Operads, quasiorders and regular languages

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Abstract

We generalize the construction of multitildes in the aim to provide multitilde operators for regular languages. We show that the underliying algebraic structure involves the action of some operads. An operad is an algebraic structure that mimics the composition of the functions. The involved operads are described in terms of combinatorial objects. These operads are obtained from more primitive objects, namely precompositions, whose algebraic counter-parts are investigated. One of these operads acts faithfully on languages in the sense that two different operators act in two different ways.

1 Introduction

Following the Chomsky-Schützenberger hierarchy [5], regular languages are defined to be the formal languages that are generated by Type-3 grammars (also called regular grammars). These particular languages have been studied from several years since they have many applications to pattern maching, compilation, verification, bioinformatics, *etc.* Their generalization as rational series links them to various algebraic or combinatorial topics: enumeration (manipulations of generating functions), rational approximation (for instance Pade approximation), representation theory (module viewed as automaton), combinatorial optimization ((max, +)-automata), *etc.*
One of their main interest is that they can be represented by various tools: regular grammars, automata, regular expressions, *etc.* Whilst regular languages can be represented by both automata and regular expressions [8], these tools are not equivalent. Indeed, Ehrenfeucht and Zeiger [6] showed a one parameter family of automata whose shortest equivalent regular expressions have a width exponentially growing with the numbers of states. Note that, it is possible to compute an automaton from a regular expression E such that the number of its states is a linear function of the alphabet width (*i.e.* the number of occurrences of alphabet symbols) of E [1, 4, 7, 13].
In the aim to increase expressiveness of expressions for a bounded length, Caron *et al.* [3] introduced the so-called multi-tilde operators and applied it to represent finite languages. Investigating the equivalence of two multi-tilde expressions, they define a natural notion of composition which endows the set of multi-tilde operators with a structure of operad. This structure has been investigated in [10].

Originating from the algebraic topology [2, 12], operad theory has been developed as a field of abstract algebra concerned by prototypical algebras that model classical properties such as commutativity and associativity [9]. Generally defined in terms of categories, this notion can be naturally applied to computer science. Indeed, an operad is just a set of operations, each one having exactly one output and a fixed finite number of inputs, endowed with the composition operation. So an operad can model the compositions of functions occurring during the execution of a program. In

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terms of theoretical computer science, this can be represented by trees with branching rules. The whole point of the operads in the context of the computer science is that this allows to use different tools and concepts from algebra (for instance: morphisms, quotients, modules *etc.*).

In the aim to illustrate this point of view, let us recall the main results of our previous paper [10]. In this paper, we first showed that the set of multi-tilde operators has a structure of operad. We used the concept of morphism in the aim to choose the operad allowing us to describe in the simplest way a given operation or a property. For instance, the original definition of the action of the multi-tildes on languages is rather complicated. But, *via* an intermediate operad based on set of boolean vectors, the action was described in a more natural way. In the same way, the equivalence problem is clearer when asked in a operad based on antisymmetric and reflexive relations which is isomorphic to the operad of multi-tildes: two operators are equivalent if and only if they have the same transitive closure. The transitive closure being compatible with the composition, we defined an operad based on partial ordered sets as a quotient of the previous operad and we showed that this representation is optimal in the sense that two different operators act in two different ways on languages. This not only helps to clarify constructions but also to ask new questions. For instance, how many different ways do k-ary multi-tildes act on languages? Precisely, the answer is the number of posets on $\{1, \dots, k+1\}$ that are compatible with the natural order on integers.

The aim of this paper is to generalize the construction to regular languages. We investigate several operads (based on double multi-tildes, antireflexive relations or quasiorders) allowing to represent a regular language as a k-ary operator O acting on a k-uplet of symbols $(\alpha_1, \dots, \alpha_k)$ where the α_i are symbols or $\emptyset$. The operators generalize the multi-tildes and the investigated properties involve the operads.

The paper is organized as follows. First we recall in Section 2 several notions concerning operad theory and multi-tilde operations. In Section 3, we remark that many of the operads involved in [10] and in this paper have some common properties. More precisely, they can be described completely by means of "shifting" operations. This leads to the definition of the category of precompositions together with a functor to the category of operads. Also we define and investigate the notion of quotient of precompositions. These structures serve as model for the operads defined in the sequel. In the aim to illustrate how to use these tools, we revisit, in Section 4, the operads defined in [10] and describe them in terms of precompositions. In Section 5, we define the double multi-tilde operad $\mathcal{DT}$ as the graded tensor square of the multi-tilde operad. We construct also an isomorphic operad ARef based on antireflexive relations and a quotient based on quasiorders QOSet. In Section 6, we describe the action of the operads on the languages. In particular, we show that any regular language can be written as $O_k(\alpha_1, \dots, \alpha_k)$ where the α_i are letters or $\emptyset$ and O_k is a k-ary operation belonging to ARef, $\mathcal{DT}$ or QOSet. Finally, we prove that the action of QOSet on regular languages is faithful, that is two different operators act in two different ways.

2 Some Combinatorial Operators in Language Theory

We recall here some basic notions about the theory of operads and set our notations for the sequel of the paper. In particular, we recall what are operads, free operads, and modules over an operad. We conclude this section by presenting the operad of multi-tildes introduced in [10].

2.1 What is an operad?

Operads are algebraic graded structures which mimic the composition of n-ary operators. Let us recall the main definitions and properties. Let $\mathfrak{P} = \bigsqcup_{n\in\mathbb{N}\setminus\{0\}} \mathfrak{P}_n$ be a graded set ($\bigsqcup$ means that the sets are disjoint); the elements of $\mathfrak{P}_n$ are called n-ary operators. The set $\mathfrak{P}$ is endowed with functions (called compositions)

$$\circ : \mathfrak{P}_n \times \mathfrak{P}_{k_1} \times \cdots \times \mathfrak{P}_{k_n} \to \mathfrak{P}_{k_1+\cdots+k_n}.$$

The pair $(\mathfrak{P}, \circ)$ is an operad if the compositions satisfy:

1. *Associativity:*

$$\mathbf{p}\circ(\mathbf{p}_1\circ(\mathbf{p}_{1,1}, \dots, \mathbf{p}_{1,k_1}), \dots, \mathbf{p}_n\circ(\mathbf{p}_{n,1}, \dots, \mathbf{p}_{n,k_n})) = (\mathbf{p}\circ(\mathbf{p}_1, \dots, \mathbf{p}_n))\circ(\mathbf{p}_{1,1}, \dots, \mathbf{p}_{1,k_1}, \dots, \mathbf{p}_{n,1}, \dots, \mathbf{p}_{n,k_n}).$$

2. *Identity:*
There exists a special element $\mathbf{1} \in \mathfrak{P}_1$ such that

$$\mathbf{p} \circ (\mathbf{1}, \ldots, \mathbf{1}) = \mathbf{1} \circ \mathbf{p} = \mathbf{p}.$$

For convenience, many authors use an alternative definition of operads involving partial compositions. A partial composition $\circ_i$ is a map (see *e.g.* [9])

$$\circ_i : \mathfrak{P}_m \times \mathfrak{P}_n \to \mathfrak{P}_{m+n-1},$$

defined by

$$\mathbf{p}_1 \circ_i \mathbf{p}_2 := \mathbf{p}_1 \circ (\overbrace{\mathbf{1}, \ldots, \mathbf{1}}^{\times i-1}, \mathbf{p}_2, \overbrace{\mathbf{1}, \ldots, \mathbf{1}}^{\times m-i})$$

for $1 \leq i \leq n$.
Let $\mathbf{p}_1 \in \mathfrak{P}_m$, $\mathbf{p}_2 \in \mathfrak{P}_n$and $\mathbf{p}_3 \in \mathfrak{P}_q$. Whence stated in terms of partial compositions, the associativity condition splits into two rules:

1. *Associativity 1*:
If $1 \leq j < i \leq n$ then
$$(\mathbf{p}_1 \circ_i \mathbf{p}_2) \circ_j \mathbf{p}_3 = (\mathbf{p}_1 \circ_j \mathbf{p}_3) \circ_{i+q-1} \mathbf{p}_2.$$

2. *Associativity 2*:
If $j \leq n$ then
$$\mathbf{p}_1 \circ_i (\mathbf{p}_2 \circ_j \mathbf{p}_3) = (\mathbf{p}_1 \circ_i \mathbf{p}_2) \circ_{i+j-1} \mathbf{p}_3.$$

Note that the compositions are recovered from the partial compositions by the formula:

$$\mathbf{p} \circ (\mathbf{p}_1, \ldots, \mathbf{p}_n) = (\ldots (\mathbf{p} \circ_n \mathbf{p}_n) \circ_{n-1} \mathbf{p}_{n-1}) \circ_2 \ldots \mathbf{p}_2) \circ_1 \mathbf{p}_1.$$

The readers could refer to [9,11] for a more complete description of the structures.

Consider two operads $(\mathfrak{P}, \circ)$ and $(\mathfrak{P}', \circ')$. A *morphism* is a graded map $\phi : \mathfrak{P} \to \mathfrak{P}'$ satisfying $\phi(\mathbf{p}_1 \circ_i \mathbf{p}_2) = \phi(\mathbf{p}_1) \circ'_i \phi(\mathbf{p}_2)$ for each $\mathbf{p}_1 \in \mathfrak{P}_m$, $\mathbf{p}_2 \in \mathfrak{P}_n$ and $1 \leq i \leq m$. Let $(\mathfrak{P}, \circ)$ be an operad, $\mathfrak{P}' = \bigcup_n \mathfrak{P}'_n$ be a graded set. Suppose that $\mathfrak{P}'$ is endowed with binary operators $\circ'_i : \mathfrak{P}'_m \times \mathfrak{P}'_n \to \mathfrak{P}'_{n+m-1}$ and there exists a surjective graded map $\eta : \mathfrak{P} \to \mathfrak{P}'$ satisfying $\eta(\mathbf{p}_1 \circ_i \mathbf{p}_2) = \eta(\mathbf{p}_1) \circ'_i \eta(\mathbf{p}_2)$. The set $\mathfrak{P}'$ is automatically endowed with a structure of operad $(\mathfrak{P}', \circ')$. Indeed, it suffices to show that the associativity rules are satisfied: Let $\mathbf{p}'_1 \in \mathfrak{P}'_m$, $\mathbf{p}'_2 \in \mathfrak{P}'_n$and $\mathbf{p}'_3 \in \mathfrak{P}'_q$. Since the η is surjective, there exist $\mathbf{p}_1 \in \mathfrak{P}_m$, $\mathbf{p}_2 \in \mathfrak{P}_n$, $\mathbf{p}_3 \in \mathfrak{P}_q$ such that $\eta(\mathbf{p}_i) = \mathbf{p}'_i$ for $i = 1 \ldots 3$. Hence,

$$\begin{array}{rcl} \mathbf{p}'_1 \circ'_i (\mathbf{p}'_2 \circ'_j \mathbf{p}'_3) & = & \eta(\mathbf{p}_1) \circ'_i (\eta(\mathbf{p}_2) \circ'_j \eta(\mathbf{p}_3)) \\ & = & \eta(\mathbf{p}_1 \circ_i (\mathbf{p}_2 \circ_j \mathbf{p}_3)) \\ & = & \eta((\mathbf{p}_1 \circ_j \mathbf{p}_3) \circ_{i+q-1} \mathbf{p}_2) \\ & = & (\eta(\mathbf{p}_1) \circ'_j \eta(\mathbf{p}_3)) \circ'_{i+q-1} \eta(\mathbf{p}_2) \\ & = & (\mathbf{p}'_1 \circ_j \mathbf{p}'_3) \circ_{i+q-1} \mathbf{p}'_2. \end{array}$$

This proves the first rule of associativity. The second rules can be proved in the same way. Furthermore the image $\eta(\mathbf{1})$ is the identity in $\mathfrak{P}'$. So $(\mathfrak{P}', \circ')$ is an operad. Remark that if η is a bijection then

$$\mathbf{p}'_1 \circ'_i \mathbf{p}'_2 = \eta(\eta^{-1}(\mathbf{p}_1) \circ_i \eta^{-1}(\mathbf{p}_2)). \quad (1)$$

If $\mathfrak{Q} \subset \mathfrak{P}$, the *suboperad* of $\mathfrak{P}$ generated by $\mathfrak{Q}$ is the smallest subset of $\mathfrak{P}$ containing $\mathfrak{Q}$ and $\mathbf{1}$ which is stable by composition. Let $\mathfrak{G} = (\mathfrak{G}_k)_k$ be a collection of sets. The set $\mathrm{Free}(\mathfrak{G})_n$ is the set of planar rooted trees with n leaves with labeled nodes where nodes with k children are labeled by the elements of $\mathfrak{G}_k$. The *free operad* on $\mathfrak{G}$ is obtained by endowing the set $\mathrm{Free}(\mathfrak{G}) = \bigcup_n \mathrm{Free}(\mathfrak{G})_n$ with the composition $\mathbf{p}_1 \circ_i \mathbf{p}_2$ which consists in grafting the ith leaf of $\mathbf{p}_1$ with the root of $\mathbf{p}_2$. Note that $\mathrm{Free}(\mathfrak{G})$ contains a copy of $\mathfrak{G}$ which is the set of the trees with only one inner node (the root) labeled by elements of $\mathfrak{G}$; for simplicity we will identify it with $\mathfrak{G}$. Clearly, $\mathrm{Free}(\mathfrak{G})$ is generated by $\mathfrak{G}$. The universality means that for any map $\varphi : \mathfrak{G} \to \mathfrak{P}$ it exists a unique operadic morphism $\phi : \mathrm{Free}(\mathfrak{G}) \to \mathfrak{P}$ such that $\phi(\mathbf{g}) = \varphi(\mathbf{g})$ for each $\mathbf{g} \in \mathfrak{G}$.

Let $\equiv$ be a graded equivalence relation on $\mathfrak{P}$. The relation $\equiv$ is a congruence, if for any $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}'_1, \mathbf{p}'_2 \in \mathfrak{P}$ we have $\mathbf{p}_1 \equiv \mathbf{p}'_1$ and $\mathbf{p}_2 \equiv \mathbf{p}'_2$ implies $\mathbf{p}_1 \circ_i \mathbf{p}_2 \equiv \mathbf{p}'_1 \circ_i \mathbf{p}'_2$. Hence, this naturally endows the quotient $\mathfrak{P}/_\equiv$ with a structure of operad. Note that if $\phi : \mathfrak{P} \to \mathfrak{P}'$ is a surjective morphism of operads then the equivalence defined by $\mathbf{p}_1 \equiv \mathbf{p}_2$ if and only if $\phi(\mathbf{p}_1) = \phi(\mathbf{p}_2)$ is a congruence.

