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SL theory gives a way of tackling certain classes of PDEs in terms of ODEs

LINEAR! (lin. diff operators, superposition principle ....)

What about NONLINEAR equations: is there any method that can reduce a NL PDE to ODE's?

it seems unlikely BUT... many important NL PDEs from physics problems admit special relations between variables that reduce to one special class of NL ODE's ... Painlevé transcendents

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Painlevé equations

Picard problem: classify

$$w'' = R(z, w, w')$$

R = rational function in w, w' (ratio of polyn's) analytic in z

w/ property that any singularity of solutions other than poles (ess. sing's) depend only on equation not on constants of integration/initial condition

movable sing.

Painlevé-type equations

Original classification by Painlevé, Gambier (1895–1910)
50 types of equations

$\boxed{2}$ (Similar property for first order eqn's : Fuchs type)

method: solutions as series of powers of small parameters (α-method)

resulting in

- 6 main Painlevé types
- 11 more that can be expressed in terms of soln's of these 6 types
- 33 remaining types that reduce to solns of linear equations of 2nd & 3rd order and elliptic functions

these are the nonlin ODE's that occur frequently in rel. to many nonlin physics-rel. PDE's

Equations of a cx variable $z \in \mathbb{C}$

(P1) $$w'' = 6w^2 + z$$

(P2) $$w'' = 2w^3 + zw + \alpha$$

(P3) $$w'' = \frac{(w')^2}{w} - \frac{1}{z}w' + \frac{1}{z}(\alpha w^2 + \beta) + \gamma w^3 + \frac{\delta}{w}$$

(P4) $$w'' = \frac{(w')^2}{2w} + \frac{3}{2}w^3 + 4zw^2 + 2(z^2 - \alpha)w + \frac{\beta}{w}$$

(P5) $$w'' = \frac{3w-1}{2w(w-1)}(w')^2 - \frac{1}{z}w' + \frac{1}{z^2}(w-1)^2\left(\alpha w + \frac{\beta}{w}\right) + \frac{\gamma w}{z} + \frac{\delta w(w+1)}{(w-1)}$$

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(P6)

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right) w' + \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)$$

Relation of Painlevé equations w/ PDE's

idea: seek solutions w/ particular relation among variables

e.g. traveling waves:

$$u(x,t) = U(x-ct)$$

Korteweg-de Vries equation (KdV)

$$u_t - 6u\,u_x + u_{xxx} = 0$$

used in hydrodynamics to describe wave evolution in shallow water

balance between for a solitary wave $\begin{cases} \text{time evol } u_t \\ \text{nonlinearity } u u_x \\ \text{dispersion } u_{xxx} \end{cases}$

(also in plasma physics: plasma waves in dispersive medium)

$\boxed{4}$ KdV admits traveling wave solutions (stationary waves)

$$u(x,t) = U(x-ct)$$

gives equation for $U(z)$

$$U'' - 3U^2 - cU = k_1$$

constant of integration

multiply by $2U'$ and integrate to get

$$(U')^2 - 2U^3 - cU^2 = 2k_1 U + k_2$$

this equation is related to the elliptic function $\wp$ (Weierstrass $\wp$-function)

$$\wp(q, \underbrace{1, \tau}_{\text{lattice } \Lambda}) = \frac{1}{q^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(q+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right)$$

$$u(x,t) = -\frac{c}{6} + 2\wp\left(x - ct - x_0, \frac{c^2}{12} - k_1, -\frac{1}{4}k_2 + \frac{c}{12}k_1 - \left(\frac{c}{6}\right)^3\right)$$

$\boxed{5}$ Weirstrass $\wp$ - function

$$\wp(z, \omega_1, \omega_2) = \wp(z, \Lambda) := \frac{1}{z^2} + \sum_{\lambda \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\lambda)^2} - \frac{1}{\lambda^2} \right)$$

$\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ lattice in $\mathbb{C}$

term needed for convergence

series converges in $\mathbb{C} \setminus \Lambda$

constructed to have a pole order 2 at each lattice pt. $\lambda \in \Lambda$

$$\underbrace{\wp(z, 1, \tau)}_{\text{also write as } \wp(z;\tau)} \qquad \mathbb{Z} + \mathbb{Z}\tau = \Lambda$$

What is this function?

