1 Sturm–Liouville

Differential operators (linear) single variable $x \in \mathbb{R}$

$$\mathcal{L} = a_n(x)\frac{\partial^n}{\partial x^n} + \dots + a_1(x)\frac{\partial}{\partial x} + a_0(x)$$

acts on a suff. smooth function $y(x)$
producing a new function $(\mathcal{L}y)(x)$

many physics-derived problems have second order diff op's (as accel. & velocity)

$$\mathcal{L} = P(x)\frac{\partial^2}{\partial x^2} + R(x)\frac{\partial}{\partial x} - Q(x) \qquad (*)$$

homog. eq. $\mathcal{L}y = 0$ two non-trivial lin indep solutions $y_1(x)$ $y_2(x)$

gen. sol'n $a y_1(x) + b y_2(x)$

Self adjoint differential operators

Sturm Liouville operators

$$\mathcal{L}y := \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)\,y \qquad (**)$$

y may be $\mathbb{C}$-valued $\Big\{$ $p(x)$ real and at least once differentiable; $q(x)$ real and at least continuous

no more restrictive than $(*)$: if $P(x) \neq 0$ on domain

$$\frac{d^2}{dx^2} + \frac{R(x)}{P(x)}\frac{d}{dx} - \frac{Q(x)}{P(x)} = e^{-\int_0^x \frac{R(t)}{P(t)}dt}\frac{d}{dx}\left(e^{\int_0^x \frac{R(t)}{P(t)}dt}\frac{d}{dx}\right) - Q(x)/P(x)$$

[2] with $p(x) = e^{\int_0^x R(t)/P(t)\, dt}$ and $q(x) = \frac{Q(x)\, p(x)}{P(x)}$ ✻

get ✻ is of form ✻✻

so general case

Advantage of form ✻✻ : operator $\mathcal{L}$ is self adjoint

$$\langle f, g\rangle = \int_a^b \overline{f(x)}\, g(x)\, dx$$

if use right boundary conditions at a, b

$$\langle \mathcal{L}f, g\rangle = \int_a^b \left(\frac{d}{dx}\left(p(x) \frac{d\bar{f}}{dx}\right) - q(x) \bar{f}(x)\right) g(x)\, dx$$

$$= p(x) \frac{d\bar{f}}{dx} \Big|_a^b - \int_a^b p(x) \frac{d\bar{f}}{dx} \frac{dg}{dx} - q(x) \bar{f}(x) g(x)\, dx$$

$$= \left(p(x) \frac{d\bar{f}}{dx} - p(x) \bar{f} \frac{dg}{dx}\right)\Big|_a^b + \int_a^b \left(\frac{d}{dx}\left(p(x) \frac{dg}{dx}\right) - q(x) g(x)\right) \bar{f}(x)\, dx$$

$$= \left(p(x) \frac{d\bar{f}}{dx} - p(x) \bar{f} \frac{dg}{dx}\right)\Big|_a^b + \langle f, \mathcal{L}g\rangle$$

so boundary condition that makes $\mathcal{L}$ self-adj (symmetric)

is

✻✻✻ $$p(x)\left(\frac{d\bar{f}}{dx} g(x) - \bar{f}(x) \frac{dg}{dx}\right)\Big|_a^b = 0$$

3

examples of boundary conditions that satisfy this are

$(b_1, b_2) \neq (0,0)$
$(c_1, c_2) \neq (0,0)$

ex 1:
$$b_1 f'(a) + b_2 f(a) = 0$$
$$c_1 f'(b) + c_2 f(b) = 0$$

for nontrivial b_i, c_i (same b_i & c_i for all f's in space of functions) $\Rightarrow$ both f & g

(***) vanishes separately at a & b

ex 2: if $p(a) = p(b)$ then periodic $f(a) = f(b)$, $f'(a) = f'(b)$ also works } all functs

ex 3: $p(a) = p(b) = 0$
special case (singular)