Let $(\mathfrak{P}, \circ)$ and $(\mathfrak{P}', \circ')$ be two operads. The graded set $\mathbb{T}(\mathfrak{P}, \mathfrak{P}') := \bigcup_{n\in\mathbb{N}} \mathbb{T}_n(\mathfrak{P}, \mathfrak{P}')$, with $\mathbb{T}_n(\mathfrak{P}, \mathfrak{P}') := \mathfrak{P}_n \times \mathfrak{P}'_n$, is naturally endowed with a structure of operad where the composition is defined by $(\mathbf{p}_1, \mathbf{p}'_1)\circ_i(\mathbf{p}_2, \mathbf{p}'_2) := (\mathbf{p}_1 \circ_i \mathbf{p}_2, \mathbf{p}'_1 \circ'_i \mathbf{p}'_2)$ with $\mathbf{p}_1 \in \mathfrak{P}_{k_1}, \mathbf{p}_2 \in \mathfrak{P}_{k_2}, \mathbf{p}'_1 \in \mathfrak{P}'_{k_1}, \mathbf{p}'_2 \in \mathfrak{P}_{k_2}$ and $1 \le i \le k_1$.Consider a set $\mathbf{S}$ together with an action of an operad $\mathfrak{P}$. That is, for each $\mathbf{p} \in \mathfrak{P}_n$ we define a map $\mathbf{p} : \mathbf{S}^n \to \mathbf{S}$. We say that $\mathbf{S}$ is a $\mathfrak{P}$-*module* if the action of $\mathfrak{P}$ is compatible with the composition in the following sense: for each $\mathbf{p}_1 \in \mathfrak{P}_m, \mathbf{p}_2 \in \mathfrak{P}_n, 1 \le i \le m, s_1, \dots, s_{m+n-1} \in \mathbf{S}$ one has:

$$\mathbf{p}_1(s_1, \dots, s_{i-1}, \mathbf{p}_2(s_i, \dots, s_{i+n-1}), s_{i+n}, \dots, s_{m+n-1}) = (\mathbf{p}_1 \circ_i \mathbf{p}_2)(s_1, \dots, s_{m+n-1}).$$

Furthermore, if for each $k > 0$ and $p \neq p' \in \mathfrak{P}_k$ there exists $a_1, \dots, a_k \in \mathbf{S}$ such that $p(a_1, \dots, a_k) \neq p'(a_1, \dots, a_k)$ then we say that the module $\mathbf{S}$ is *faithful*.

2.2 Multi-tildes and related operads

In [10], we have defined several operads. Let us recall briefly the main constructions. First we defined the operad $\mathcal{T} = \bigsqcup_n \mathcal{T}_n$ of multi-tildes. A multi-tilde of $\mathcal{T}_n$ is a subset of $\{(x, y) : 1 \le x \le y \le n\}$. Note that $\bigsqcup_n$ means that the same set belonging in two different graded components $\mathcal{T}_n$ and $\mathcal{T}_m$ are considered as different operators. For any pair (x, y) we define

1. $\overset{k}{\ggg}(x, y) = (x + k, y + k)$
2. $\overset{n,k}{\circ\!\!\to}(x, y) = \begin{cases} (x, y) & \text{if } y < k, \\ (x, y + n - 1) & \text{if } x \le k \le y, \\ \overset{n-1}{\ggg}(x, y) & \text{otherwise.} \end{cases}$

The actions of the two operators are extended to the set of pairs by

1. $\overset{k}{\ggg}(E) = \{\overset{k}{\ggg}(x, y) : (x, y) \in E\}$,
2. $\overset{n,k}{\circ\!\!\to}(E) = \{\overset{n,k}{\circ\!\!\to}(x, y) : (x, y) \in E\}$.

We shown the following result:

Theorem 1 ([10]). *The set $\mathcal{T}$ endowed with the partial compositions*

$$\circ_i : \begin{cases} \mathcal{T}_m \times \mathcal{T}_n & \to & \mathcal{T}_{n+m-1} \\ T_1 \circ_i T_2 & = & \overset{n,i}{\circ\!\!\to}(T_1) \cup \overset{i-1}{\ggg}(T_2), \end{cases}$$

is an operad.

We also define the operators

$$\overset{n,i}{\diamond\!\!\to}(x) = \begin{cases} x & \text{if } x \le i, \\ x + n - 1 & \text{otherwise} \end{cases}$$

$$\overset{n,i}{\diamond\!\!\to}(x, y) = (\overset{n,i}{\diamond\!\!\to}(x), \overset{n,i}{\diamond\!\!\to}(y)) \text{ and } \overset{n,i}{\diamond\!\!\to}(E) = \{\overset{n,i}{\diamond\!\!\to}(x, y) : (x, y) \in E\}.$$

The operad $(\mathcal{T}, \circ)$ is isomorphic to another operad $(\mathrm{RAS}, \diamond)$ whose underlying set is the set $\mathrm{RAS} = \bigsqcup_n \mathrm{RAS}_n$ where RAS_n denotes the set of Reflexive and Antisymmetric Subrelations of the natural order $\le$ on $\{1, \dots, n + 1\}$. The partial compositions of RAS are defined by

$$R_1 \diamond_i R_2 = \overset{n,i}{\diamond\!\!\to}(R_1) \cup \overset{i-1}{\ggg}(R_2),$$

if $R_1 \in \mathrm{RAS}_m$ and $R_2 \in \mathrm{RAS}_n$. The isomorphism between $\mathcal{T}$ and RAS sends $T \in \mathcal{T}_n$ to $\{(x, y + 1) : (x, y) \in T\} \cup \{(x, x) : x \in \{1, \dots, n + 1\}\}$. See [10] for more details.

3 Breaking operads

The objective of this section is to introduce new algebraic objects, namely the *precompositions*. We present here a functor from the category of precompositions to the category of operads. We shall use this functor in the sequel to reconstruct some already known operads and to construct new ones.

3.1 Precompositions

We consider the monoid $\square$ defined by generators $\{\overset{i,k}{\square\!\!\Rightarrow} : i \in \mathbb{Z}, k \in \mathbb{N} \setminus \{0\}\}$ and relations:

$$\overset{i,k}{\square\!\!\Rightarrow} = \overset{0,k}{\square\!\!\Rightarrow} \text{ for any } i < 0. \tag{2}$$

$$\overset{i,1}{\square\!\!\Rightarrow} = \overset{0,1}{\square\!\!\Rightarrow} = \mathbf{1}_{\square} \text{ for any } i. \tag{3}$$

$$\overset{i,k}{\square\!\!\Rightarrow}\overset{j,k'}{\square\!\!\Rightarrow} = \overset{j+k-1,k'}{\square\!\!\Rightarrow}\ \overset{i,k}{\square\!\!\Rightarrow} \text{ if } i \le j \text{ or } i, j \le 0, \tag{4}$$

$$\overset{i+j,k}{\square\!\!\Rightarrow}\overset{i,k'}{\square\!\!\Rightarrow} = \overset{i,k+k'-1}{\square\!\!\Rightarrow} \text{ if } 0 \le j < k'. \tag{5}$$

Let $(\mathcal{S}, \oplus)$ be a commutative monoid endowed with a filtration $\mathcal{S} = \bigcup_{n \in \mathbb{N}\setminus\{0\}} \mathcal{S}_n$ with $\mathcal{S}_1 \subset \cdots \subset \mathcal{S}_n \subset \cdots$ and a unity $\mathbf{1}_{\mathcal{S}} \in \mathcal{S}_1$.
A precomposition is a monoid morphism $\circ : \square \to Hom(\mathcal{S}, \mathcal{S})$ satisfying:

$$\circ\,(\overset{i,k}{\square\!\!\Rightarrow}) : \mathcal{S}_n \to \mathcal{S}_{n+k-1} \tag{6}$$

$$\circ\left(\overset{i,k}{\square\!\!\Rightarrow}\right)|_{\mathcal{S}_n} = Id_{\mathcal{S}_n} \text{ if } n < i \tag{7}$$

where $|_{\mathcal{S}_n}$ denotes the restriction to $\mathcal{S}_n$. For simplicity we denote $\overset{i,k}{\circ\!\!\to} := \circ(\overset{i,k}{\square\!\!\Rightarrow})$. Let $\circ : \square \to Hom(\mathcal{S}, \mathcal{S})$ and $\triangleright : \square \to Hom(\mathcal{S}', \mathcal{S}')$ be two precompositions. A map $\phi : \mathcal{S} \to \mathcal{S}'$ is a precomposition morphism from $\circ$ to $\triangleright$ if and only if it is a monoid morphism satisfying

$$\phi : \mathcal{S}_n \to \mathcal{S}'_n \tag{8}$$

$$\overset{k,n}{\triangleright\!\!\to}(\phi(x)) = \phi\left(\overset{k,n}{\circ\!\!\to}(x)\right). \tag{9}$$

We denote by $Hom(\circ, \triangleright)$ the set of precomposition morphism from $\circ$ to $\triangleright$.
Let $\circ : \square \to Hom(\mathcal{S}, \mathcal{S})$, $\triangleright : \square \to Hom(\mathcal{S}', \mathcal{S}')$ and $\diamond : \square \to Hom(\mathcal{S}'', \mathcal{S}'')$ be three precompositions together with $\phi \in Hom(\circ, \triangleright)$ and $\varphi \in Hom(\triangleright, \diamond)$. Remark that the composition $\varphi\phi : \mathcal{S} \to \mathcal{S}''$ is a morphism sending $\mathcal{S}_n$ to $\mathcal{S}''_n$ and satisfying

$$\varphi\phi\left(\overset{i,k}{\circ\!\!\to}(x)\right) = \varphi\left(\overset{i,k}{\triangleright\!\!\to}\left(\phi(x)\right)\right) = \overset{i,k}{\diamond\!\!\to}\left(\varphi\phi(x)\right)$$

for each $x \in \mathcal{S}$ and each $i \in \mathbb{Z}$ and $k \in \mathbb{N} \setminus \{0\}$. Hence, $\varphi\phi \in Hom(\circ, \diamond)$.
For each precomposition $\circ : \square \to Hom(\mathcal{S}, \mathcal{S})$ we define $Id_\circ := Id_{\mathcal{S}}$. Clearly, $Id_\circ \in Hom(\circ, \circ)$ and for each $\phi \in Hom(\circ, \circ)$ we have $\phi Id_\circ = Id_\circ \phi = \phi$.
Now, if $\phi \in Hom(\circ, \triangleright)$, $\varphi \in Hom(\triangleright, \diamond)$ and $\psi \in Hom(\diamond, \square)$ then we have, straightforwardly, $(\psi\varphi)\phi = \psi(\varphi\phi)$. Hence:

Proposition 1. *The family* PreComp *of precompositions endowed with the arrows* $Hom(\circ, \triangleright)$ *for each* $\circ, \triangleright \in$ PreComp *is a category.*

3.2 From precompositions to operads

We consider a precomposition $\circ : \square \to Hom(\mathcal{S}, \mathcal{S})$. For simplicity we denote $\overset{k}{\ggg}_\circ := \overset{0,k+1}{\circ\!\!\to}$ ($\overset{k}{\ggg}$ for short when there is no ambiguity). From $\mathcal{S}$ we define $\mathbb{S}_k := \{a_s^{(k)} : s \in \mathcal{S}_k\}$ and $\mathbb{S} := \bigcup_k \mathbb{S}_k$. For each $a_s^{(k)} \in \mathcal{S}_k$ we set

$$\overset{i,k'}{\circ\!\!\to}(a_s^{(k)}) := \begin{cases} \overset{i,k'}{\circ\!\!\to}(s) & \text{if } i \le k \\ s & \text{otherwise,} \end{cases}$$

and $\overset{k'}{\ggg}(a_s^{(k)}) = \overset{0,k'+1}{\circ\!\!\to}(a_s^{(k)})$.
Now for each $1 \le i \le k$ we define the binary operator $\circ_i : \mathbb{S}_k \times \mathbb{S}_{k'} \to \mathbb{S}_{k+k'-1}$ by $a_s^{(k)} \circ_i a_{s'}^{(k')} := a_{s''}^{(k+k'-1)}$ where $s'' = \overset{i,k'}{\circ\!\!\to}(a_s^{(k)}) \oplus \overset{i-1}{\ggg}(a_{s'}^{(k')}) \in \mathcal{S}_{k+k'-1}$.

Proposition 2. *The set $\mathbb{S}$ endowed with the partial compositions $\circ_i$ is an operad.*

Proof. First remark that the identity of the structure is $\mathbf{1}_{\mathbb{S}} := a_{\mathbf{1}_{\mathcal{S}}}^{(1)}$. Indeed:

1. We have $\mathbf{1}_{\mathbb{S}} \circ_1 a_s^{(k)} = a_{s'}^{(k)}$ with $s' = \overset{1,k}{\circ\!\!\to}(a_{\mathbf{1}_{\mathcal{S}}}^{(1)}) \oplus \overset{0}{\ggg}(a_s^{(k)})$. But, $\overset{i,k}{\circ\!\!\to}(a_{\mathbf{1}_{\mathcal{S}}}^{(1)}) = \overset{i,k}{\circ\!\!\to}(\mathbf{1}_{\mathcal{S}}) = \mathbf{1}_{\mathcal{S}}$ and $\overset{0}{\ggg}(a_s^{(k)}) = s$ (because $\overset{0}{\ggg} = \circ\left(\mathbf{1}_{\square}\right)$). Hence, $s' = s$ and $\mathbf{1}_{\mathbb{S}} \circ_1 a_s^{(k)} = a_s^{(k)}$.
2. Let $1 \le i \le k$. We have $a_T^{(k)} \circ_i \mathbf{1}_{\mathbb{S}} = a_{T'}^{(k)}$ with $s' = \overset{i,1}{\circ\!\!\to}(a_s^{(k)}) \oplus \overset{k-1}{\ggg}(a_{\mathbf{1}_{\mathcal{S}}}^{(1)})$. But, $\overset{i,1}{\circ\!\!\to}(a_s^{(k)}) = \overset{0,1}{\circ\!\!\to}(a_s^{(k)}) = s$ (because $\overset{0,1}{\square\!\!\Rightarrow} = \mathbf{1}_{\square}$) and $\overset{k-1}{\ggg}(a_{\mathbf{1}_{\mathcal{S}}}^{(1)}) = \mathbf{1}_{\mathcal{S}}$. Hence, $s' = s$ and $a_s^{(k)} \circ_i \mathbf{1}_{\mathbb{S}} = a_s^{(k)}$.