Elliptic curves

$$E = \{ (x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2 x - g_3 \}$$

cubic equation $\quad g_2, g_3 \in \mathbb{C}$

$$g_2^3 - 27 g_3^2 \neq 0$$

Elliptic funct's:

$$u(z) = -\int_z^\infty \frac{ds}{\sqrt{4s^3 - g_2 s - g_3}}$$

$u^{-1}(z)$ inverse is $\wp(z, \tau)$

where $g_2(\tau), g_3(\tau)$

<ins>Analogy:</ins> circle

$$C = \{ (x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1 \}$$

quadratic eq.

parameterization by trig. functions

$$a(x) = \int_0^x \frac{dy}{\sqrt{1-y^2}}$$

$y = \sin t$

$s = \arcsin x \qquad \int_0^s dt = s = \arcsin x$

trig funct. as inverse of an integral

$a^{-1}(x) = \sin x$

$\boxed{6}$ $\downarrow$ fixed τ

$\wp(z)$ satisfies diff eq

$(u^{-1})' = (4x^3 - g_2 x - g_3)^{1/2}$

$(u^{-1}(x))' = \dfrac{1}{u'(u^{-1}(x))}$

$$(\wp'(z))^2 = 4\wp^3(z) - g_2\,\wp(z) - g_3$$

dependence of g_2, g_3 on τ $\longleftrightarrow$ descr. of ell curves

$$\mathbb{C}/_{\Lambda = \mathbb{Z} + \mathbb{Z}\tau} \simeq E \simeq \{(x,y) \in \mathbb{C}^2 \mid y^2 = 4x^3 - g_2(\tau)x + g_3(\tau)\}$$

half-periods: $e_1 = \wp(\frac{\omega_1}{2})$ for $\Lambda = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$

$\left.\begin{array}{l} e_2 = \wp(\frac{\omega_2}{2}) \\ e_3 = \wp(\frac{\omega_1+\omega_2}{2}) \end{array}\right\}$ roots of $4\wp(z)^3 - g_2\wp(z) - g_3$

with $e_1 + e_2 + e_3 = 0$

$$(\wp'(z))^2 = 4(\wp(z) - e_1)(\wp(z) - e_2)(\wp(z) - e_3)$$

this from map

$p : \mathbb{C}/\Lambda \to E$ $\quad z \mapsto (\wp(z), \wp'(z))$

$z \neq 0$

(compactified for $z = 0$ w/ pt ∞ (in $\mathbb{P}^2$))

have an explicit Fourier series in $q = e^{2\pi i \tau}$

This will be useful in describing solutions of (P6) Painlevé equation

$\boxed{7}$ Also in KdV equation

if substitute $\quad u(x,t) = -w(z) + \lambda t$

$$z = x + 3\lambda t^2$$

obtain an equation for z-variable

$$w'''(z) = -6w(z)\,w'(z) + \lambda$$

that can be integrated to give

$$w'' = -3w^2 + \lambda z + K$$

a form of first Painleve' equation (P1)

$\gamma w(\alpha z + \beta)$

version of KdV with

$$2u_t + 2uu_x + u_{xxx} = 0$$

substitution

$$\begin{cases} u = \dfrac{\beta}{2\alpha} + (3\alpha t + \delta)^{-2/3} f(\tau) \\ \tau = (3\alpha t + \delta)^{-1/3}\left(x - \beta t/(2\alpha) + \gamma/\alpha - \beta\delta/(2\alpha^2)\right) \end{cases}$$

gives $\quad f''' + 2ff' - 2\alpha\tau f' - 4\alpha f = 0$

further substitute $\quad f = 3\alpha\tau - 6\alpha^{2/3} F \quad z = -\alpha^{1/3}\tau$

gives equation

$$FF'' = \frac{(F')^2}{2} + 4F^3 + 2zF^2 + k$$

integration constant

further substitution

$$F = (w^2 - w' - z)/2$$ gives

$$w'' = 2w^3 - 2zw + \alpha \qquad \alpha = (-8k)^{1/2} - 1$$

modified version of Painlevé 2

Another rel. to Painlevé:

start w/ KdV

apply Miura transformation $u = v_x + v^2$

get modified KdV (mKdV)

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

as indeed get

$$2vv_t + v_{xt} - 6(v^2 + v_x)(2vv_x + v_{xx})$$
$$+ 6 v_x v_{xx} + 2v v_{xxx} + v_{xxxx} = 0$$

that can be written as

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$$\left(2v + \frac{\partial}{\partial x}\right) M(v) = \left(2v + \frac{\partial}{\partial x}\right)\left(v_t - 6v^2 v_x + v_{xxx}\right) = 0$$

So if v satisfies mKdV

$$v_t - 6v^2 v_x + v_{xxx} = 0$$

also satisfies KdV ([illegible] form)

then can see another relation to Painlevé 2:

$$v(x,t) = (3t)^{-1/3} w(z) \qquad z = x(3t)^{-1/3}$$

(Ablowitz-Segur)

gives $\quad w''' - 6w^2 w' - (zw)' = 0$

that gives (P2) by integration

Cylindrical KdV: (cKdV)

$$2u_t + \frac{1}{t} u + 2uu_x + u_{xxx} = 0$$

reduces to Painlevé (P1) by transf:

$$\tau = x t^{-1/2} + \left(\delta + 2\beta t^{1/2}\right) / (2\gamma t)$$

$$u(x,t) = \frac{x}{2t} + \delta t^{-3/2} / 2\gamma + \beta / 2\gamma t + f(\tau) / t$$