Weight functions: instead of inner product
$$\langle f, g \rangle = \int_a^b \bar{f}(x)\, g(x)\, dx$$

consider more general inner products with a weight function $w(x) > 0$ ← more gen w only fin many zeros in $[a,b]$

$$\langle f, g \rangle = \int_a^b \bar{f}(x)\, g(x)\, w(x)\, dx$$

Examples:
with $w(x) \equiv 1$ and $[a,b] = [0,\pi]$ the functions

$f_n(x) = \sin(nx)$, $n \in \mathbb{N}$ are pairwise orthogonal

4 Example: $f_0(x)=1$, $f_1(x)=2x$, $f_2(x)=4x^2-1$

$f_3(x)=8x^3-4x$ $\quad f_4(x)=16x^4-12x^2+1$

$f_5(x)=32x^5-32x^3+6x$ ... orthogonal

with $w(x)=\sqrt{1-x^2}$ and $[a,b]=[-1,1]$

(Chebyshev polynomials of second kind)

Consider the eigenvalue problem for $\mathcal{L}$

but written in the form (weighted)

$$\mathcal{L}y(x)=\lambda\, w(x)\, y(x)$$

note: if $y(x)$ an eigenfunction of this problem (have $w(x)>0$) then

$\tilde{y}(x)=\sqrt{w(x)}\cdot y(x)$ eigenfunction of usual eigenvalue problem for

$$\tilde{\mathcal{L}}\tilde{y}=\lambda\tilde{y} \qquad \tilde{\mathcal{L}}\tilde{y}=\frac{1}{\sqrt{w}}\mathcal{L}\left(\frac{\tilde{y}}{\sqrt{w}}\right)$$

- if $\mathcal{L}y=\lambda w y$ then

$$\lambda\langle f,f\rangle_w=\langle f,\lambda w f\rangle=\langle f,\mathcal{L}f\rangle=\langle\mathcal{L}f,f\rangle$$

$$=\bar{\lambda}\langle f,f\rangle_w \implies \lambda\in\mathbb{R}$$

weighted inn prod

- then if y eigenfunction $\bar{y}$ also (p,q,w real, λ real)

[5]

- eigenfunctions w/ different eigenvalue $\lambda_i \neq \lambda_j$ but same weight are orthogonal (w-inn prod)

$$\lambda_j < y_i, y_j >_w = < y_i, \mathcal{L} y_j > = < \mathcal{L} y_i, y_j >$$
$$= \lambda_i < y_i, y_j >_w$$

$$\lambda_j \neq \lambda_i \Rightarrow < y_i, y_j >_w = 0$$

- orthonormal eigenfunctions

$$Y_i(x) = y_i(x) \Big/ \underbrace{\sqrt{\int_a^b |y_i(x)|^2 w(x) dx}}_{= \| y_i \|_w}$$

weighted norm

in particular interested in cases where $\exists$ sequence of eigenvalues

$$\lambda_1, \lambda_2, \ldots, \lambda_n, \ldots \qquad |\lambda_n| \to \infty$$

and set $\{ Y_i \}_{i \geq 1}$ of orthonormal eigenfunctions

that give a basis of funct. space w/ a partic. choice of boundary cond. at a, b

i.e. can expand functions

$$f(x) = \sum_{n=1}^{\infty} a_n Y_n(x)$$ with

generalization of Fourier series

$$a_n = \int_a^b \overline{Y_n(x)} f(x) w(x) dx = < Y_n, f >_w$$

[6] eigenvalue / eigenfunction problem
is problem of solution of "regular Sturm Liouville problem"

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + (\lambda w(x) - q(x))y = 0 \quad \text{for } x \in [a,b]$$

$$\begin{cases} b_1 \, y(a) + b_2 y'(a) = 0 \\ c_1 \, y(b) + c_2 y'(b) = 0 \end{cases}$$

$(b_1, b_2) \neq (0,0)$
$(c_1, c_2) \neq (0,0)$
p, p', q, w contin on $[a,b]$
$p, w > 0$ on $[a,b]$