Now, let us prove the two associativity rules:

1. Let k, k', k'', i, j be five integers such that $1 \le i < j \le k$. Consider also $s \in \mathcal{S}_k$, $s' \in \mathcal{S}_{k'}$ and $s'' \in \mathcal{S}_{k''}$. Applying the definition of the composition $\circ_i$, we find: $(a_s^{(k)} \circ_j a_{s'}^{(k')}) \circ_i a_{s''}^{k''} = a_{s^{(3)}}^{(k+k'+k''-2)}$ where
$$s^{(3)} = \overset{i,k''}{\circ\!\!\to}(a_{s^{(4)}}^{(k+k'-1)}) \oplus \overset{i-1}{\ggg}(a_{s''}^{(k'')}),$$
and $s^{(4)} = \overset{j,k'}{\circ\!\!\to}(a_s^{(k)}) \oplus \overset{j-1}{\ggg}(a)_{s'}^{(k')}$.
Since $i \le k$, we have $\overset{i,k''}{\circ\!\!\to}(a_{s^{(4)}}^{(k+k'-1)}) = \overset{i,k''}{\circ\!\!\to}\left(s^{(4)}\right)$. Furthermore $s^{(4)} = \overset{j,k'}{\circ\!\!\to}(s) \oplus \overset{j-1}{\ggg}(s')$ since $j \le k$ and $0 \le k'$. Hence,
$$\begin{aligned}\overset{i,k''}{\circ\!\!\to}(a_{s^{(4)}}^{(k+k'-1)}) &= \overset{i,k''}{\circ\!\!\to}(\overset{j,k'}{\circ\!\!\to}(s) \oplus \overset{j-1}{\ggg}(s')) = \overset{i,k''}{\circ\!\!\to}\left(\overset{j,k'}{\circ\!\!\to}(s)\right) \oplus \overset{i,k''}{\circ\!\!\to}\left(\overset{j-1}{\ggg}(s')\right)\\ &= \circ\left(\overset{i,k''}{\square\!\!\Rightarrow}\overset{j,k'}{\square\!\!\Rightarrow}\right)(s) \oplus \circ\left(\overset{i,k''}{\square\!\!\Rightarrow}\overset{0,j}{\square\!\!\Rightarrow}\right)(s')\end{aligned}$$
From (4), we have $\overset{i,k''}{\square\!\!\Rightarrow}\overset{j,k'}{\square\!\!\Rightarrow} = \overset{j+k''-1,k'}{\square\!\!\Rightarrow}\overset{i,k''}{\square\!\!\Rightarrow}$. In the same way, (4) gives $\overset{i,k''}{\square\!\!\Rightarrow}\overset{0,j}{\square\!\!\Rightarrow} = \overset{0,j}{\square\!\!\Rightarrow}\overset{i-j+1,k''}{\square\!\!\Rightarrow}$ and since $i - j + 1 \le 0$, the rule (2) gives $\overset{i,k''}{\square\!\!\Rightarrow}\overset{0,j}{\square\!\!\Rightarrow} = \overset{0,k''}{\square\!\!\Rightarrow}\overset{0,j}{\square\!\!\Rightarrow} = \overset{0,j+k''-1}{\square\!\!\Rightarrow} = \overset{j+k''-2}{\square\!\!\Rightarrow}$ from (5). One deduces
$$s^{(3)} = \circ\left(\overset{j+k''-1,k'}{\square\!\!\Rightarrow}\overset{i,k''}{\square\!\!\Rightarrow}\right)(s) \oplus \left(\overset{j+k''-2}{\ggg}\right)(T') \oplus \left(\overset{i-1}{\ggg}\right)(T''). \tag{10}$$
Now examine $(a_s^{(k)} \circ_i a_{s''}^{(k'')}) \circ_{j+k''-1} a_{s'}^{k'} = a_{\tilde{s}^{(3)}}^{k+k'+k''-2}$ with
$$\tilde{s}^{(3)} = \overset{j+k''-1,k'}{\circ\!\!\to}(a_{\tilde{s}^{(4)}}^{(k+k''-1)}) \oplus \overset{j+k''-2}{\ggg}(a_{s'}^{(k')})$$
and $\tilde{s}^{(4)} = \overset{i,k''}{\circ\!\!\to}(a_s^{(k)}) \oplus \overset{i-1}{\ggg}(a_{s''}^{(k'')})$. Since $i \le k$ and $0 \le k''$ we deduce $\tilde{s}^{(4)} = \overset{i,k''}{\circ\!\!\to}(s) \oplus \overset{i-1}{\ggg}(s')$. Furthermore, since $j \le k$ and $0 \le k''$, we have
$$\tilde{s}^{(3)} = \overset{j+k''-1,k'}{\circ\!\!\to}(\tilde{s}^{(4)}) \oplus \overset{j+k''-2}{\ggg}(T') = \circ\left(\overset{j+k''-1,k'}{\square\!\!\Rightarrow}\overset{i,k''}{\square\!\!\Rightarrow}\right)(s) \oplus \circ\left(\overset{j+k''-1,k'}{\square\!\!\Rightarrow}\overset{0,i}{\square\!\!\Rightarrow}\right)(s'') \oplus \overset{j+k''-2}{\ggg}s'.$$
But $\overset{j+k''-1,k'}{\square\!\!\Rightarrow}\overset{0,i}{\square\!\!\Rightarrow} = \overset{0,i}{\square\!\!\Rightarrow}\overset{j-i+k'',k'}{\square\!\!\Rightarrow}$ and $\overset{j-i+k'',k'}{\circ\!\!\to}(s'') = s''$ from (7) since $j > i$. Hence, we obtain
$$\tilde{s}^{(3)} = \circ\left(\overset{j+k''-1,k'}{\square\!\!\Rightarrow}\overset{i,k''}{\square\!\!\Rightarrow}\right)(s) \oplus \overset{i-1}{\ggg}(s'') \oplus \overset{j+k''-2}{\ggg}(s') = s^{(3)}.$$
Hence,
$$(a_s^{(k)} \circ_j a_{s'}^{(k')}) \circ_i a_{s''}^{k''} = (a_s^{(k)} \circ_i a_{s''}^{(k'')}) \circ_{j+k''-1} a_{s'}^{k'}.$$

2. Let k, k', k'', i, j be five integers such that $1 \leq i \leq k'$, $1 \leq j \leq k$ and $1 \leq k, k', k''$. Consider $s \in \mathcal{S}_k$, $s' \in \mathcal{S}_{k'}$ and $s'' \in \mathcal{S}_{k''}$. Applying the definition of $\circ_i$, one has

$$a_s^{(k)} \circ_j (a_{s'}^{(k')} \circ_i a_{s''}^{(k'')}) = a_{s^{(3)}}^{(k+k'+k''-2)}.$$

where $s^{(3)} = \overset{j,k'+k''-1}{\circ\!\!\rightarrow}(a_s^{(k)}) \oplus \overset{j-1}{\ggg}(a_{s^{(4)}}^{(k'+k''-1)})$ and $s^{(4)} = \overset{i,k''}{\circ\!\!\rightarrow}(a_{s'}^{(k')}) \oplus \overset{i-1}{\ggg}(a_{s''}^{(k'')})$. Since $i \leq k'$ and $0 \leq k''$, we obtain $s^{(4)} = \overset{i,k''}{\circ\!\!\rightarrow}(s') \oplus \overset{i-1}{\ggg}(s'')$. Furthermore, $j \leq k$ and $0 \leq k + k' - 1$ imply

$$\begin{aligned} s^{(3)} &= \overset{j,k'+k''-1}{\circ\!\!\rightarrow}(s) \oplus \overset{j-1}{\ggg}(s^{(4)}) \\ &= \overset{j,k'+k''-1}{\circ\!\!\rightarrow}(s) \oplus \circ\left(\overset{0,j}{\square\!\!\Rightarrow}\overset{i,k''}{\square\!\!\Rightarrow}\right)(s') \oplus \circ\left(\overset{0,j}{\square\!\!\Rightarrow}\overset{0,i}{\square\!\!\Rightarrow}\right)(s'') \\ &= \overset{j,k'+k''-1}{\circ\!\!\rightarrow}(s) \oplus \circ\left(\overset{i+j-1,k''}{\square\!\!\Rightarrow}\ \overset{0,j}{\square\!\!\Rightarrow}\right)(s') \oplus \overset{i+j-2}{\ggg}(s''). \end{aligned}$$

Now, let us examine: $(a_s^{(k)} \circ_j a_{s'}^{(k')}) \circ_{i+j-1} a_{s''}^{(k'')} = a_{\tilde{s}^{(3)}}^{k+k'+k''-2}$ with $\tilde{s}^{(3)} = \overset{i+j-1,k''}{\circ\!\!\rightarrow}(a_{\tilde{s}^{(4)}}^{(k+k'-1)}) \oplus \overset{i+j-2}{\ggg}(a_{s''}^{(k'')})$ and $\tilde{s}^{(4)} = \overset{j,k'}{\circ\!\!\rightarrow}(a_s^{(k)}) \oplus \overset{j-1}{\ggg}(a_{s'}^{(k')})$. Since $j \leq k$ and $0 \leq k'$ we have $\tilde{s}^{(4)} = \overset{j,k'}{\circ\!\!\rightarrow}(s) \oplus \overset{j-1}{\ggg}(s')$. Since, $i + j - 1 \leq k + k' - 1$ and $0 \leq k''$, we obtain

$$\tilde{s}^{(3)} = \circ\left(\overset{i+j-1,k''}{\square\!\!\Rightarrow}\ \overset{j,k'}{\square\!\!\Rightarrow}\right)(s) \oplus \circ\left(\overset{i+j-1,k''}{\square\!\!\Rightarrow}\ \overset{j-1}{\square\!\!\Rightarrow}\right)(s') \oplus \overset{i+j-2}{\ggg}(s'')$$

But $i - 1 < k'$ implies $\overset{i+j-1,k''}{\square\!\!\Rightarrow}\ \overset{j,k'}{\square\!\!\Rightarrow} = \overset{j,k'+k''-1}{\square\!\!\Rightarrow}$ (eq. (5). Hence,

$$\tilde{s}^{(3)} = \overset{j,k'+k''-1}{\circ\!\!\rightarrow}(s) \oplus \circ\left(\overset{i+j-1,k''}{\square\!\!\Rightarrow}\ \overset{0,j}{\square\!\!\Rightarrow}\right)(s') \oplus \overset{i+j-2}{\ggg}(s'') = s^{(3)}$$

Hence,

$$a_s^{(k)} \circ_j (a_{s'}^{(k')} \circ_i a_{s''}^{(k'')}) = (a_s^{(k)} \circ_j a_{s'}^{(k')}) \circ_{i+j-1} a_{s''}^{(k'')}.$$

The compositions $\circ_i$ satisfy the two assertions rules and admit a unity. The set $\mathbb{S}$ has a structure of operad. □

We define $OP(\circ) := (\mathbb{S}, \circ_i)$ as defined in the construction. Let $\phi \in Hom(\circ, \triangleright)$, we define

$$\phi^{OP} : OP(\circ) \to OP(\triangleright)$$

by

$$\phi^{OP}(a_s^{(k)}) = a_{\phi(s)}^{(k)}.$$

Theorem 2. *The arrow OP : PreComp → Operad which associates with each precomposition $\circ$ the operad $OP(\circ)$ and to each homomorphism $\phi \in Hom(\circ, \triangleright)$ the operadic morphism ϕ^{OP} is a functor.*

Proof. We have to prove three properties

1. OP satisfies the equality:

$$Id_\circ^{OP} = Id_{OP(\circ)}.$$

   This is straightforward from the definition.

2. Each ϕ^{OP} is a morphism of operad. Indeed, let $\phi \in Hom(\circ, \triangleright)$, it suffices to compute $\phi^{OP}(a_{s_1}^{(k_1)} \circ_i a_{s_2}^{(k_2)})$ for $s_1 \in \mathcal{S}_{k_1}$, $s_2 \in \mathcal{S}_{k_2}$ and $1 \leq i \leq k_1$. We have

$$\phi^{OP}(a_{s_1}^{(k_1)} \circ_i a_{s_2}^{(k_2)}) = a_{s_3}^{(k_1+k_2-1)}$$

   where

$$\begin{aligned} s_3 &= \phi\left(\overset{i,k_1}{\circ\!\!\rightarrow}(T_1) \oplus \overset{0,i-1}{\circ\!\!\rightarrow}(T_2)\right) \\ &= \overset{i,k_1}{\triangleright\!\!\!\rightarrow}\left(\phi(s_1)\right) \oplus \overset{0,i-1}{\triangleright\!\!\!\rightarrow}\left(\phi(s_2)\right). \end{aligned}$$

Hence

$$\begin{aligned}\phi^{OP}(a_{s_1}^{(k)} \circ_i a_{T_2}^{(k')}) &= a_{\phi(s_1)}^{(k)} \triangleright_i a_{\phi(s_2)}^{(k')}\\ &= \phi^{OP}(a_{s_1}^{(k)}) \triangleright_i \phi^{OP}(a_{\phi(s_2)}^{(k')}).\end{aligned}$$

We deduce that ϕ^{OP} is an operadic morphism.

3. OP is compatible with the composition of homomorphisms. Indeed, let $\phi \in Hom(\circ, \triangleright)$ and $\varphi \in Hom(\triangleright, \diamond)$. For any $s \in \mathcal{S}_k$, we have

$$\varphi^{OP}\phi^{OP}(a_s^{(k)}) = \varphi^{OP}\left(a_{\phi(s)}^{(k)}\right) = a_{\varphi(\phi(s))}^{(k)} = (\varphi\phi)^{OP}\left(a_s^{(k)}\right).$$

We have then shown that $(\varphi\phi)^{OP} = \varphi^{OP}\phi^{OP}$.

Hence, the arrow OP satisfies the three required properties to be a functor. □

3.3 Quotients of precompositions

Let $\circ : \square \to Hom(\mathcal{S}, \mathcal{S})$ be a precomposition and $\gamma : \mathcal{S} \to \mathcal{S}$ be an idempotent ($\gamma^2 = \gamma$) monoid morphism sending $\mathcal{S}_k$ to $\mathcal{S}_k$ and satisfying: $\overset{i,k}{\circ\!\!\rightarrow}\gamma = \gamma\overset{i,k}{\circ\!\!\rightarrow}$.
We define $\gamma : \mathbb{S} \to \mathbb{S}$ by $\gamma a_s^{(k)} = a_{\gamma s}^{(k)}$.

Proposition 3. *The two following conditions hold:*

1. *For each $s \in \mathcal{S}_k$, $s' \in \mathcal{S}_{k'}$ and $1 \le i \le k$:*

$$\gamma\left(\gamma(a_s^{(k)}) \circ_i \gamma(a_{s'}^{(k')})\right) = \gamma\left(a_s^{(k)} \circ_i a_{s'}^{(k')}\right)$$

2. *$\gamma(s_1) = \gamma(s')$ and $\gamma(s_2) = \gamma(s'_2)$ implies $\gamma(a_{s_1}^{(k)} \circ_i a_{s_2}^{(k')}) = \gamma(a_{s'_1}^{(k)} \circ_i a_{s'_2}^{k'})$*

Proof. 1. We have $\gamma(a_s^{(k)}) \circ_i \gamma(a_{s'}^{(k')}) = a_{\gamma s}^{(k)} \circ_i a_{\gamma s'}^{(k')} = a_{s''}^{(k+k'-1)}$ with

$$s'' = \overset{i,k'}{\circ\!\!\rightarrow}(a_{\gamma s}^{(k)}) \oplus \overset{0,i}{\circ\!\!\rightarrow}(a_{\gamma s'}^{(k')}) = \overset{i,k'}{\circ\!\!\rightarrow}(\gamma s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(\gamma s') = \gamma\left(\overset{i,k'}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')\right).$$

Hence

$$\begin{aligned}\gamma\left(\gamma(a_s^{(k)}) \circ_i \gamma(a_{s'}^{(k')})\right) &= \gamma\left(\gamma\left(\overset{i,k'}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')\right)\right) = \gamma\left(\overset{i,k'}{\circ\!\!\rightarrow}(s) \oplus \overset{0,i}{\circ\!\!\rightarrow}(s')\right)\\ &= \gamma\left(\overset{i,k'}{\circ\!\!\rightarrow}(a_s^{(k)}) \oplus \overset{0,i}{\circ\!\!\rightarrow}(a_{s'}^{(k')})\right) = \gamma\left(a_s^{(k)} \circ_i a_{s'}^{k'}\right).\end{aligned}$$

2. Suppose $\gamma(s_1) = \gamma(s')$ and $\gamma(s_2) = \gamma(s'_2)$ then we have

$$\gamma(a_{s_1}^{(k)} \circ_i a_{s_2}^{k'}) = \gamma\gamma(a_{s_1}^{(k)} \circ_i a_{s_2}^{k'}) = \gamma(a_{\gamma s_1}^{(k)} \circ_i a_{\gamma s_2}^{k'}) = \gamma(a_{\gamma s'_1}^{(k)} \circ_i a_{\gamma s'_2}^{k'}) = \gamma(a_{s'_1}^{(k)} \circ_i a_{s'_2}^{k'}).$$

□

Consider now the equivalence relation $\sim_\gamma$ on $\mathcal{S}$ defined for any $s, s' \in \mathcal{S}$ by $s \sim_\gamma s'$ if and onlyif $\gamma(s) = \gamma(s')$. By definition of γ, $\sim_\gamma$ is a monoid congruence of $\mathcal{S}$ and hence, $\mathcal{S}/_{\sim_\gamma}$ is a monoid. Consider also the equivalence relation $\equiv_\gamma$ on $OP(\circ)$ defined for any $a_s^{(k)}, a_{s'}^{(k)} \in OP(\circ)$ by $a_s^{(k)} \equiv_\gamma a_{s'}^{(k)}$ if and only if $s \sim_\gamma s'$.Proposition 3 shows that $\equiv_\gamma$ is actually an operadic congruence and hence, that $OP(\circ)/_{\equiv_\gamma}$ is an operad.