(Tajiri-Kawamoto)

(integration) $\Rightarrow$ $f'' + f^2 - \frac{\delta \tau}{2\gamma} = k$ integration constant

or reduces to Painlevé (P2) by transf:

$$z = \alpha t + \frac{2\gamma t^{3/2}}{3}$$

$$u(x,t) = (\gamma x + (3\beta - 2\gamma\delta/\alpha)) t^{1/2} (3z)^{-1} + z^{-2/3} f(\tau)$$

which in ckdV gives

$$f''' + 2ff' - \frac{2}{3}\alpha\tau f' - \frac{\alpha}{3} f = 0$$

using transf

$$W^2 = -6^{-1/3}\alpha^{-2/3} f \qquad \xi = \left(\frac{\alpha}{6}\right)^{1/3}\tau$$

gives a solution

$w(\xi)$ of (P2) w/ $\alpha = 0$

Boussinesq and Kadomtsev-Petviashvili equations

Boussinesq:

$$\boxed{u_{tt} = c^2\left(u_{xx} + \frac{3}{2}\left(\frac{u^2}{h}\right)_{xx} + \frac{1}{3}h^2 u_{xxxx}\right)}$$

[11] also coming from hydrodynamics
describing Propagation of long water waves
(with speed c) in shallow water of depth h

Kadomtsev-Petviashvili:

$$(u_t - 6uu_x + u_{xxx})_x + 3\mu^2 u_{yy} = 0$$

slowly varying NL waves in a dispersive medium
2D generalization of KdV

normalization: (and $\mu^2 = 1$)

$$v_{tx} + (v^2 + v_{xx})_{xx} + v_{yy} = 0$$

then transf:

$$x \mapsto x - \tfrac{1}{3}y - \tfrac{10}{9}t$$

$$t \mapsto y + \tfrac{2}{3}t$$

$$v \mapsto u$$

gives $\quad u_{tt} - u_{xx} + (u^2 + u_{xx})_{xx} = 0 \qquad (*)$

replacing $z = x - ct$ in ↑ gives

$$(c^2-1)u + (u^2 + u'') = k_1 z + k_2 \quad \text{int. constants}$$

sol'ns in term of PI sol'ns & Weierstrass $\wp$

- if in ⊗ substitute instead

$$\xi = x - \frac{\delta}{2\theta} t^2 \qquad f = \left(\frac{\delta}{\theta} t\right)^2 + 2u$$

and integrate once

$$f''' + f f' - f' - \left(\frac{\delta}{\theta}\right) f - 2\left(\frac{\delta}{\theta}\right)^2 \xi = k_1$$

then using

$$z = -\left(\frac{\delta}{2\theta}\right)^{1/3}\left(\xi + \frac{1}{2}\frac{\theta}{\delta}\left(1 + k_1 \frac{\theta}{\delta}\right)\right)$$

$$F = -\frac{1}{12}\left(2\frac{\theta}{\delta}\right)^{2/3}\left(f - 2\frac{\delta}{\theta}\xi - 2 - k_1 \frac{\delta}{\theta}\right)$$

$$\Rightarrow \quad F F'' = \frac{(F')^2}{2} + 4F^3 + 2zF^2 + k_1$$

$\rightsquigarrow$ again related to Painlevé (P2)

Generalized Boussinesq equation

$$u_{xxxx} + p\, u_t u_{xx} + q\, u_x u_{xt} + r\, u_x^2 u_{xx} + u_{tt} = 0$$

$p, q, r \neq 0$ constants

substitute

$$u(x,t) = w(z) + \lambda_1 \log t \qquad z = x t^{-1/2}$$

$$W(z) = w'(z) \qquad \text{gives}$$

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$$W''' - \frac{1}{2}(p+q) z W W' + r W^2 W'$$
$$+ \left(\frac{1}{4} z^2 + p\lambda_1\right) W' - \frac{1}{2} q W^2$$
$$+ \frac{3}{4} z W - \lambda_1 = 0$$

further replace

$$W(z) = -\left(3^{3/4} y(x) - z\right)/p$$

$$x = -\frac{1}{2} 3^{1/4} z$$

with $q=0$ and $r = -\frac{1}{2} p^2$

gives Painlevé (P4) for

$y(x)$ w/ $\alpha = p\lambda_1/\sqrt{3}$ $\quad \beta$ integration constant

- again for $q=0$ and $r = -\frac{1}{2} p^2$

take

$$u(x,t) = w(z) + 2\lambda_1 x t + \lambda_2 \lg t$$
$$z = x t^{-1/2} + \lambda_1 p t^{3/2}$$

