G(x,y,t-\tau)\, f(y,\tau)\, dy\, d\tau$$

$$+ \int_0^\infty G(x,y,t)\, g(y)\, dy$$

where $G(x,y,t)$ is G_D or G_N

$$G_D(x,y,t) = G_0(x-y,t) - G_0(x+y,t)$$

$$G_N(x,y,t) = G_0(x-y,t) + G_0(x+y,t)$$

$$G_0(\alpha, t) = \frac{1}{2\sqrt{k\pi t}}\, e^{-\alpha^2/4kt}$$ as before

Multidimensional heat equation

$$u_t = k\,\Delta u \qquad \text{in } \mathbb{R}^n$$

$$G_n(x, x', t) = (4\pi k t)^{-n/2}\, e^{-\|x-x'\|^2/4kt}$$

satisfies the n-dim heat equations

with $\lim_{t \to 0^+} G_n(x, x', t) = \delta(x - x')$ as distrib.

rotational symmetry in radial dir. (spheres around x')

(as heat eq. for diffusion in isotropic medium k = constant)