Let the precomposition

$$\odot : \square \to Hom\left(\mathcal{S}/_{\sim_\gamma}, \mathcal{S}/_{\sim_\gamma}\right) \tag{11}$$

defined for any $\sim_\gamma$-equivalence class $[s]_{\sim_\gamma}$ by $\overset{i,k}{\odot\!\!\rightarrow}\left([s]_{\sim_\gamma}\right) := [\overset{i,k}{\circ\!\!\rightarrow}(s)]_{\sim_\gamma}$. We then have

Corollary 1. *The operads $OP(\circ)/_{\equiv_\gamma}$ and $OP(\odot)$ are isomorphic.*

Proof. Let us denote by $\circ_i^\gamma$ the composition map of $OP(\circ)/_{\equiv_\gamma}$. Let the map

$$\phi : OP(\circ)/_{\equiv_\gamma} \to OP(\odot) \tag{12}$$

defined for any $\equiv_\gamma$-equivalence class $[a_s^{(k)}]_{\equiv_\gamma}$ by

$$\phi([a_s^{(k)}]_{\equiv_\gamma}) := a^{(k)}_{[s]_{\sim_\gamma}}. \tag{13}$$

Let us show that ϕ is an operad morphism. For that, let $[a_s^{(k)}]_{\equiv_\gamma}$ and $[a_{s'}^{(k')}]_{\equiv_\gamma}$ be two $\equiv_\gamma$ equivalence classes. One has

$$\phi([a_s^{(k)}]_{\equiv_\gamma} \circ_i^\gamma [a_{s'}^{(k')}]_{\equiv_\gamma}) \;=\; \phi([a_s^{(k)} \circ_i a_{s'}^{(k')}]_{\equiv_\gamma}) \;=\; \phi([a_{s''}^{(k+k'-1)}]_{\equiv_\gamma}) \;=\; a^{(k+k'-1)}_{[s'']_{\sim_\gamma}}, \tag{14}$$

where $s'' := \overset{i,k'}{\circ\!\!\to}(s) \oplus \overset{0,i}{\circ\!\!\to}(s')$. We moreover have

$$\phi([a_s^{(k)}]_{\equiv_\gamma}) \odot_i \phi([a_{s'}^{(k')}]_{\equiv_\gamma}) \;=\; a^{(k)}_{[s]_{\sim_\gamma}} \odot_i a^{(k')}_{[s']_{\sim_\gamma}} \;=\; a^{(k+k'-1)}_{[s''']_{\sim_\gamma}}, \tag{15}$$

where $[s''']_{\sim_\gamma} := \overset{i,k'}{\odot\!\!\to}([s]_{\sim_\gamma}) \oplus \overset{0,i}{\odot\!\!\to}([s']_{\sim_\gamma})$. Now, by using the fact that $\sim_\gamma$ is a monoid congruence, one has

$$\begin{aligned}
[s''']_{\sim_\gamma} &= \overset{i,k'}{\odot\!\!\to}([s]_{\sim_\gamma}) \oplus \overset{0,i}{\odot\!\!\to}([s']_{\sim_\gamma}) \\
&= [\overset{i,k'}{\circ\!\!\to}(s)]_{\sim_\gamma} \oplus [\overset{0,i}{\circ\!\!\to}(s')]_{\sim_\gamma} \\
&= [\overset{i,k'}{\circ\!\!\to}(s) \oplus \overset{0,i}{\circ\!\!\to}(s')]_{\sim_\gamma} \\
&= [s'']_{\sim_\gamma}.
\end{aligned} \tag{16}$$

This shows that (14) and (15) are equal and hence, that ϕ is an operad morphism.

Furthermore, the definitions of $\sim_\gamma$ and $\equiv_\gamma$ imply that ϕ is a bijection. Therefore, ϕ is an operad isomorphism. □

4 Multi-tildes and precompositions

In [10], we investigated several operads allowing to describe the behaviour of the multi-tilde operators. In this section, we show that some of them admit an alternative definition using the notion of precomposition.

4.1 The operad $\mathcal{T}$ revisited

We consider the sets $\mathcal{S}_n^{\mathcal{T}} = 2^{\{(x,y):1\le x\le y\le n\}}$ for each $n > 0$. Noting that $\mathcal{S}_n^{\mathcal{T}} \subset \mathcal{S}_{n+1}^{\mathcal{T}}$ we define $\mathcal{S}^{\mathcal{T}} := \bigcup_{n\in\mathbb{N}\setminus\{0\}} \mathcal{S}_n^{\mathcal{T}}$. Considering the binary operation $\cup$ as a product, the pair $(\mathcal{S}^{\mathcal{T}}, \cup)$ defines a commutative monoid whose unity is $\mathbf{1}_{\mathcal{S}^{\mathcal{T}}} = \emptyset \in \mathcal{S}_1^{\mathcal{T}}$. This is a commutative monoid generated by the set $\{\{(x,y)\}_{1\le x\le y}\}$.
Now define $\circ : \square \to \mathrm{Hom}(\mathcal{S}^{\mathcal{T}}, \mathcal{S}^{\mathcal{T}})$ by

$$\circ(\overset{i,k}{\square\!\!\Rightarrow}) := \overset{i,k}{\circ\!\!\to}$$

where each homomorphism $\overset{i,k}{\circ\!\!\to}$ is defined by its values on the generators:

$$\overset{i,k}{\circ\!\!\to}(\{(x,y)\}) = \begin{cases} \{(x,y)\} & \text{if } y < i, \\ \{(x, y+k-1)\} & \text{if } x \le i \le y, \\ \{(x+k-1, y+k-1\} & \text{otherwise.} \end{cases}$$

Remark that $\circ$ is a monoid morphism. Indeed,

1. The set of the homomorphisms $\overset{i,k}{\circ\!\!\to}$ generates a submonoid of $\mathrm{Hom}(\mathcal{S}^{\mathcal{T}}, \mathcal{S}^{\mathcal{T}})$ (which unity is $\mathrm{Id}_{\mathcal{S}^{\mathcal{T}}}$)

2. By construction, $\overset{i,k}{\circ\!\!\to} : \mathcal{S}^{\mathcal{T}}_n \to \mathcal{S}^{\mathcal{T}}_{n+k-1}$ and $\overset{i,k}{\circ\!\!\to}|_{\mathcal{S}^{\mathcal{T}}_n} = \mathrm{Id}_{\mathcal{S}^{\mathcal{T}}_n}$ if $n < i$.

3. The operators $\overset{i,k}{\circ\!\!\to}$ satisfy (see [10])

   - $\overset{i,k}{\circ\!\!\to} = \overset{0,k}{\circ\!\!\to}$ for each $i < 0$,
   - $\overset{i,1}{\circ\!\!\to} = \overset{0,1}{\circ\!\!\to} = \mathrm{Id}_{\mathcal{S}^{\mathcal{T}}}$ for each i
   - $\overset{i,k}{\circ\!\!\to}\overset{j,k'}{\circ\!\!\to} = \overset{j+k-1,k'}{\circ\!\!\to}\ \overset{i,k}{\circ\!\!\to}$ if $i \le j$ or $i, j \le 0$
   - $\overset{i+j,k}{\circ\!\!\to}\overset{i,k'}{\circ\!\!\to} = \overset{i,k+k'-1}{\circ\!\!\to}$ if $0 \le j < k'$.

Hence $\circ$ is a precomposition. More precisely, the operad $\mathcal{T}$ can be seen as the operad constructed from the precomposition $\circ$:

Proposition 4. *The operads $\mathcal{T}$ and $\mathrm{OP}(\circ)$ are isomorphic.*

Proof. The isomorphism is given by the map from $\mathcal{T}_k$ to $\mathbb{S}_k$ sending any element T to $a^{(k)}_T$. □

4.2 The operad RAS revisited

In [10], we considered an operad RAS on reflexive and antisymmetric relations that are compatible with the natural order on integers (*i.e.* $(x, y) \in \mathrm{RAS}$ implies $x \le y$). Since the elements (x, x) do not play any role in the construction, we propose here an alternative construction based on antireflexive and antisymmetric relations.

Consider the sets $\mathcal{S}^{\diamond}_n = 2^{\{(x,y):1\le x<y\le n+1\}}$ for each $n > 0$. By construction we have $\mathcal{S}^{\diamond}_n \subset \mathcal{S}^{\diamond}_{n+1}$. Endowed with the binary operation $\cup$ the set $\mathcal{S}^{\diamond} := \bigcup_{n\in\mathbb{N}\setminus\{0\}} \mathcal{S}^{\diamond}_n$ is a commutative monoid generated by $\{\{(x, y)\}_{1\le x<y}\}$.

Let us define $\diamond : \square \to \mathrm{Hom}(\mathcal{S}^{\diamond}, \mathcal{S}^{\diamond})$ by $\diamond(\overset{i,k}{\square\!\!\Rightarrow}) := \overset{i,k}{\diamond\!\!\to}$ with

$$\overset{i,k}{\diamond\!\!\to}(\{(x, y)\}) = \begin{cases} \{(x, y)\} & \text{if } y \le i, \\ \{(x, y + k - 1)\} & \text{if } x \le i < y, \\ \{(x + k - 1, y + k - 1\} & \text{otherwise.} \end{cases} \tag{17}$$

Similarly to Section 4.1, we consider the submonoid of $\mathrm{Hom}(\mathcal{S}^{\diamond}, \mathcal{S}^{\diamond})$ generated by the elements $\overset{i,k}{\diamond\!\!\to}$. We have $\overset{i,k}{\diamond\!\!\to} : \mathcal{S}^{\diamond}_n \to \mathcal{S}^{\diamond}_{n+k-1}$ and $\overset{i,k}{\diamond\!\!\to}|_{\mathcal{S}^{\diamond}_n} = \mathrm{Id}_{\mathcal{S}^{\diamond}_n}$ when $n < i$. Furthermore, the elements $\overset{i,k}{\diamond\!\!\to}$ satisfy the properties

- $\overset{i,k}{\diamond\!\!\to} = \overset{0,k}{\diamond\!\!\to}$ for each $i < 0$,
- $\overset{i,1}{\diamond\!\!\to} = \overset{0,1}{\diamond\!\!\to} = \mathrm{Id}_{\mathcal{S}^{\diamond}}$ for each i
- $\overset{i,k}{\diamond\!\!\to}\overset{j,k'}{\diamond\!\!\to} = \overset{j+k-1,k'}{\diamond\!\!\to}\ \overset{i,k}{\diamond\!\!\to}$ if $i \le j$ or $i, j \le 0$
- $\overset{i+j,k}{\diamond\!\!\to}\overset{i,k'}{\diamond\!\!\to} = \overset{i,k+k'-1}{\diamond\!\!\to}$ if $0 \le j < k'$.

The map $\diamond$ is a monoid morphism and so a precomposition. We set $\mathrm{ARAS} := \mathrm{OP}(\diamond) = (\mathbb{S}^{\diamond}, \diamond)$. The operad ARAS is an alternative closed construction for the operad RAS as shown by:

Proposition 5. *The operads* RAS *and* ARAS *are isomorphic.*

Proof. The isomorphism is given by the map from RAS_k to $\mathbb{S}^{\diamond}_k$ sending any element R to $a^{(k)}_{R\setminus\Delta}$, where $\Delta = \{(x, x) : x \in \mathbb{N}\}$. □

4.3 The operad POSet revisited

The operad POSet is defined as a quotient of the operad RAS. In [10], we showed that POSet is optimal in the sense that two of its operators have two different actions on languages.

Denote by $\gamma : S^\diamond \to S^\diamond$ the transitive closure. Remarking that $\gamma(R) : S_k^\diamond \to S_k^\diamond$ and $\overset{i,k}{\diamond\!\!\to}\gamma = \gamma\overset{i,k}{\diamond\!\!\to}$, we apply the result of Section 3.3 and define the precomposition $\diamond : \square \to \mathrm{Hom}(S^\diamond_{/\equiv_\gamma}, S^\diamond_{/\equiv_\gamma})$ by setting $\overset{i,k}{\diamond\!\!\to}([R]) = \left[\overset{i,k}{\diamond\!\!\to}(R)\right]$ where $[] : S^\diamond \to S^\diamond_{/\equiv_\gamma}$ denotes the natural morphism sending each element R of $S^\diamond$ to its class $[R]$.

The operad $\mathrm{OP}(\diamond)$ gives an alternative way to define the operad POSet using precompositions.

Proposition 6. *The operads* POSet, $\mathrm{OP}(\diamond)$ *and* $\mathrm{ARAS}_{/\equiv_\gamma}$ *are isomorphic.*

Proof. The isomorphism is given by the map from POSet_k to $\mathbb{S}_k^\diamond$ sending any element P to $a^{(k)}_{[(P\setminus\Delta)]}$, where $\Delta = \{(x,x) : x \in \mathbb{N}\}$. □

5 The operad of double multi-tildes

In [10], we proved that the action of $\mathcal{T}$ on symbols allows us to denote all finite languages. In this section, we propose an extension of the operad $\mathcal{T}$ in order to represent infinite languages. New operators are required in order to describe the Kleene star operation *. In the last section of [10], we introduced an operad $\mathcal{T}^*$ generated by $\mathcal{T}$ together with an additional operator $\star$ (denoting the Kleene star *). Albeit this operad allows the manipulation of regular languages, the equivalence of the operators, w.r.t. the action over languages, is difficult to model. In this section, we introduce a new operad $\mathcal{DT}$ which is composed of two kinds of multi-tildes: right and left multi-tildes. The * operation will be realized by a combination of right and left multi-tildes operations. Furthermore, we show that the expressiveness of these operators is higher than operators of $\mathcal{T}^\star$ for a given number of symbols. We start by considering that the two types of operators are independently composed. More precisely,

$$\mathcal{DT} := \mathbb{T}(\mathcal{T}, \mathcal{T}). \tag{18}$$

We mimic the construction of [10] linking multi-tildes and reflexive antisymmetric relations in order to construct a new operad ARef, which elements are antireflexive relations, isomorphic to $\mathcal{DT}$.

