Ordinary Differential Equations

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Ma142: Ordinary and Partial Differential Equations
Caltech, Winter 2024

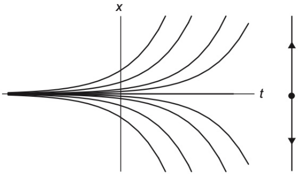


Figure 1.1 The solution graphs and phase line for $x' = ax$ for $a > 0$. Each graph represents a particular solution.

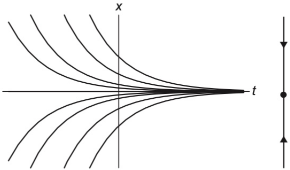


Figure 1.2 The solution graphs and phase line for $x' = ax$ for $a < 0$.

stability of qualitative behavior of solutions with respect to real parameter a

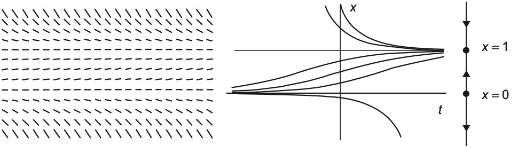


Figure 1.3 Slope field, solution graphs, and phase line for $x' = ax(1-x)$.

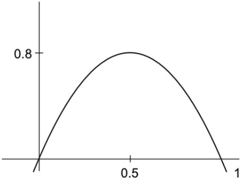


Figure 1.4 The graph of the function $f(x) = ax(1-x)$ with $a = 3.2$.

Logistic population growth model

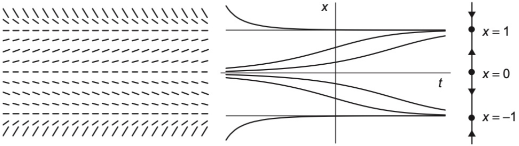


Figure 1.5 Slope field, solution graphs, and phase line for $x' = x - x^3$.

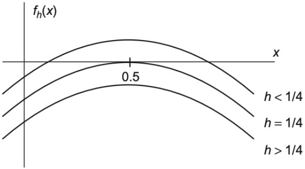


Figure 1.6 The graphs of the function $f_{h(x)} = x(1 - x) - h$.

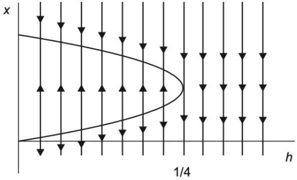


Figure 1.7 The bifurcation diagram for $f_{h(x)} = x(1 - x) - h$.

Phase portrait of plane linear systems

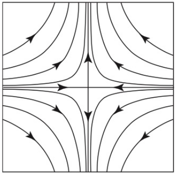


Figure 3.1 Saddle phase portrait for $x' = -x$, $y' = y$.

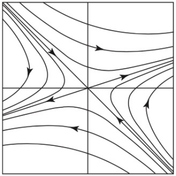


Figure 3.2 Saddle phase portrait for $x' = x + 3y$, $y' = x - y$.

Saddle points

Phase portrait of plane linear systems

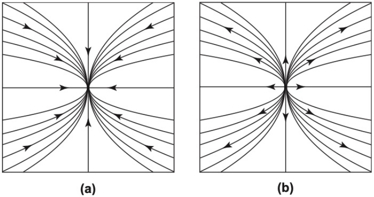


Figure 3.3 Phase portraits for a sink and a source.

Sinks and sources

Phase portrait of plane linear systems

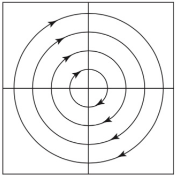


Figure 3.4 Phase portrait for a center.

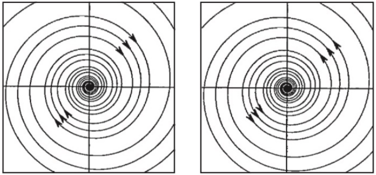


Figure 3.5 Phase portraits for a spiral sink and a spiral source.

Centers and unstable/stable spirals

Phase portrait of plane linear systems

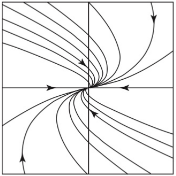
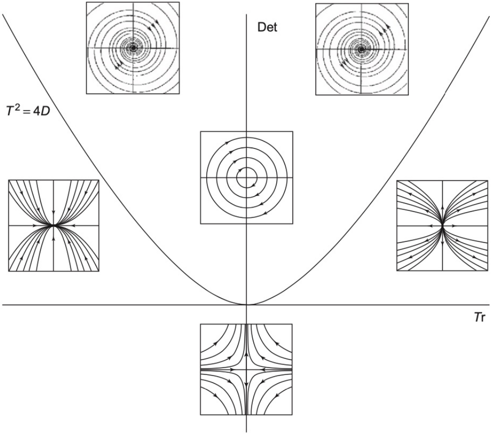


Figure 3.6 Phase portrait for a system with repeated negative eigenvalues.