"separated" boundary conditions

Example of regular Sturm-Liouville problem:

$$y'' + \lambda y = 0 \qquad 0 < x < L$$

$$y(0) = y(L) = 0 \qquad \left(\begin{array}{c} p(x) = w(x) \equiv 1 \\ q(x) = 0 \end{array}\right)$$

then $\lambda_n = \dfrac{n^2\pi^2}{L^2}$ and

$$y_n(x) = \sin\frac{n\pi x}{L} \qquad \text{eigenfunctions } n = 1, 2, 3, \ldots$$

---

other example: start w/ equation

$$\frac{1}{2} H'' - x H' = -\lambda H \qquad x \in \mathbb{R}$$

[7] and look for solutions that grow at most polynomially fast at $|x| \to \infty$

make the change of variables to put in Sturm-Liouville form

$$p(x) = -\int_0^x 2t\,dt = -x^2$$

then

$$\frac{d}{dt}\left(e^{-x^2}\frac{dH}{dt}\right) = -2\lambda e^{-x^2} H(x)$$

is the equivalent Sturm-Liouville form

Hermite equation

self-adjointness condition

zero boundary term because $e^{-x^2}|f(x)| \to 0$ at $x \to \pm\infty$

$$\int_{\mathbb{R}} \bar{f}(x) \frac{d}{dx}\left(e^{-x^2}\frac{dg}{dx}\right)dx = \int_{\mathbb{R}} \frac{d\bar{f}}{dx} e^{-x^2}\frac{dg}{dx}\,dx$$

and convergence of integrals by decay cond.

$$= \int_{\mathbb{R}} \frac{d}{dx}\left(e^{-x^2}\frac{d\bar{f}}{dx}\right) g(x)\,dx$$

$H_0(x) = 1 \quad H_1(x) = 2x$

$H_2(x) = 4x^2 - 2 \quad \ldots$

$$H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$$

eigenfunctions w/ eigenvalues $n \geq 0$ non-neg integers

$\boxed{8}$ orthogonal

$$\langle H_m, H_n \rangle_{e^{-x^2}} = \int_{\mathbb{R}} H_m(x)\, H_n(x)\, e^{-x^2} dx$$

$$= \delta_{m,n} \cdot 2^n \cdot \sqrt{\pi} \cdot n!$$

Regular Sturm–Liouville problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - \left(q(x) - \lambda w(x)\right) y = 0 \qquad x \in [a, b]$$

$$\begin{cases} b_1 y'(a) + b_2 y(a) = 0 \\ c_1 y'(b) + c_2 y(b) = 0 \end{cases}$$

Thm:

$\Rightarrow$ eigenvalues form an increasing sequence

$$\lambda_0 < \lambda_1 < \lambda_2 < \lambda_3 < \cdots$$

$$\lim_{n \to \infty} \lambda_n = \infty$$

unique eigenfunction y_n (up to normalization) of λ_n and y_n has exactly $n-1$ zeros for $a < x < b$ interval

(9) cases that are not regular

singular Sturm-Liouville problems

Example:

$$x^2 y'' + x y' + (\lambda^2 x^2 - m^2) y = 0 \qquad 0 < x < a$$
$$y(a) = 0$$

not regular because $\quad p(x) = w(x) = x \quad q(x) = -\frac{m^2}{x}$

- $p(x)$ & $w(x)$ not strictly positive: zero at $x = 0$
- $q(x)$ not continuous at $x = 0$
- missing boundary condition at $x = 0$

Notation J interval

$L(J, \mathbb{C}) :=$ vect sp of $\mathbb{C}$-valued Lebesgue measurable funct's on J

$L_{loc}(J, \mathbb{C}) :=$ " on all compact subintervals $[\alpha, \beta] \subseteq J$

if J compact $L_{loc} = L$

10

Notation $A, B \in M_{2\times 2}(\mathbb{C})$

$A : B \in M_{2\times 4}(\mathbb{C})$ $\quad (\underbrace{A \;\; B}_{\text{columns}})$