5.1 $\mathcal{DT}$ and antireflexive relations

We consider the graded set

$$\mathcal{S}^{\mathrm{ARef}} := \bigcup_n \mathcal{S}_n^{\mathrm{ARef}} \text{ with } S_n^{\mathrm{ARef}} = 2^{\{(x,y):1\le x\neq y\le n+1\}}$$

where 2^E denotes the set of the subsets of E. Endowed with the binary operation $\cup$, the set $\mathcal{S}^{\mathrm{ARef}}$ is a commutative monoid generated by $\{(x,y) : x \neq y\}$. We define the map $\diamond : \square \to Hom(\mathcal{S}^{\mathrm{ARef}}, \mathcal{S}^{\mathrm{ARef}})$ by $\diamond(\overset{i,k}{\square\!\!\Rightarrow}) = \overset{ik}{\diamond\!\!\to}$ where

$$\overset{i,k}{\diamond\!\!\to}(\{(x,y)\}) = \begin{cases} \{(x,y)\} & \text{if } x, y \le i, \\ \{(x, y+k-1)\} & \text{if } x \le i \text{ and } i < y, \\ \{(x+k-1, y)\} & \text{if } i < x \text{ and } y \le i, \\ \{(x+k-1, y+k-1\} & \text{otherwise.} \end{cases}$$

We easily check that $\diamond$ is a precomposition and we set $\mathrm{ARef} := \mathrm{OP}(\diamond)$.

Proposition 7. *The operad* ARef *is isomorphic to* $\mathbb{T}(\mathrm{ARAS}, \mathrm{ARAS})$.

Proof. Let us denote $\mathrm{rev}(x,y) = (y,x)$. If $R \in \mathrm{ARef}$ we will denote $\mathrm{rev}(R) = \{\mathrm{rev}(x,y) : (x,y) \in R\}$, $R^< = \{(x,y) \in R : x < y\}$ and $R^> = \{(x,y) \in R : x > y\} = \mathrm{rev}((\mathrm{rev}(R))^<)$ (note that $R = R^< \cup R^>$). Let $\Phi : \mathbb{T}(\mathrm{ARAS}, \mathrm{ARAS}) \to \mathrm{ARef}$ the map defined by $\Phi(a^{(k)}_{R_1}, a^{(k)}_{R_2}) = a^{(k)}_{R_1\cup\mathrm{rev}(R_2)}$. This map is a bijection

which inverse is $\Phi^{-1}(a_R^{(k)}) = (a_{R^<}^{(k)}, a_{\mathrm{rev}(R^>)}^{(k)})$.
Let us prove that $a_R^{(k)} \diamond_i a_{R'}^{(k')} = \Phi(\Phi^{-1}(a_R^{(k)}) \diamond_i \Phi^{-1}(a_{R'}^{(k')}))$. We have

$$\Phi(\Phi^{-1}(a_R^{(k)}) \diamond_i \Phi^{-1}(a_{R'}^{(k')})) = \Phi(\Phi^{-1}(a_{R''}^{(k+k'-1)})) = \Phi(a_{R''^<}^{(k+k'-1)}, a_{\mathrm{rev}(R''^>)}^{(k+k'-1)}),$$

where $R'' = \overset{i,k'}{\diamond\!\!\to}(R) \cup \overset{0,i}{\diamond\!\!\to}(R')$. Let $* \in \{<,>\}$. Since, $\overset{i,k'}{\diamond\!\!\to}(R^*) = (\overset{i,k'}{\diamond\!\!\to}(R))^*$ and $\overset{0,i}{\diamond\!\!\to}(R'^*) = (\overset{0,i}{\diamond\!\!\to}(R'))^*$, we have $(\overset{i,k'}{\diamond\!\!\to}(R^*) \cup \overset{0,i}{\diamond\!\!\to}(R'^*)) = (R'')^*$. In other words: $R''^* = \overset{i,k'}{\diamond\!\!\to}(R^*) \cup \overset{0,i}{\diamond\!\!\to}(R'^*)$. Hence,

$$\Phi(\Phi^{-1}(R) \diamond_i \Phi^{-1}(R')) = \Phi(a_{R''^<}^{(k+k'-1)}, a_{\mathrm{rev}(R''^>)}^{(k+k'-1}) = a_{R''^< \cup R''^>}^{(k+k'-1} = a_{R''}^{(k+k'-1)} = a_R^{(k)} \diamond_i a_{R'}^{(k')}.$$

This proves that ARef is an operad isomorphic to $\mathbb{T}(\mathrm{ARAS}, \mathrm{ARAS})$. □

Corollary 2. *The operads $\mathcal{DT}$, ARef, $\mathbb{T}(\mathrm{RAS}, \mathrm{RAS})$ and $\mathbb{T}(\mathrm{ARAS}, \mathrm{ARAS})$ are isomorphic.*

In the aim to illustrate the isomorphism between ARef and $\mathcal{DT}$, we recall that the graded map $\zeta : \mathcal{T}_k \to \mathrm{RAS}_k$ defined by $\zeta(R) = \{(x, y+1) : (x, y) \in R\} \cup \{(1,1), \ldots, (k+1, k+1)\}$ is an isomorphism of operad. According to the definition of ARAS, we obtain explicitly an isomorphism from $\mathcal{T}$ to ARAS by slight modification of ζ: $\zeta^A(R) = a_{\zeta(R)\setminus\Delta}^{(k)}$. Since ARAS and $\mathcal{T}$ are isomorphic, this is also the case for $\mathcal{DT}$ and ARef (because ARef is isomorphic to $\mathbb{T}(\mathrm{ARAS}, \mathrm{ARAS})$). From the construction described in Proposition 7, the map $\xi : \mathcal{DT} \to \mathrm{ARef}$ defined by $\xi(R_1, R_2) = a_{\zeta^A(R_1) \cup \mathrm{rev}(\zeta^A(R_2))}^{(k)}$, when $(R_1, R_2) \in \mathcal{DT}_k$, explicits the isomorphism.

Example 1. Consider $P_1 = (\{(1,3), (2,2), (3,4)\}, \{(2,3)\}) \in \mathcal{DT}_5$ and $P_2(\{(2,3), (3,4)\}, \{(1,2), (3,4)\}) \in \mathcal{DT}_4$. We have

$$\xi(P_1) = a_{\{(1,4),(2,3),(3,5),(4,2)\}}^{(5)} \text{ and } \xi(P_2) = a_{\{(2,4),(3,5),(3,1),(5,3)\}}^{(4)}.$$

Remark that

$$\begin{aligned} P_1 \circ_2 P_2 &= (\{(1,3), (2,2), (3,4)\} \circ_2 \{(2,3), (3,4)\}, \{(2,3)\} \circ_2 \{(1,2), (3,4)\}) \\ &= (\{(1,6), (2,5), (6,7), (3,4), (4,5)\}, \{(2,6), (2,3), (4,5)\}), \end{aligned}$$

and then

$$\xi(P_1 \circ_2 P_2) = a_{\{(1,7),(2,6),(6,8),(3,5),(4,6),(7,2),(4,2),(6,4)\}}^{(8)}.$$

Let us now compute $\xi(P_1) \diamond_2 \xi(P_2)$:

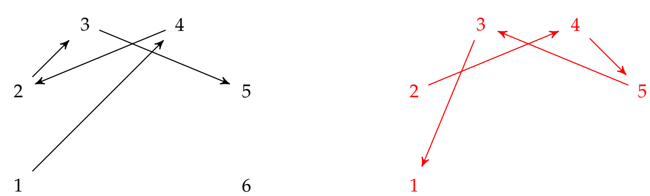
$$\xi(P_1) \diamond_2 \xi(P_2) = a_{\{(1,4),(2,3),(3,5),(4,2)\}}^{(5)} \diamond_2 a_{\{(2,4),(3,5),(3,1),(5,3)\}}^{(4)} = a_R^{(8)}$$

with

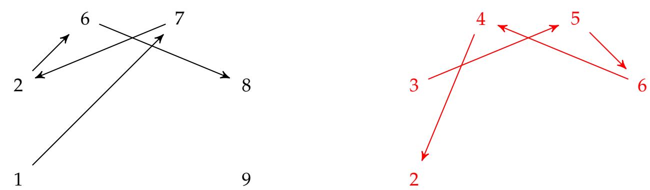
$$\begin{aligned} R &= \overset{2,4}{\diamond\!\!\to}(\{(1,4), (2,3), (3,5), (4,2)\}) \cup \overset{0,2}{\diamond\!\!\to}(\{(2,4), (3,5), (3,1), (5,3)\}) \\ &= \{(1,7), (2,6), (6,8), (7,2), (3,5), (4,6), (4,2), (6,4)\}. \end{aligned}$$

We observe that $\xi(P_1 \circ_2 P_2) = \xi(P_1) \diamond_2 \xi(P_2)$.

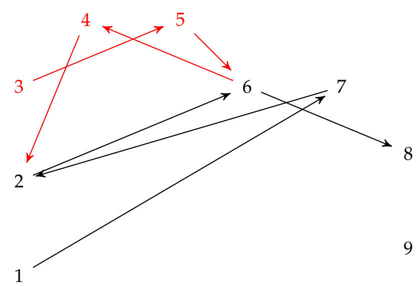
Graphically, the composition $\diamond_i$ can be illustrated in two steps corresponding to the operators $\overset{ik'}{\diamond\!\!\to}$ and $\overset{0,i}{\diamond\!\!\to}$ by drawing the graph of the relations. For instance, we start with the two graphs of the relations $\{(1,4), (2,3), (3,5), (4,2)\}$ and $\{(2,4), (3,5), (3,1), (5,3)\}$:



We rename the vertices $3 \to 6, 4 \to 7, \ldots, 6 \to 9$ in the graphs of $\{(1,4), (2,3), (3,5), (4,2)\}$ and the vertices $1 \to 2, \ldots, 5 \to 6$ in the graph of $\{(2,4), (3,5), (3,1), (5,3)\}$.



Then we identify the vertices which have the same label in the two graphs:



5.2 An operad on quasiorders

A quasiorder is a reflexive and transitive relation. If R is a relation we denote by $\gamma(R)$ its transitive closure. We also set $\gamma^{\mathrm{A}}(R) = R \setminus \{(n,n) : n \in \mathbb{Z}\}$ and $\gamma^{\mathrm{R}}(R) = R \cup \{(n,n) : n \in \mathbb{Z}\}$. Note that $\gamma^{\mathrm{R}}(R)$ is the smallest quasiorder which contains R. Since $\gamma^{\mathrm{A}} : \mathcal{S}^{\mathrm{ARef}} \to \mathcal{S}^{\mathrm{ARef}}$ is an idempotent monoid morphism sending S_k^{ARef} to S_k^{ARef} and satisfying $\overset{ik}{\diamond\!\!\rightarrow}\gamma^A = \gamma^A\overset{ik}{\diamond\!\!\rightarrow}$, following Section 3.3, we construct the precomposition $\diamond : \square \to Hom(\mathcal{S}^{\mathrm{ARef}}/_{\equiv_{\gamma^A}}, \mathcal{S}^{\mathrm{ARef}}/_{\equiv_{\gamma^A}})$ defined by $\overset{i,k}{\diamond\!\!\rightarrow}([R]) = \left[\overset{i,k}{\diamond\!\!\rightarrow}(R)\right]$ where $[]$ denotes the natural morphism $\mathcal{S}^{\mathrm{ARef}} \to \mathcal{S}^{\mathrm{ARef}}/_{\equiv_{\gamma^A}}$ sending each relation to its class. Hence, we consider the operad $\mathrm{OP}(\diamond)$.

Alternatively, consider the set QOSET_n of quasiorder of $\{1,\dots,n+1\}$ and $\mathrm{QOSET} := \bigcup_n \mathrm{QOSET}_n$. Consider also the partial composition defined by $Q\diamond_i Q' = \gamma(\overset{i,k'}{\diamond\!\!\rightarrow}(Q) \cup \overset{0,i}{\diamond\!\!\rightarrow}(Q'))$ if $Q \in \mathrm{QOSET}_k$, $Q' \in \mathrm{QOSET}_{k'}$ and $i \le k$.

Theorem 3. *The pair* $(\mathrm{QOSET}, \diamond)$ *is an operad isomorphic to* $\mathrm{OP}(\diamond)$.

Proof. Consider the map $\eta : \mathrm{QOSET} \to \mathrm{OP}(\diamond)$ given by $\eta(Q) = a^{(k)}_{[Q\setminus\Delta]}$. The map η is a graded bijection and its inverse is given by $\eta^{-1}(a^{(k)}_{[R]}) = \gamma^{\mathrm{R}}(R)$. Remarking that

$$\eta^{-1}(a^{(k)}_{[R']})\diamond_i a^{(k)}_{[R'']})) = \gamma^{\mathrm{R}}(R'\diamond_i R'') = \gamma(\overset{i,k'}{\diamond\!\!\rightarrow}(\gamma^{\mathrm{R}}(R'))\cup\overset{0,i}{\diamond\!\!\rightarrow}(\gamma^{\mathrm{R}}(R''))) = \gamma^{\mathrm{R}}(R')\diamond_i\gamma^{\mathrm{R}}(R'') = \eta^{-1}(a^{(k)}_{[R']})\diamond_i\eta^{-1}(a^{(k)}_{[R'']}),$$

we prove that the set QOSET inherits from $\mathrm{OP}(\diamond)$ of a structure of operad. □

Example 2. Let us give an example. Consider, as in Example 1, the antireflexive relations $R_1 = \{(1,4),(2,3),(3,5),(4,2)\}$ and $R_2 = \{(2,4),(3,5),(3,1),(5,3)\}$. We have

$$\gamma^{\mathrm{R}}(R_1) = \{(1,4),(2,3),(3,5),(4,2),(1,2),(2,5),(4,3),(1,3),(4,5),(1,5),(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

and

$$\gamma^{\mathrm{R}}(R_2) = \{(2,4),(3,5),(3,1),(5,3),(5,1),(1,1),(2,2),(3,3),(4,4),(5,5)\}.$$

We have

$$\begin{array}{rcl}\gamma^{\mathrm{R}}(R) & = & \gamma^{\mathrm{R}}(\overset{2,4}{\diamond\!\!\rightarrow}(R_1)\cup\overset{0,2}{\diamond\!\!\rightarrow}(R_2))\\ & = & \gamma^{\mathrm{R}}(\{(1,7),(2,6),(6,8),(7,2),(3,5),(4,6),(4,2),(6,4)\})\\ & = & \{(1,7),(2,6),(6,8),(7,2),(3,5),(4,6),(4,2),(6,4),\\ & & (1,2),(2,8),(2,4),(4,8),(7,8),(7,4),(6,2),(1,8),(1,4),(7,6),(1,6)\\ & & (1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)\}.\end{array}$$

Also:

$$\begin{array}{rcl}\gamma(\overset{2,4}{\diamond\!\!\to}(\gamma^R(R_1)) \cup \overset{0,2}{\diamond\!\!\to}(\gamma^R(R_2))) & = & \gamma(\{(1,7),(2,6),(6,8),(7,2),(1,2),(2,8),(7,6),(1,6),(7,8),(1,8),\\ & & (1,1),(2,2),(6,6),(7,7),(8,8),(9,9)\}\\ & & \cup\{(3,5),(4,6),(4,2),(6,4),(6,3),(2,2),(3,3),(4,4),(5,5),(6,6)\})\\ & = & \{(1,7),(2,6),(6,8),(7,2),(1,2),(2,8),(7,6),(1,6),(7,8),(1,8),(3,5),(4,6),\\ & & (4,2),(6,4),(6,2),(2,4),(4,8),(7,4),(1,4),\\ & & (1,1),(2,2),(3,3),(4,4),(5,5),(6,6),(7,7),(8,8),(9,9)\}\\ & = & \gamma^R(R)\end{array}$$

6 Action on languages

The aim of this section is to describe regular languages by using the operads defined above. More precisely, we show that the set of regular languages is a module on each of these operads. Furthermore, we prove that each regular language is denoted by an operator acting on symbols or $\emptyset$. Finally, we show that the operad `QOSet` is optimal in the sense that its action is faithful.