Trace-Determinant Map



3D examples

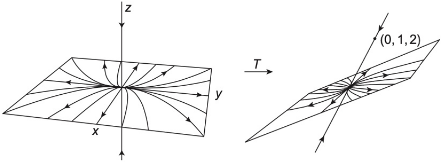


Figure 6.1 Stable and unstable subspaces of a saddle in dimension 3. On the left, the system is in canonical form.

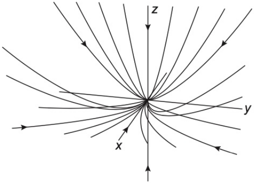


Figure 6.2 A sink in three dimensions.

3D examples

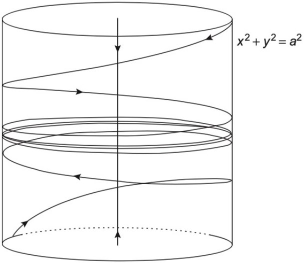


Figure 6.3 Phase portrait for a spiral center.

3D examples

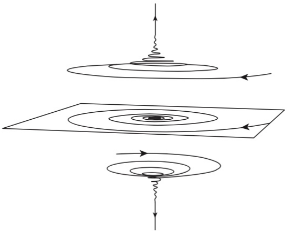


Figure 6.5 Typical spiral saddle solutions tend to spiral toward the unstable line.

Kronecker foliation on the torus

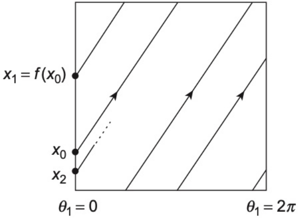


Figure 6.8 Poincaré map on the circle $\theta_1 = 0$ in the $\theta_1\theta_2$-torus.

Nonlinear systems: equilibria

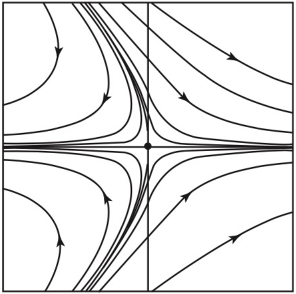


Figure 8.1 Phase plane for $x' = x + y^2$, $y' = -y$. Note the stable curve tangent to the y-axis.

Nonlinear systems: equilibria

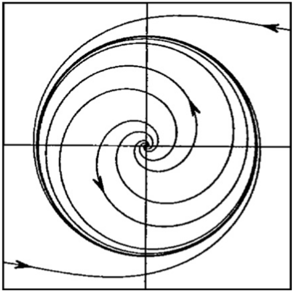


Figure 8.2 Phase plane for
$r' = \frac{1}{2}(r - r^3)$, $\theta' = 1$.

Nonlinear systems: equilibria

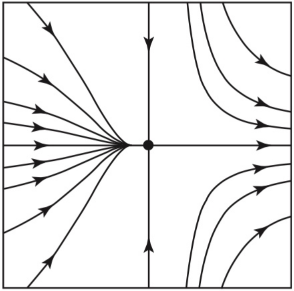


Figure 8.3 Phase plane for
$x' = x^2, y' = -y$.

Bifurcations

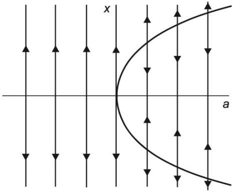


Figure 8.7 Bifurcation diagram for a pitchfork bifurcation.

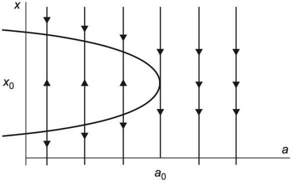


Figure 8.6 Bifurcation diagram for a saddle-node bifurcation.

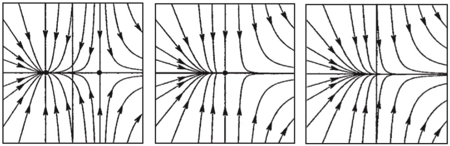


Figure 8.8 Saddle-node bifurcation when $a < 0$, $a = 0$, and $a > 0$.

Bifurcations

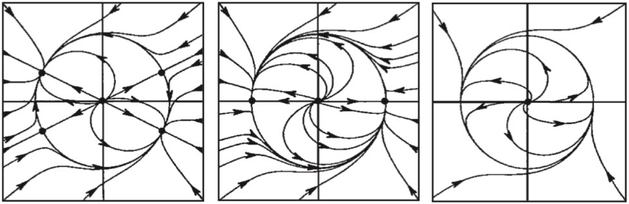


Figure 8.9 Global effects of saddle-node bifurcations when $a < 0$, $a = 0$, and $a > 0$.