Sturm Liouville problem (Regular)

$$-(py')' + qy = \lambda w y \qquad J = (a,b)$$

w/ conditions at boundary:

$$A\,Y(a) + B\,Y(b) = 0 \qquad Y = \begin{pmatrix} y \\ py' \end{pmatrix}$$

"quasi derivative" of y

SLP (*)

$$A, B \in M_2(\mathbb{C}) \qquad AEA^* = BEB^* \qquad E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

$$\operatorname{rk}(A:B) = 2$$

$$r = \frac{1}{p},\; q,\; w \in L^1(J, \mathbb{R})$$

$$\underline{w > 0}$$

(Note: r, q, w can vanish on subintervals
⤳ these counted as "single zero")

"trivial solution" on $I \subset J$ $\quad y|_I \equiv 0, \; py'|_I \equiv 0$

$f \in L^1$ real val. changes sign on I if $\exists\, A, B \subset I$ pos meas:
$f|_A > 0 \quad f|_B < 0$

$\boxed{11}$ write SL eq also as $Y' = (P - \lambda W) Y$

$Y = \begin{pmatrix} y \\ py' \end{pmatrix}$ $P = \begin{pmatrix} 0 & 1/p \\ q & 0 \end{pmatrix}$ $W = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$

boundary conditions $AY(a) + BY(b) = 0$

cases: (1) separated

simpler case like discussed before:

$a_i, b_i \in \mathbb{R}$ $\quad a_1 y(a) + a_2 py'(a) = 0$

$(a_1, a_2) \neq (0,0)$ $\quad b_1 y(b) + b_2 py'(b) = 0$

$(b_1, b_2) \neq (0,0)$

since ∂-condition invariant under multipl. by a ~~[illegible]~~ nonsing matrix

here can rescale and parameterize by

$0 \leq \alpha < \pi$ $\quad \cos\alpha \; y(a) - \sin(\alpha) \; py'(a) = 0$

$0 < \beta \leq \pi$ $\quad \cos\beta \; y(b) - \sin(\beta) \; py'(b) = 0$

(for convenience in some computations)

(2) real coupled

$Y(b) = K Y(a)$ $\quad K \in SL_2(\mathbb{R})$

(3) complex coupled

$Y(b) = e^{i\gamma} K Y(a)$ $\quad K \in SL_2(\mathbb{R})$

$-\pi < \gamma < 0$ or $0 < \gamma < \pi$

Fact Any SLP ⊛ is always equiv. to ①, ②, or ③

12 all self-adjoint SLP w/ $w>0$ weight

Thm: for SLP ⊛ (w/ $w>0$ a.e. on J)

(no sign restriction on p, q)

(1) all eigenvalues are real, isolated, countably ∞ set w/ no (finite) accumulation pts,

(2) if p changes sign on J
$\Rightarrow$ seq. of eigenvalues unbounded below and above

$$\cdots \le \lambda_{-m} \le \cdots \le \lambda_{-1} \le \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

$\lambda_n \to \infty$, $\lambda_{-n} \to -\infty$

each λ_k either simple or double (not more)

($\Rightarrow$ for each fixed $\lambda = \lambda_k$ exactly 2 lin. indep. solutions of eq.)

(3) if p changes sign on J
and ∂-conditions are separated
then all λ_k are simple

(4) if $p>0$ on J and ∂-cond are real coupled
then eigenvalues are bounded below

$$-\infty < \lambda_0 \le \lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \le \cdots$$

$\lambda_n \to \infty$

w/ eigenv. simple or double

$\{u_n\}$ real valued eigenfunctions for λ_n
(for real coupled ∂-cond.)