6.1 Action of ARef

We associate to each relation $a_R^{(k)} \in \mathrm{ARef}_k$, a list of productions $\mathtt{P}(a_R^{(k)})$ defined by

1. $\mathtt{S}_i \to \mathtt{a}_i\mathtt{S}_{i+1}$ for each $1 \le i \le k$
2. $\mathtt{S}_i \to \mathtt{S}_{i'}$ if $(i,i') \in R$
3. $\mathtt{S}_{k+1} \to \varepsilon$,

and we construct the grammar $\mathbf{G}_R^{(k)} := (\mathtt{A}_k, \Gamma_k, \mathtt{S}_1, \mathtt{P}(a_R^{(k)}))$ with $\mathtt{A}_k := \{\mathtt{a}_i : 1 \le i \le k\}$ and $\Gamma_k := \{\mathtt{S}_i : 1 \le i \le k+1\}$.

Example 3. Let $a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}}$, we have

$$\mathtt{P}(a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}}) = \begin{cases} \mathtt{S}_1 \to \mathtt{a}_1\mathtt{S}_2\\ \mathtt{S}_1 \to \mathtt{S}_4\\ \mathtt{S}_2 \to \mathtt{a}_2\mathtt{S}_3\\ \mathtt{S}_2 \to \mathtt{S}_3\\ \mathtt{S}_3 \to \mathtt{a}_3\mathtt{S}_4\\ \mathtt{S}_3 \to \mathtt{S}_5\\ \mathtt{S}_4 \to \mathtt{a}_4\mathtt{S}_5\\ \mathtt{S}_4 \to \mathtt{S}_2\\ \mathtt{S}_5 \to \mathtt{a}_5\mathtt{S}_6\\ \mathtt{S}_6 \to \varepsilon. \end{cases}$$

We have

Lemma 1. *The language $\mathbb{L}(G_R^{(k)})$ denoted by the grammar $\mathbf{G}_R^{(k)}$ is regular.*

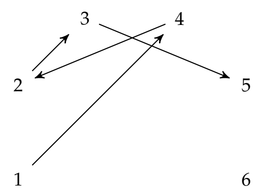
Proof. It suffices to remark that the language is recognized by the ε-automaton $\mathcal{A}(a_R^{(k)}) = (\Gamma_k, \mathtt{A}_k, \delta_R^{(k)}, \mathtt{S}_1, \{\mathtt{S}_{k+1}\})$ where the transitions $\delta_R^{(k)}$ are

$$\begin{array}{l}\mathtt{S}_i \xrightarrow{\mathtt{a}_i} \mathtt{S}_{i+1} \text{ for each } 1 \le i \le k\\ \mathtt{S}_i \xrightarrow{\varepsilon} \mathtt{S}_j \text{ for each } (i,j) \in R\end{array}$$

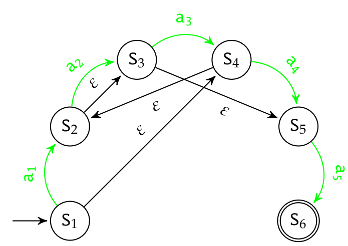
□

Note that the automaton $\mathcal{A}(a_R^{(k)})$ is just an reinterpretation of the relation R by adding transition.

Example 4. We obtain the automaton $\mathcal{A}(a^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}})$ from the graph of the relation $\{(1,4),(2,3),(3,5),(4,2)\}$



by adding transitions:



If $L_1, \ldots, L_k$ are k languages we define $\mathbf{G}_R^{(k)}(L_1, \ldots, L_k) = \mathbb{L}(\mathbf{G}_R^{(k)})|_{\mathsf{a}_i = L_i}$ that is the language $\mathbb{L}(G_R^{(k)})$ denoted by the grammar $\mathbf{G}_R^{(k)}$ where each letter a_i is replaced by the language L_i.

Example 5. Using the same relation than in Example 4 we find

$$\mathbb{L}(G_R^{(5)}) = (\mathsf{a}_1 + \varepsilon)(\mathsf{a}_3 + \mathsf{a}_2\mathsf{a}_3)^*(\mathsf{a}_5 + \mathsf{a}_2\mathsf{a}_5 + (\mathsf{a}_3 + \mathsf{a}_2\mathsf{a}_3)\mathsf{a}_4\mathsf{a}_5) + \mathsf{a}_4\mathsf{a}_5.$$

So if $L_1, \ldots, L_5$ are five languages:

$$G_R^{(5)}(L_1, \ldots, L_5) = (L_1 + \varepsilon)(L_3 + L_2L_3)^*(L_5 + L_2L_5 + (L_3 + L_2L_3)L_4L_5) + L_4L_5.$$

It is easy to see that this construction is compatible with the partial compositions:

$$\mathbf{G}_{R\circ_i R'}^{(k+k'-1)}(L_1, \ldots, L_{k+k'-1}) = \mathbf{G}_R^{(k)}(L_1, \ldots, L_{i-1}, \mathbf{G}_{R'}^{(k')}(L_i, \ldots, L_{i+k'-1}), L_{i+k'}, \ldots, L_{k+k'-1}), \tag{19}$$

for each $a_R^{(k)} \in \mathrm{ARef}_k$, $a_{R'}^{(k')} \in \mathrm{ARef}_{k'}$ and $i \leq k$. Indeed

$$\mathsf{P}(a_{R\circ_i R'}^{(k+k-1)}) = \begin{cases} \mathsf{S}_j \to \mathsf{a}_j\mathsf{S}_{j+1} \text{ for each } 0 \leq j \leq k + k' - 1 \\ \mathsf{S}_\ell \to \mathsf{S}_{\ell'} \text{ if } (\ell, \ell') = \overset{i,k'}{\diamond\!\!\to}((j, j')) \text{ for } (j, j') \in R \\ \mathsf{S}_\ell \to \mathsf{S}_{\ell'} \text{ if } (\ell, \ell') = \overset{0,i}{\diamond\!\!\to}((j, j')) \text{ for } (j, j') \in R' \\ \mathsf{S}_{k+k'} \to \varepsilon \end{cases}$$

Hence, we have

$$\begin{array}{rcl} \mathsf{P}(a_{R\circ_i R'}^{(k+k-1)}) & = & \{\mathsf{S}_j \to \mathsf{a}_j\mathsf{S}_{j+1} : 0 \leq j \leq k + k' - 1\} \cup \\ & & \{\mathsf{S}_\ell \to \mathsf{S}_{\ell'} : \mathsf{S}_j \to \mathsf{S}_{j'} \in \mathsf{P}(a_R^{(k)}) \text{ and } (\ell, \ell') = \overset{i,k'}{\diamond\!\!\to}((j, j))\} \cup \\ & & \{\mathsf{S}_\ell \to \mathsf{S}_{\ell'} : \mathsf{S}_j \to \mathsf{S}_{j'} \in \mathsf{P}(a_{R'}^{(k')}) \text{ and } (\ell, \ell') = \overset{0,i}{\diamond\!\!\to}((j, j))\} \\ & & \cup \{\mathsf{S}_{k+k'-1} \to \varepsilon\} \end{array}$$

We deduce that

$$\mathbb{L}(\mathbf{G}_{R\circ_i R'}^{(k+k'-1)}) = \mathbf{G}_R^{(k)}(\mathsf{a}_1, \ldots, \mathsf{a}_{i-1}, \mathbf{G}_{R'}^{(k')}(\mathsf{a}_i, \ldots, \mathsf{a}_{i+k'-1}), \mathsf{a}_{i+k'}, \ldots, \mathsf{a}_{k+k'-1})$$

which implies (19).

Remark 1. Alternatively, the construction can be described in terms of automata. The automaton $\mathcal{A}(a_{R\circ_i R'}^{(k+k'-1)})$ is obtained by replacing the transition $\mathsf{S}_i \xrightarrow{\mathsf{a}_i} \mathsf{S}_{i+1}$ in $\mathcal{A}(a_R^{(k)})$ by a copy of the automata $\mathcal{A}(a_{R'}^{(k')})$ and hence relabeling the vertices and edges.

Setting $a_R^{(k)}.(L_1,\dots,L_k) = \mathbf{G}_R^{(k)}(L_1,\dots,L_k)$ we define an action of the operad ARef on languages.

Theorem 4. *The sets 2^{Σ^*} and $Reg(\Sigma)$ are* ARef*-module.*

Proof. The fact that 2^{Σ^*} is a ARef-module is a direct consequence of (19).
Remarking that $\mathbb{L}(G_R^{(k)}) \in Reg(\mathtt{A}_k^*)$ (Lemma 1), we deduce that $\mathbf{G}_R^{(k)}(L_1,\dots,L_k) \in Reg(\Sigma)$ when $L_1,\dots,L_k \in Reg(\Sigma)$. Equivalently, $Reg(\Sigma)$ is ARef-module. □

Note that the action can be defined directly from $\mathcal{DT}$. Let $(a_{T_1}^{(k)}, a_{T_2}^{(k)}) \in \mathcal{DT}_n$ we construct the grammar $\mathbf{G}_{T_1,T_2}^{(k)} := (\mathtt{A}_k, \Gamma_k, \mathtt{S}_1, \mathtt{P}_{DT}(a_R^{(k)}))$ R where the production rules $\mathtt{P}_{DT}\left(a_R^{(k)}\right)$ are given by

1. $\mathtt{S}_i \to \mathtt{a}_i\mathtt{S}_{i+1}$ for each $1 \le i \le k$
2. $\mathtt{S}_i \to \mathtt{S}_{i'}$ if $(i', i-1) \in T_2$ or $(i, i'-1) \in T_1$
3. $\mathtt{S}_{k+1} \to \varepsilon$.

Example 6. Let $((13)(24)(34), (23)) \in \mathcal{DT}_5$. The grammar $\mathbf{G}^{(5)}_{(13)(22)(34),(23)}$ is given by

$$\left\{\begin{array}{lll}
\mathtt{S}_1 & \to & \mathtt{a}_1\mathtt{S}_2, \\
\mathtt{S}_1 & \to & \mathtt{S}_4, \\
\mathtt{S}_2 & \to & \mathtt{a}_2\mathtt{S}_3, \\
\mathtt{S}_2 & \to & \mathtt{S}_3, \\
\mathtt{S}_3 & \to & \mathtt{a}_3\mathtt{S}_4, \\
\mathtt{S}_3 & \to & \mathtt{S}_5, \\
\mathtt{S}_4 & \to & \mathtt{a}_4\mathtt{S}_5, \\
\mathtt{S}_4 & \to & \mathtt{S}_2, \\
\mathtt{S}_5 & \to & \mathtt{a}_5\mathtt{S}_6, \\
\mathtt{S}_6 & \to & \varepsilon
\end{array}\right. \tag{20}$$

Note that we recover the grammar $\mathbf{G}^{(5)}_{\{(1,4),(2,3),(3,5),(4,2)\}}$.
In general we have

Proposition 8. *For each $(a_{T_1}^{(k)}, a_{T_2}^{(k)}) \in \mathcal{DT}_n$, $\mathbf{G}_{T_1,T_2}^{(k)} = \mathbf{G}_{\xi(T_1,T_2)}^{(k)}$.*

6.2 Operadic expressions for regular languages

The following proposition shows that any regular language admits an expression involving an operator of ARef, symbols of the alphabet and $\emptyset$.

Proposition 9. *Each regular language $L \in Reg(\Sigma)$ can be written as*

$$L = a_R^{(k)}(\alpha_1,\dots,\alpha_k)$$

for some $k > 0$, $a_R^{(k)} \in \mathrm{ARef}_k$ and $\alpha_1,\dots,\alpha_k \in \{\{\mathtt{a}\} : \mathtt{a} \in \Sigma\} \cup \{\emptyset\}$.

Proof. First note that $\{\mathtt{a}\} = a_\emptyset^{(1)}(\{\mathtt{a}\})$, $\{\varepsilon\} = a^{(1)}_{\{(1,2)\}}(\emptyset)$ and $\emptyset = a_\emptyset^{(1)}(\emptyset)$.
Suppose now that $L, L' \in Ref(\Sigma)$ are two regular languages satisfying

$$L = a_R^{(k)}(\alpha_1,\dots,\alpha_k) \text{ and } L' = a_{R'}^{(k')}(\alpha'_1,\dots,\alpha'_{k'})$$

for some $k' > 0$, $a_R^{(k)} \in \mathrm{ARef}_k$, $a_{R'}^{(k')} \in \mathrm{ARef}_{k'}$ and $\alpha_1,\dots,\alpha_k,\alpha'_1,\dots,\alpha'_{k'} \in \{\{\mathtt{a}\} : \mathtt{a} \in \Sigma\} \cup \{\emptyset\}$. We have

1. $L + L' = a_{R''}^{(k+k'+1)}(\alpha_1,\dots,\alpha_k,\emptyset,\alpha'_1,\dots,\alpha'_{k'})$ with $R'' = R \cup \overset{0,k+1}{\diamond\!\!\to} R' \cup \{(1,k+2),(k+1,k+k'+2)\}$.
2. $LL' = a_{R''}^{(k+k'+1)}(\alpha_1,\dots,\alpha_k,\emptyset,\alpha'_1,\dots,\alpha'_{k'})$ with $R'' = R \cup \overset{0,k+1}{\diamond\!\!\to} R' \cup \{(k+1,k+2)\}$.
3. $L^* = a^{(k)}_{R\cup\{(k+1,1),(1,k+1)\}}(\alpha_1,\dots,\alpha_k)$.

The property is obtained by a straightforward induction. □

Remark 2. Note that in Formula 2 of the proof, the symbol $\emptyset$ is important for the computation of the catenation. For instance, we have $\mathtt{a}^+\mathtt{b}^+ = a^{(1)}_{\{(2,1)\}}(\mathtt{a})\cdot a^{(1)}_{\{(2,1)\}}(\mathtt{b}) = a^{(3)}_{\{(2,1),(2,3),(4,3)\}}(\mathtt{a},\emptyset,\mathtt{b}) \neq a^{(2)}_{\{(2,1),(3,2)\}}(\mathtt{a},\mathtt{b}) = (\mathtt{a}^+\mathtt{b}^+)^+$. But in some cases it may be omitted. For instance $a^{(1)}_{\{(1,2)\}}(\mathtt{a})\cdot a^{(1)}_{\{(1,2)\}}(\mathtt{b}) = a^{(2)}_{\{(1,2),(2,3)\}}(\mathtt{a},\mathtt{b}) = \varepsilon+\mathtt{a}+\mathtt{b}+\mathtt{ab}$.

Let us give few examples. First we illustrate the construction described in the proof of Proposition 9.