Bifurcations

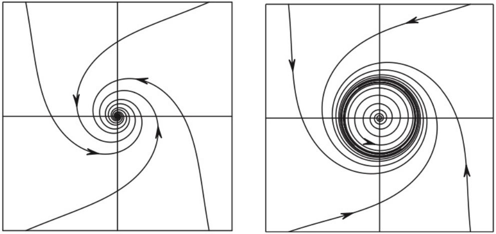


Figure 8.10 Hopf bifurcation for $a < 0$ and $a > 0$.

Nonlinear systems: nullclines

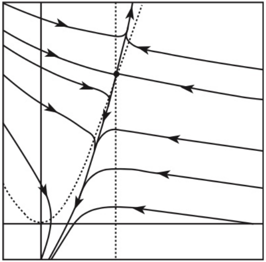


Figure 9.3 Nullclines and phase portrait for $x' = y - x^2$, $y' = x - 2$.

Nonlinear systems: nullclines

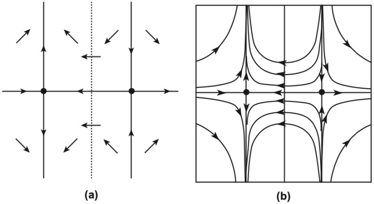


Figure 9.4 Nullclines and phase portrait for $x' = x^2 - 1$, $y' = -xy$.

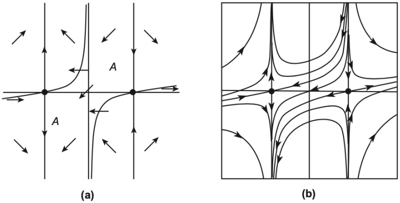


Figure 9.5 Nullclines and phase plane when $a > 0$ after the heteroclinic bifurcation.

Nonlinear pendulum

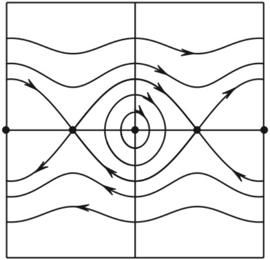


Figure 9.6 Phase portrait for the ideal pendulum.

Gradient flows

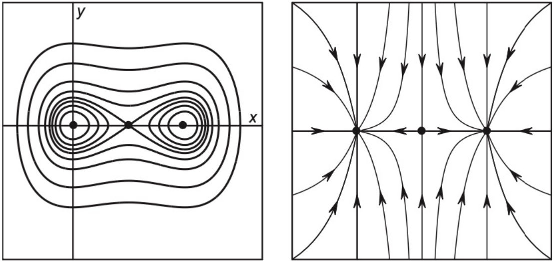


Figure 9.10 Level sets and phase portrait for the gradient system determined by $V(x,y) = x^2(x-1)^2 + y^2$.

Hamiltonian systems

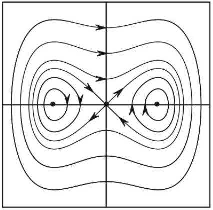


Figure 9.11 Phase portrait for $x' = y$, $y' = -x^3 + x$.

A Hamiltonian function is

$$H(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{4}.$$

Nonlinear systems: closed orbits and limit sets

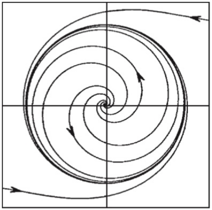


Figure 10.1 The phase plane for $r' = \frac{1}{2}(r - r^3)$, $\theta' = 1$.

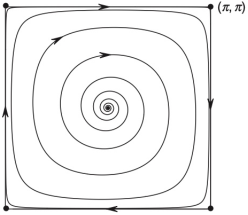


Figure 10.2 The ω-limit set of any solution emanating from the source at $(\pi/2, \pi/2)$ is the square bounded by the four equilibria and the heteroclinic solutions.

Nonlinear systems: return maps

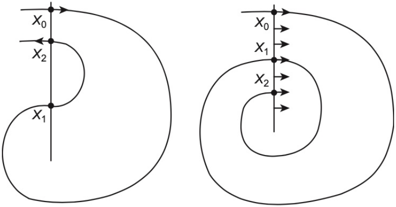


Figure 10.6 Two solutions crossing a straight line. On the left, X_0, X_1, X_2 is monotone along the solution but not along the straight line. On the right, X_0, X_1, X_2 is monotone along both the solution and the line.

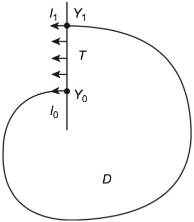


Figure 10.7 Solutions exit the region D through T.

Nonlinear systems: return maps

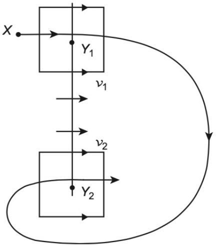


Figure 10.8 The solution through X cannot cross $\mathcal{V}_1$ and $\mathcal{V}_2$ infinitely often.

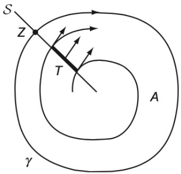


Figure 10.11 The region A is positively invariant.