$\Rightarrow$ # zeros of u_n in open $\mathring{J}$ is

- 0 or 1 for $n=0$
- $n-1$, n, or $n+1$ for $n \geq 1$

(5) if $p>0$ and ∂-cond is complex coupled
all eigenvalues are simple
if $\{u_n\}$ eigenfunctions for λ_n
number of zeros of $\mathrm{Re}(u_n)$ on $[a,b)$

- 0 or 1 for $n=0$
- $n-1, n, n+1$ for $n \geq 1$

Same for $\mathrm{Im}(u_n)$

but zeros not in same places
u_n no zero on $[a,b]$

(6) if $p>0$ and ∂-conditions
are separated then eigenvalues are simple

$\{u_n\}$ eigenfunctions : u_n's are unique
(up to normalization)
and u_n has exactly n zeros on $\mathring{J}$

14

(7) if $p>0$ any ∂-condition (1), (2), (3)
have asymptotic formula:

$$\frac{\lambda_n}{n^2} \xrightarrow{n\to\infty} c = \pi^2 \left(\int_a^b \sqrt{\frac{w}{p}} \right)^{-2}$$

(8) if p changes sign $p^{\pm} = p$ where $\gtrless 0$ and 0 else

$$\frac{\lambda_n}{n^2} \to \pi^2 \left(\int_a^b \sqrt{\frac{w}{p_+}} \right)^{-2} \quad n \to \infty$$

$$\frac{\lambda_n}{n^2} \to \pi^2 \left(\int_a^b \sqrt{\frac{w}{p_-}} \right)^{-2} \quad n \to -\infty$$

Main ideas in the proof of this thm:

- operator $\mathcal{L}$ and spectrum (eigenvalue probl.) in a Hilbert space $L^2(J, w)$ weighted Leb. meas $w(x)\,dx$

(see below) $\Rightarrow$ eigenvalues are real and there is an ∞ seq. of them

- eigenvalues isolated w/ no accumulation point in $\mathbb{R}$:
  characterization as zeros of an entire function

15

$Y(t)$ matrix solution of SLP ⊛ (2×2)

$\operatorname{rk} Y(t) = \operatorname{rk} Y(u) \quad \forall t, u \in J$

for fixed $u \in J$:

primary fundamental matrix $\phi(t,u) = Y(t)\,Y(u)^{-1}$

$\phi(u,u) = I$

(also write as $\phi(t,u,\underline{\underline{P}})$ or $\phi(t,u,\underline{\underline{\lambda}})$)

$$\phi' = (P - \lambda W)\,\phi \quad \text{on } J \qquad \phi(u,u,\lambda) = I$$

$a \le u \le b$

can def for arbitrary $\lambda \in \mathbb{C}$

<u>characteristic function</u>:

$$\delta(\lambda) = \det\big(A + B\,\phi(a,b,\lambda)\big)$$

is an <u>entire function</u> of λ

(for fixed a, b, A, B, P)

Note: $\exists!$ also implies continuous dependence of solution on parameters so cont. in a, b ...

because ϕ is entire in $\lambda \in \mathbb{C}$

$\frac{d}{d\lambda}$: obtained from <u>variation of parameters</u> formula

for $Y' = PY + F$

ϕ for $Y' = PY$ ~~primary~~ fund matrix

C constant matrix, then

ϕC solution of $Y' = PY$

$$Y(t) = \phi(t,u,P)\,C + \int_u^t \phi(t,s,P)\,F(s)\,ds$$

solution of ... w/ $Y(u) = C$

$\boxed{16}$

$$P \mapsto P + zW$$

$$\underbrace{\left(Y(t,u,P+(z+h)W) - Y(t,u,P+zW) \right)}$$

$$= \underbrace{\left(\phi(t,u,P+(z+h)W) - \phi(t,u,P+zW) \right)} C$$

$$+ \int_u^t \left(\phi(t,r,P+(z+h)W) - \phi(t,r,P+zW) \right) F(r)\,dr$$

$$S(z) = \phi^{-1}(\cdot, u, P+zW)\, W(\cdot)\, \phi(\cdot, u, P+zW)$$

$$= \underbrace{\phi(t,u,P+(z+h)W)} \left(\phi(t,u,hS(z)) - I \right)$$

$$= \qquad \cdot \left(h \int_u^t S(z) + o(h) \right)$$

$$= h\, \phi(t,u,P+(z+h)W) \int_u^t S(z) + o(h)$$

$$= h \left(\phi(t,u,P+zW) \left(\int_u^t S(z) \right) \cdot C \right.$$

$$\left. + \int_u^t \phi(t,r,P+zW) \left(\int_r^t S(z) \right) F(r)\,dr \right)$$