Example 7. Consider the languages $L = \mathtt{b}(\mathtt{ab}^*)+\mathtt{a}^*$. We have $\{\mathtt{a}\} = a^{(1)}_{\emptyset}(\mathtt{a})$, $\{\mathtt{b}\} = a^{(1)}_{\emptyset}(\mathtt{b})$. So $\mathtt{b}^* = a^{(1)}_{(2,1),(1,2)}(\mathtt{b})$, $\mathtt{ab}^* = a^{(3)}_{(4,3),(3,4)}(\mathtt{a},\emptyset,\mathtt{b})$ and $\mathtt{b}(\mathtt{ab}^*) = a^{(5)}_{(6,5),(5,6)}(\mathtt{b},\emptyset,\mathtt{a},\emptyset,\mathtt{b})$. On the other hand $\mathtt{a}^* = a^{(1)}_{(2,1),(1,2)}(\mathtt{a})$, hence$L = a^{(7)}_{(6,5),(5,6),(8,7),(7,8),(6,8),(1,7)}(\mathtt{b},\emptyset,\mathtt{a},\emptyset,\mathtt{b},\emptyset,\mathtt{a})$.

Manipulating the relations allows to obtain some languages from others. We give here few constructions.

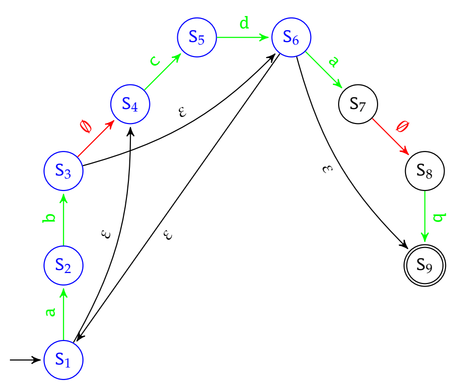
Example 8.

- Consider a language $L = a^{(k)}_R(\alpha_1,\dots,\alpha_k)$ with $R \in \mathrm{ARef}_k$ and $\alpha_i \in \{\{\mathtt{a}\} : \mathtt{a}\in\Sigma\}$. We define $R_P := R \cup \{(i,k+1) : 1 \le i \le k\}$. The language $a^{(k)}_{R_P}(\alpha_1,\dots,\alpha_k)$ is the set of the prefixes of L.

  For instance, consider $L = a^{(3)}_{(4,1),(1,4)}(\mathtt{a},\mathtt{b},\mathtt{c}) = (\mathtt{abc})^*$ we have $L = a^{(3)}_{(4,1),(1,4),(2,4),(3,4)}(\mathtt{a},\mathtt{b},\mathtt{c}) = (\mathtt{abc})^*\{\varepsilon,\mathtt{a},\mathtt{ab}\}$.
- For a more general regular language L, Proposition 9 implies that there exists $k>0$, $R\in\mathrm{ARef}_k$ and $\alpha_i \in \{\{\mathtt{a}\} : \mathtt{a}\in\Sigma\}\cup\{\emptyset\}$ satisfying $L = a^{(k)}_R(\alpha_1,\dots,\alpha_k)$. An *admissible position* is an integer $1 \le i \le k+1$ such that there exists a path $i_1 = 1 \xrightarrow{\beta_1} i_2 \xrightarrow{\beta_2} i_3 \cdots i_{p-1} \xrightarrow{\beta_p} i_p = i_{k+1}$ in $\mathcal{A}(a^{(k)}_R)$ with either $\beta_i = \varepsilon$ either $\beta_i = \mathtt{a}_i$ with $\alpha_i \neq \emptyset$ such that $i_\ell = i$ for some $1 \le \ell \le p-1$. The set of admissible positions is denoted by $\mathrm{Adm}(R;\alpha_1,\cdots,\alpha_k)$. We define $R_P := R\cup\{(i,k+1) : i \in \mathrm{Adm}(R;\alpha_1,\cdots,\alpha_k), i\neq k+1\}$. The language $a^{(k)}_{R_P}(\alpha_1,\dots,\alpha_k)$ is the set of the prefixes of L.

  For instance consider $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b})$. We have $L = (\mathtt{ab}+\mathtt{cd})^+$,

$$\mathcal{A}(a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}) =$$



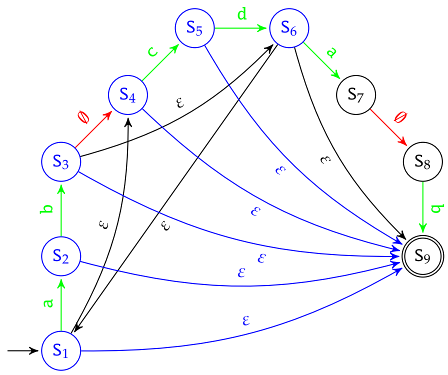
  and

$$\mathrm{Adm}(\{(1,4),(3,6),(6,1),(6,9)\};\mathtt{a},\mathtt{b},\emptyset,\mathtt{c},\mathtt{d},\mathtt{a},\emptyset,\mathtt{b}) = \{1,2,3,4,5,6\}.$$

  So $R_P = \{(1,4),(3,6),(6,1),(6,9),(1,9),(2,9),(3,9),(4,9),(5,9)\}$. We verify $a^{(8)}_{R_P}(\mathtt{a},\mathtt{b},\emptyset,\mathtt{c}) = (\mathtt{ab}+$

cd)*(ε + a + c) = Pref(L). Graphically

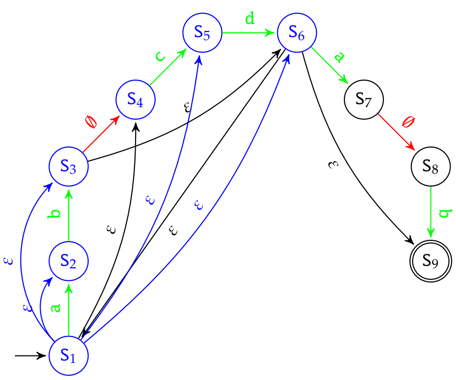
$\mathcal{A}(a^{(8)}_{R_p})$ =



Indeed the language recognized by this automaton is $(\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5 + \mathsf{a}_1\mathsf{a}_2 + \mathsf{a}_4\mathsf{a}_5)^*(\varepsilon + \mathsf{a}_1 + \mathsf{a}_1\mathsf{a}_2 + \mathsf{a}_1\mathsf{a}_2\mathsf{a}_3 + \mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4 + \mathsf{a}_4 + (\mathsf{a}_1\mathsf{a}_2\mathsf{a}_3\mathsf{a}_4\mathsf{a}_5 + \mathsf{a}_1\mathsf{a}_2 + \mathsf{a}_4\mathsf{a}_5)(\varepsilon + \mathsf{a}_6\mathsf{a}_7\mathsf{a}_8))$. Setting $\mathsf{a}_i = \alpha_i$ in this expression, we find $(\mathsf{ab} + \mathsf{cd})^*(\varepsilon + \mathsf{a} + \mathsf{ab} + \mathsf{c} + (\mathsf{ab} + \mathsf{cd})) = (\mathsf{ab} + \mathsf{cd})^*(\varepsilon + \mathsf{a} + \mathsf{c})$ as expected.

- Symmetrically, the language of the suffixes of L is obtained by considering the relation $R_S := R \cup \{(1, i) : i \in \mathrm{Adm}(R; \alpha_1, \cdots, \alpha_k),\ i \neq 1\}$. From the example above we obtain $R_S = \{(1,4),(3,6),(6,1),(6,9),(1,2),(1,3),(1,4),(1,5),(1,6)\}$. Graphically:

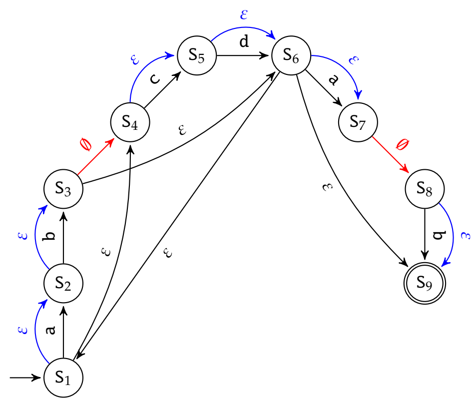
$\mathcal{A}(a^{(8)}_{R_S})$ =



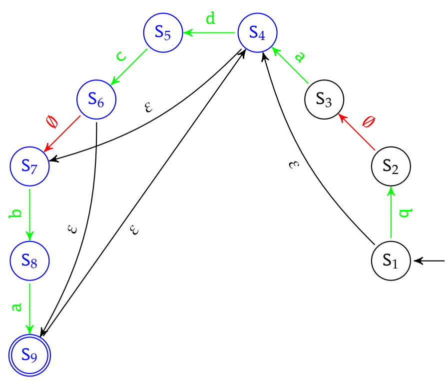
- The language of the factors of L is obtained by first computing the prefixes and hence the suffixes. Applying this construction to $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathsf{a}, \mathsf{b}, \emptyset, \mathsf{c}, \mathsf{d}, \mathsf{a}, \emptyset, \mathsf{b})$, we find that the set of the factors of L is denoted by $a^{(8)}_{R_F}(\mathsf{a}, \mathsf{b}, \emptyset, \mathsf{c}, \mathsf{d}, \mathsf{a}, \emptyset, \mathsf{b})$ with

$$R_F = \{(1,4),(3,6),(6,1),(6,9),(1,9),(2,9),(3,9),(4,9),(5,9),(1,2),(1,3),(1,4),(1,5),(1,6)\}.$$

- The subwords of L are denoted by the expressions $a^{(k)}_S(\alpha_1, \ldots, \alpha_k)$ where $S = R \cup \{(i, i+1) : \alpha_i \neq \emptyset\}$. Applying the construction to $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathsf{a}, \mathsf{b}, \emptyset, \mathsf{c}, \mathsf{d}, \mathsf{a}, \emptyset, \mathsf{b})$, the language of the subwords of L is $a^{(8)}_{(1,4),(3,6),(6,1),(6,9),(1,2),(2,3),(4,5),(5,6),(6,7),(8,9)}(\mathsf{a}, \mathsf{b}, \emptyset, \mathsf{c}, \mathsf{d}, \mathsf{a}, \emptyset, \mathsf{b})$. The associated automaton is

$\mathcal{A}(a_S^{(8)}) =$ 

- The mirror image of L is obtained by computing $a_M^{(k)}(\alpha_k, \dots, \alpha_1)$ where $M = \{(k+2-j, k+2-i) : (i,j) \in R\}$. Let us again illustrate the construction on $L = a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{a}, \mathtt{b}, \emptyset, \mathtt{c}, \mathtt{d}, \mathtt{a}, \emptyset, \mathtt{b})$. The mirror image of L is $a^{(8)}_{(1,4),(3,6),(6,1),(6,9)}(\mathtt{b}, \emptyset, \mathtt{a}, \mathtt{d}, \mathtt{c}, \emptyset, \mathtt{b}, \mathtt{a})$. Graphically:

$\mathcal{A}(a_M^{(8)}) =$ 

The language recognized by $\mathcal{A}(a_M^{(8)})$ is $(\varepsilon + \mathtt{a}_1\mathtt{a}_2\mathtt{a}_3)(\mathtt{a}_4\mathtt{a}_5(\varepsilon + \mathtt{a}_6\mathtt{a}_7\mathtt{a}_8) + \mathtt{a}_7\mathtt{a}_8)^+$. Specializing to $\mathtt{a}_1 = \mathtt{b}$, $\mathtt{a}_2 = \emptyset$, $\mathtt{a}_3 = \mathtt{a}$, $\mathtt{a}_4 = \mathtt{d}$, $\mathtt{a}_5 = \mathtt{c}$, $\mathtt{a}_6 = \emptyset$, $\mathtt{a}_7 = \mathtt{b}$ and $\mathtt{a}_8 = \mathtt{a}$, we recover the language $(\mathtt{dc} + \mathtt{ba})^+$ that is the mirror image of L.

Few other examples:

Example 9. Let $\mathtt{a}_1, \dots, \mathtt{a}_k$ be k letters. We have

- $a^{(k)}_{\{(k+1,1),(1,k+1)\}}(\mathtt{a}_1, \dots, \mathtt{a}_k) = (\mathtt{a}_1 \cdots \mathtt{a}_k)^*$.
- $a^{(k)}_{\{(i,j):i \neq j\}}(\mathtt{a}_1, \dots, \mathtt{a}_k) = (\mathtt{a}_1 + \cdots + \mathtt{a}_k)^*$.
- $a^{(k)}_{\{(k+1,1)\} \cup \{(i,k+1):1 \leq i \leq k\}}(\mathtt{a}_1, \dots, \mathtt{a}_k) = (\mathtt{a}_1 + \mathtt{a}_1\mathtt{a}_2 + \cdots + \mathtt{a}_1 \cdots \mathtt{a}_k)^*$.
- $a^{(k)}_{\{(k+1,1)\} \cup \{(1,i+1):1 \leq i \leq k\}}(\mathtt{a}_1, \dots, \mathtt{a}_k) = (\mathtt{a}_k + \mathtt{a}_{k-1}\mathtt{a}_k + \cdots + \mathtt{a}_1 \cdots \mathtt{a}_k)^*$.
- $a^{(k)}_{\{(i+1,i):1 \leq i \leq k\}}(\mathtt{a}_1, \dots, \mathtt{a}_k) = \{w \in \{\mathtt{a}_1, \dots, \mathtt{a}_k\}^* : w = \mathtt{a}_1 w' \mathtt{a}_k \text{ and } w = u\mathtt{a}_i\mathtt{a}_j v \text{ implies } j \leq i+1\}$.

6.3 Action of $QOSet$

Let $a_R^{(k)} \in \mathrm{ARef}_k$. If we compare the grammars $G_R^{(k)}$ and $G_{\gamma R}^{(k)}$ we observe that $S_i \to S_\ell \in P(a_{\gamma R}^{(k)})$ implies there exists $i_1 = i, i_2, \dots, i_p = \ell$ such that $S_{i_h} \to S_{h+1} \in P(a_R^{(k)})$ for each $1 \leq h < \ell$. Hence, the languages $\mathbb{L}(G_R^{(k)})$ and $\mathbb{L}(G_{\gamma R}^{(k)})$ are equal.

Example 10. Consider $R = \{(1,2),(2,3)\}$, we have $\tilde{\gamma}(R) = \{(1,2),(2,3),(1,3)\}$. We have

$$P(a_R^{(2)}) = \begin{cases} \mathsf{S}_1 \to \mathsf{a}_1\mathsf{S}_2 \\ \mathsf{S}_1 \to \mathsf{S}_2 \\ \mathsf{S}_2 \to \mathsf{a}_2\mathsf{S}_3 \\ \mathsf{S}_2 \to \mathsf{S}_3 \\ \mathsf{S}_3 \to \varepsilon \end{cases} \quad \text{and } P(a_{\tilde{\gamma}R}^{(2)}) = \begin{cases} \mathsf{S}_1 \to \mathsf{a}_1\mathsf{S}_2 \\ \mathsf{S}_1 \to \mathsf{S}_2 \\ \mathsf{S}_1 \to \mathsf{S}_3 \\ \mathsf{S}_2 \to \mathsf{a}_2\mathsf{S}_3 \\ \mathsf{S}_2 \to \mathsf{S}_3 \\ \mathsf{S}_3 \to \varepsilon \end{cases}$$

Hence, $\mathbb{L}(G_R^{(k)}) = \{\varepsilon, \mathsf{a}_1, \mathsf{a}_1\mathsf{a}_2, \mathsf{a}_2\} = \mathbb{L}(G_{\tilde{\gamma}R}^{(k)})$.