17 So eigenvalues λ_k are zeros of

$$\delta(\lambda) = \det(A + B\,\phi(a,b,\lambda))$$

i.e. $\phi(t,u,\lambda)$ sol'n of problem SL w/ λ
and $\delta(\lambda)=0 \Rightarrow$ boundary conditions are satisfied

zeros of an entire function $\Rightarrow$ discrete set (no accumulation pt.)

- The unboundedness below/above when p has $p_- \neq 0$ (resp. $p_+ \neq 0$) is a result of Möller (paper added to course website)

- for # of zeros of eigenfunctions
Kong, Wu, Zettl "Dependence of n-th Sturm-Liouville eigenvalue on the problem" JDE '99

- eigenvalues simple or double: at most two lin indep sol'n (∂-cond. can fix only one sol'n)
- simple eigenvalues for separated boundary condition $\Longleftrightarrow$ uniqueness of solution w/ those boundary cond. for fixed λ (if existence)

- for complex coupled ∂-cond. w/ $\mathcal{L}$ self adjoint

have $\phi(a,b,\lambda_n) = e^{i\gamma}K$
(∂-condition)
again then fix unique sol'n

real
$\gamma = 0$ or $\gamma = \pi$

18 Spectral thm for unbounded self adjoint operators on a Hilbert space

- spectral thm for bounded self adjoint op's

$T \in B(\mathcal{H})$ lin op. $\|T\| = \sup \frac{\|Tx\|}{\|x\|} < \infty$

and $T = T^*$

$R_\lambda(T) = (T - \lambda I)^{-1}$

$\sigma(T) = \mathrm{Spec}(T)$ complement of reg set

λ regular value if

i) $R_\lambda(T)$ exists

ii) $R_\lambda(T)$ def on dense dom.

iii) $R_\lambda(T)$ bounded

$\rho(T)$ set reg. values

discrete sp.: pt spec if this fails (i)

essential sp.: residual spec if this fails (ii): R_λ bounded on some non-dense dom.

Cont. sp. if this fails (iii) (R_λ exists on dense dom. not bounded)

if $T \in B(\mathcal{H})$ $\Longrightarrow$ no residual spectrum

Spec(T) closed contained in disk $\|T\|$ radius

$\exists$ family of projections in $\mathcal{H}$ (spectral projections) E_λ $\lambda \in \mathbb{R}$

$E_\lambda E_\mu = E_\mu E_\lambda = E_\lambda$ for $\lambda < \mu$

$\lim_{\lambda \to -\infty} E_\lambda x = 0$ $\quad \lim_{\lambda \to +\infty} E_\lambda x = Ix = x$

$E_\lambda x = \lim_{\mu \to \lambda^+} E_\mu x$

[on an interval if $E_\lambda = 0$ $\lambda < a$, $E_\lambda = I$ $\lambda > b$]

$u_{xy}(\lambda) := \langle E_\lambda x, y \rangle$

Spectral thm for <u>bounded</u> <u>self adjoint</u> T: $\exists E_\lambda$ s.t. $\forall x, y \in \mathcal{H}$

$$\langle Tx, y \rangle = \int_a^b \lambda \, du_{x,y}(\lambda)$$

$$\langle f(T) x, y \rangle = \int_a^b f(\lambda) \, du_{x,y}(\lambda)$$

f polyn. (& more gen. class of cont fn's)

Spectral thm for <u>Unitary</u> operators

$U \in B(\mathcal{H})$ $\quad UU^* = U^*U = I$

$\sigma(U)$ closed subset of $S^1 = \{|\lambda| = 1\}$ $\quad \lambda = e^{2\pi i \theta}$

$\exists$ spectral family E_θ on $[-\pi, \pi]$ s.t. $\forall x, y \in \mathcal{H}$

$$\langle f(U) x, y \rangle = \int_{-\pi}^{\pi} f(e^{i\theta}) \, du_{x,y}(\theta)$$

polyn. funct.

$u_{x,y}(\theta) = \langle E_\theta x, y \rangle$

log $U = e^{iS}$ for S bounded self-adjoint w/ $\sigma(S) \subset [-\pi, \pi]$

Spectral thm for <u>unbounded</u> <u>self-adj.</u> op's

$T : D(T) \to \mathcal{H}$ dense domain

$T^* = T$

Cayley transform

$$U = (T - iI)(T + iI)^{-1}$$

$\implies$ U unitary operator

(Möbius transf. that maps real axis to unit circle)

20 can invert $T = i(I+U)(I-U)^{-1}$ ($\Rightarrow 1$ not in $\sigma(U)$ as inverse exists)

$\exists$ spectral family E_θ (for $-U$)

$\forall x \in \mathrm{Dom}(T)$ $\quad \mu_{xx}(\theta) = \langle E_\theta x, x \rangle$

$$\langle Tx, x \rangle = \int_{-\pi}^{\pi} \tan\left(\frac{\theta}{2}\right) d\mu_{xx}(\theta)$$

$$= \int_{\mathbb{R}} \lambda \, d\nu(\lambda)$$

$\nu(\lambda) = \langle F_\lambda x, x \rangle$

$F_\lambda = E_{2\arctan(\lambda)}$

$\mathcal{L}$ self-adjoint op. on $L^2(J, w)$ (unbounded)

$\rightsquigarrow$ spectral theorem

but from spectral theorem can have discrete and continuous spectrum

cases w/ only discrete spectrum:

can ensure only discrete spectrum for T if resolvent $R_\lambda(T)$ is a compact operator

- compact operators :
  - $\forall B \in \mathcal{H}$ bounded set $\overline{T(B)}$ in $\mathcal{H}$ compact set (e.g. image of unit ball)
  - closure in $B(\mathcal{H})$ of finite rank op's (matrices)
  - $\{\psi_k\}$ bounded seq in $\mathcal{H}$ $\{T\psi_k\}$ has conv. subseq.

<u>spectrum of compact operators</u> :

- no essential spectrum
- $0 \in \mathrm{Spec}(T)$ always (otherwise would get I comp. impossible on $\dim \mathcal{H} = \infty$)
- 0 not nec. eigenvalue
  can be limit pt.
  can have ∞-multipl. if eigenvalue
- $\lambda \neq 0 \in \mathrm{Spec}(T)$ isolated pt
  finite multiplicity

$\mathrm{Spec}(T) \setminus \{0\}$ either finite or sequence $\lambda_k \to 0$

- if resolvent $R_z(T)$ is compact it is so for all $z \in \rho(T)$

$$(T - zI)^{-1} = (T - z_0 I)^{-1} - (z - z_0)\underbrace{(T - z_0 I)^{-1}(T - z)^{-1}}$$

& comp op $\mathcal{K} \subset B(\mathcal{H})$ ideal so compact if $(T - z_0 I)^{-1}$ is

if $0 \notin \sigma(T)$ can just take $R_{z=0}(T) = T^{-1}$ compact

then $\sigma(T^{-1})$ is sequence w/ only accum. pt at 0

$\Rightarrow \{\lambda_k^{-1} \mid \lambda_k \in \sigma(T)\} \Rightarrow \sigma(T)$ discrete seq no accum. pt. except at ∞

<u>unbounded</u>

w/ $|\lambda_k| \to \infty$