This allows to consider the action of $\mathrm{OP}(\diamondsuit)$ defined by $a_{[R]}^{(k)}(L_1,\dots,L_k) := a_R^{(k)}(L_1,\dots,L_k)$. Alternatively, the action of QOSet is defined by $Q(L_1,\dots,L_k) = a_{Q\setminus\Delta}^{(k)}(L_1,\dots,L_k)$. Observing that the operads Q*OSet* and $\mathrm{OP}(\diamondsuit)$ are isomorphic and that the isomorphism η satisfies $\eta(Q)(L_1,\dots,L_k) = a_{[Q\setminus\Delta]}^{(k)}(L_1,\dots,L_k) = a_{Q\setminus\Delta}^{(k)}(L_1,\dots,L_k) = Q(L_1,\dots,L_k)$, the action of QOSet is compatible with the partial compositions. Hence, Theorem 4 implies

Corollary 3. *The sets 2^{Σ^*} and $Reg(\Sigma)$ are* QOSet*-module.*

Now, we prove that the operad QOSet is optimal in the sense that two different operators act in two different ways on regular languages. That is:

Theorem 5. *$Reg(\Sigma)$ is a faithful* QOSet*-module.*

Proof. Let $Q_1 \neq Q_2 \in \mathrm{QOSet}_k$ be two quasiorders. Without loss of generalities, we suppose that there exists $(i,j) \in Q_1$ such that $(i,j) \notin Q_2$. The constructions above show that the word $\mathsf{a}_1 \dots \mathsf{a}_{i-1}\mathsf{a}_j\mathsf{a}_{j+1}\dots\mathsf{a}_k$ belongs to $Q_1(\{a_1\},\dots,\{a_k\})$ but not to $Q_2(\{a_1\},\dots,\{a_k\})$. This shows the result. □

Note that the number of elements of QOSet_k is known up to $k = 17$ (see [14] `sequence A000798`):

$$4, 29, 355, 6942, 209527, 9535241, 642779354, 63260289423, \dots$$

Example 11.

- Let us examine the four operators of QOSet_1:

  $Q_1 = \{(1,1),(2,2)\}, Q_2 = \{(1,1),(1,2),(2,2)\}, Q_3 = \{(1,1),(2,1),(2,2)\}, Q_4 = \{(1,1),(1,2),(2,1),(2,2)\}$,

  The four languages are $Q_1(\mathsf{a}_1) = \mathsf{a}_1$, $Q_2(\mathsf{a}_1) = \varepsilon + \mathsf{a}_1$, $Q_3 = \mathsf{a}_1^+$ and $Q_4 = \mathsf{a}_1^*$.
- Let us examine the 29 operators of QOSet_2:

$Q\setminus\Delta$	$Q(\mathsf{a}_1,\mathsf{a}_2)$	$Q\setminus\Delta$	$Q(\mathsf{a}_1,\mathsf{a}_2)$	$Q\setminus\Delta$	$Q(\mathsf{a}_1,\mathsf{a}_2)$
$\emptyset$	$\mathsf{a}_1\mathsf{a}_2$	$\{(1,2)\}$	$\mathsf{a}_2 + \mathsf{a}_1\mathsf{a}_2$	$\{(1,3)\}$	$\varepsilon + \mathsf{a}_1\mathsf{a}_2$
$\{(2,3)\}$	$\mathsf{a}_1 + \mathsf{a}_1\mathsf{a}_2$	$\{(2,1)\}$	$\mathsf{a}_1^+\mathsf{a}_2$	$\{(3,1)\}$	$(\mathsf{a}_1\mathsf{a}_2)^+$
$\{(3,2)\}$	$\mathsf{a}_1\mathsf{a}_2^+$	$\{(1,2),(2,1)\}$	$\mathsf{a}_1^*\mathsf{a}_2$	$\{(1,3),(3,1)\}$	$(\mathsf{a}_1\mathsf{a}_2)^*$
$\{(2,3),(3,2)\}$	$\mathsf{a}_1\mathsf{a}_2^*$	$\{(1,2),(3,2)\}$	$(\varepsilon + \mathsf{a}_1)\mathsf{a}_2^+$	$\{(2,1),(2,3)\}$	$a_1^+(\varepsilon + \mathsf{a}_2)$
$\{(1,3),(2,3)\}$	$(\varepsilon + \mathsf{a}_1 + \mathsf{a}_2)$	$\{(3,1),(3,2)\}$	$(\mathsf{a}_1\mathsf{a}_2^+)^+$	$\{(3,1),(2,1)\}$	$(\mathsf{a}_1^+\mathsf{a}_2)^+$
$\{(1,3),(1,2)\}$	$\varepsilon + \mathsf{a}_2 + \mathsf{a}_1\mathsf{a}_2$	$\{(1,2),(2,3),(1,3)\}$	$\varepsilon + \mathsf{a}_1 + \mathsf{a}_2 + \mathsf{a}_1\mathsf{a}_2$	$\{(2,1),(3,2),(3,1)\}$	$(\mathsf{a}_1^+\mathsf{a}_2^+)^+$
$\{(1,3),(3,2),(1,2)\}$	$(\varepsilon + \mathsf{a}_1)\mathsf{a}_2^+ + \varepsilon$	$\{(3,1),(2,3),(2,1)\}$	$(\mathsf{a}_1^+(\varepsilon + \mathsf{a}_2))^+$	$\{(2,1),(1,3),(2,3)\}$	$\varepsilon + \mathsf{a}_1^+(\varepsilon + \mathsf{a}_2)$
$\{(1,2),(3,1),(3,2)\}$	$((\mathsf{a}_1 + \varepsilon)\mathsf{a}_2^+)^+$				

$Q\setminus\Delta$	$Q(\{\mathsf{a}_1\},\{\mathsf{a}_2\})$	$Q\setminus\Delta$	$Q(\{\mathsf{a}_1\},\{\mathsf{a}_2\})$
$\{(1,2),(2,1),(2,3),(1,3)\}$	$\varepsilon + \mathsf{a}_1^*\mathsf{a}_2$	$\{(1,2),(2,1),(3,2),(3,1)\}$	$(\mathsf{a}_1^*\mathsf{a}_2^+)^+$
$\{(1,3),(3,1),(1,2),(3,2)\}$	$((\varepsilon + \mathsf{a}_1)\mathsf{a}_2^+)^*$	$\{(1,3),(3,1),(2,1),(2,3)\}$	$(\mathsf{a}_1^2(\varepsilon + \mathsf{a}_2))^*$
$\{(2,3),(3,2),(2,1),(3,1)\}$	$(\mathsf{a}_1^+\mathsf{a}_2^*)^+$	$\{(2,3),(3,2),(1,2),(1,3)\}$	$(\varepsilon + \mathsf{a}_1)\mathsf{a}_2^*$
$\{(1,2),(1,3),(2,3),(2,1),(2,3),(3,1)\}$	$(\mathsf{a}_1 + \mathsf{a}_2)^*$		

We illustrate the proof of Theorem 5. Remarking that $(3,2) \in \{(2,3),(3,2),(2,1),(3,1)\}$, $(3,2) \notin \{(2,1),(1,3),(2,3)\}$, we have $\mathsf{a}_1\mathsf{a}_2\mathsf{a}_2 \in (\mathsf{a}_1^+\mathsf{a}_2^*)^+ = \{(2,3),(3,2),(2,1),(3,1)\}(\mathsf{a}_1,\mathsf{a}_2)$ and $\mathsf{a}_1\mathsf{a}_2\mathsf{a}_2 \notin \varepsilon + \mathsf{a}_1^+(\varepsilon + \mathsf{a}_2) = \{(2,1),(1,3),(2,3)\}(\mathsf{a}_1,\mathsf{a}_2)$.

6.4 Back to (simple) multi-tildes

The purpose of this section is to show that the restriction of the action to (simple) multi-tildes is compatible with the action described in [10]. In this paper, the action of multi-tildes involve another operad: the operad of sets of boolean vectors $\mathcal{B} = \bigcup_n \mathcal{B}_n$ with $\mathcal{B}_n = 2^{\mathbb{B}^n}$ and $\mathbb{B} = \{0,1\}$. The composition is defined by

$$E \circ_k F = \{[e_1, \ldots, e_{k-1}, e_k f_1, \ldots, e_k f_n, e_{k+1}, \ldots, e_m] : [e_1, \ldots, e_m] \in E, [f_1, \ldots, f_n] \in F\}$$

for $E \in \mathcal{B}_m$ and $F \in \mathcal{B}_n$. The action on the languages is defined by

$$E(L_1, \ldots, L_m) = \bigcup_{[e_1,\ldots,e_m]\in E} L_1^{e_1} \cdots L_m^{e_m}.$$

We denote $[x,z] = \{y : x \leq y \leq z\}$. For each $T \in \mathcal{T}_k$ we set $\mathcal{F}(T) = \{S \subset T : (x,y),(z,t) \in S$ implies $[x,y] \cup [z,t] = \emptyset\}$. Finally we define $V(T) = \{v(S) : S \in \mathcal{F}(T)\}$ with $v(S) = (v_1, \ldots, v_k)$ where $v_j = 0$ if $j \in \bigcup_{(x,y)\in S}[x,y]$ and 1 otherwise. In [10] we proved that V is an operadic morphism and defined the action $T(L_1, \ldots, L_k) = V(T)(L_1, \ldots, L_k)$.
Remark that $\mathcal{T}$ is isomorphic to the suboperad of $\mathcal{DT}$ generated by $(a_T^{(k)}, a_\emptyset^{(k)})$ (the isomorphism sends each T to $(a_T^{(k)}, a_\emptyset^{(k)})$. So we have to prove that $T(L_1, \ldots, L_k) = (a_T^{(k)}, a_\emptyset^{(k)})(L_1, \ldots, L_k)$. Equivalently,

$$T(\mathsf{a}_1, \ldots, \mathsf{a}_k) = \mathbb{L}(\mathbf{G}_{T,\emptyset})(\mathsf{a}_1, \ldots, \mathsf{a}_k).$$

To this aim, we associate a set of boolean vectors to each grammar $\mathbf{G}_{T,\emptyset}$ in the following way: we consider the grammar $\mathbf{G}_{0,1}(T)$ which is obtained from $\mathbf{G}_{T,\emptyset}$ by substituting to each rule $\mathsf{S}_i \to \mathsf{a}_i\mathsf{S}_{i+1}$ the rule $\mathsf{S}_i \to 1\mathsf{S}_{i+1}$ and to each rule $\mathsf{S}_i \to \mathsf{S}_j$ the rule $\mathsf{S}_i \to 0^{j-i}\mathsf{S}_j$. Denote $\mathbb{L}_{0,1}(T) = \mathbb{L}(\mathbf{G}_{0,1}(T))$. Each word of $\mathbb{L}_{0,1}(T)$ has a length equal to k. Remark that

$$\mathbb{L}(\mathbf{G}_{T,\emptyset})(\mathsf{a}_1, \ldots, \mathsf{a}_k) = \{\mathsf{a}_1^{e_1} \cdots \mathsf{a}_k^{e_k} : e_1 \ldots e_k \in \mathbb{L}_{0,1}(T)\}.$$

Assimilating each word $e_1 \ldots e_k \in \mathbb{L}_{0,1}(T)$ to the boolean vector $(e_1, \ldots, e_k)$ we prove the following result:

Proposition 10. *Let $a_T^{(k)} \in \mathcal{T}_k$, we have $a_T^{(k)}(L_1, \ldots, L_k) = (a_T^{(k)}, a_\emptyset^{(k)})(L_1, \ldots, L_k)$.*

Proof. Let us first recall that a *closed* multitilde is a multitilde T satisfying

$$(i,j),\ (j+1,\ell) \in T \Rightarrow (i,\ell) \in T.$$

The *normal form* $\widetilde{T}$ of a multitilde T is the smallest closed multitilde containing T as a subset (see *e.g.* [3]). From the definition of the action of $\mathcal{T}$, we have $a_T^{(k)}(L_1, \ldots, L_k) = a_{\widetilde{T}}^{(k)}(L_1, \ldots, L_k)$. From the construction of $\mathbf{G}_{T,\emptyset}$ we observe that $\mathbb{L}_{0,1}(\tilde{T}) = \mathbb{L}_{0,1}(T)$. Indeed, it suffices to remark that one can add the rule $\mathsf{S}_i \to 0^{l-i}\mathsf{S}_i$ in $\mathbf{G}_{0,1}(T)$, when $\mathsf{S}_i \to 0^{j-i}\mathsf{S}_j$ and $\mathsf{S}_j \to 0^{l-j}$ are two rules of $\mathbf{G}_{0,1}(T)$, without modifying the language.
So, we have to prove $a_T^{(k)}(L_1, \ldots, L_k) = (a_T^{(k)}, a_\emptyset^{(k)})(L_1, \ldots, L_k)$ for any closed multitilde T. That is $v = 0^{i_1}10^{i_2} \cdots 10^{i_p} \in V(T)$ (considering the vector as a word) if and only if $v \in \mathbb{L}_{0,1}(T)$. The case when $p = 1$ means that $v = 0^{i_1} = 0^k$. For convenience, we set $i_0 = 1$. Obviously $(i_0, i_1), (i_0 + i_1 + 1, i_0 + i_1 + i_2 + 1), \ldots, (i_0 + i_1 + \cdots + i_{\ell-1} + 2(\ell - 1) + 1, i_0 + i_1 + \cdots + i_\ell + 2(\ell - 1) + 1) \in T$ if and only if $\mathsf{S}_1 \xrightarrow{*} 0^{i_1}1 \ldots 10^{i_\ell}\mathsf{S}_{i_0+i_1+\cdots+i_\ell+2\ell}$ for any $0 \leq \ell \leq p$ (here $E \xrightarrow{*} w$ means that we can produce the word w from E by applying a finite sequence of rules). Equivalently $v \in V(T)$ if and only if $\mathsf{S}_1 \xrightarrow{*} v\mathsf{S}_{k+1} \to v$. This proves the result. □

7 Conclusion and perspectives

We have described a faithful action of a combinatorial operad on regular languages. This means that we describe countable operations providing a new kind of expressions for denoting regular languages. One of the interest of the construction is that we propose expressions which are close to the representation by automata. The obtained expressions are more expressive in the sense that most of the complexity of the denoted language is concentrated at the operator. So this allows to define several measures of the complexity of a language. For instance, let us define $rk_w(L) = \min\{k : \exists Q \in \text{QOSet}_k,\ \alpha_1, \ldots, \alpha_k \in \Sigma \cup \{\emptyset\}$ such that $L = Q(\alpha_1, \ldots, \alpha_k)\}$ and $rk_h(L) = \min\{h : \exists k \geq 1, O \in \mathcal{DT}_k,\ \alpha_1, \ldots, \alpha_k \in \Sigma \cup \{\emptyset\}$ such that $L = O(\alpha_1, \ldots, \alpha_k)$ and $\#O = h\}$. The two ranks rk_w and rk_h can be respectively interpreted as the width and the height of a language. The first one (rk_w) is the minimal number of occurrences of symbols or $\emptyset$ in the expression. The rank rk_h expresses the minimal complexity of an operator involved for denoting the languages. These measures will be investigated; in particular a parallel with the size of a minimal (in terms of states or transitions) automaton should be established.

The operads considered in this paper are `SET`-operads, that are operads that can be constructed from the category `SET`. We can also consider linear combinations of operators which consists to use `VECT`-operads based on the category of the vector spaces. By this way, we expect to construct an adapted notion of multitildes for rational series.

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