$$W(z) = w'(z)$$

gives

$$W''' - \frac{1}{2}p^2 W^2 W' - \frac{1}{2} p z W W' + \left(\frac{1}{4} z^2 + p\lambda_2\right) W' + \frac{3}{4} z W - \frac{\lambda}{2} = 0$$

then again $W(z) = -\left(3^{3/4} y(x) - z\right)/p$

$$x = -\frac{1}{2} 3^{1/4} z$$

gives for $y(z)$ Painlevé (P4)

with $\alpha = p\lambda_2/\sqrt{3}$ & β const int

- if $r = \frac{1}{2} q(p+q) \neq 0$ and take

$$u(x,t) = w(z) - 4\lambda_1 x t + \lambda_2 t - \frac{8}{3}\lambda_1^2 q t^3$$

$$z = x + \lambda_1 q t^2$$

$W(z) = w'(z)$ $\qquad$ $\lambda_1 \neq 0$, λ_2 arbitrary constants

$$\Rightarrow \quad W''' + \frac{1}{4} q(p+q) W^2 W' + (-4\lambda_1 p z + \lambda_2 p) W' - 2\lambda_1 q W = 0$$

if $q = 2p$ $\quad r = 3p^2/2$ this reduces to a Painlevé after integration

$$W'' + p^2 W^3/2 + (\lambda_2 p - 4\lambda_1 p z) W = \lambda_3$$

int. const.

modif. of Painlevé (P2)

Spherical Boussinesq

$$u_{tt} + \frac{2}{t} u_t + \frac{\gamma}{2t} u_{xx} - 3(u^2)_{xx} - u_{xxxx} = 0$$

$$u(x,t) = -\frac{1}{4t}\left(\frac{dw}{dz} + w^2 + 2zw \right)$$

$$z = x/2\sqrt{t}$$

$\Rightarrow$ Painlevé (P4) for $w(z)$ w/ $\alpha = 1 - \frac{1}{2}\gamma$

Sine Gordon equation

arising in geometry of surfaces of constant neg. curv.
nonlinear optics, solid state physics

typical case of sine-Gordon:

$$u_{xt} = \exp(u)$$

if $w(z)$ solution of

$$\overbrace{w'' = \frac{(w')^2}{w} - \frac{1}{z} w' + \frac{1}{z} w^2}^{(P3) \text{ with } \alpha = 1,\ \beta=\gamma=\delta=0}$$

then $u = \log w(z)$ $\quad z = xt$ $\quad$ is sol'n of

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- other sine-Gordon equation

$$u_{XX} - \frac{1}{c^2} u_{TT} = \sin(u)$$

$$x = \frac{1}{2}(X - cT)$$

$$t = \frac{1}{2}(X + cT)$$

invariant under $u \mapsto u + 2\pi n$

gives $u_{xt} = \sin(u)$ $\circledast$

using:

$$\phi(\tau, \eta) = u(x,t)$$

$$2x = \tau + \eta$$

$$2t = \tau - \eta$$

turns into

$$\phi_{\tau\tau} - \phi_{\eta\eta} = \sin\phi$$

$$u(x,t) = -i \log w(z)$$

$$z = xt$$

if $w(z)$ solves $(P3)$

$$w'' = \frac{(w')^2}{w} - \frac{1}{z} w' + \frac{1}{zz}(w^2 - 1)$$

then $u(x,t)$ solves

$u_{xt} = \sin(u)$

sine-Gordon

- other form of sine-Gordon eq

$$u_{xt} = \nu \sin(u) + \mu x t \sin(2u) \qquad (**)$$

ν, μ parameters

reduces to (*) under
$$u_1 = 2u$$
$$x_1 = x^2$$
$$t_1 = t^2 \mu/2$$

with $z = xt$ $\mu = 2$

if $u(z)$ soln of (**)
then $w(z) = \exp(-i\, u(z))$
soln of (P3) w/ $2\alpha = -2\beta = \nu$
$\gamma = -\delta = 1$

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hyperbolic sine-Gordon

$$u_{xt} = \sinh(u)$$ also related to Painlevé (P3)

if $W(z)$ soln of (P3) same param. as above for sine-Gordon

using $u(x,t) = \log w(z)$ (instead of $u(x,t) = -i \log w(z)$) gives sol'n

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2D sine-Gordon :

γ, μ param.

$$u_{tt} - u_{xx} - u_{yy} + \nu \sin(u) + \mu f(x,y,t) \sin 2u = 0$$

$f(x,y,t)$ function chosen differently to ensure particular properties of solutions

e.g. $f(x,y,t) = \xi \qquad \xi = \xi_1 - \xi_2^2/h$

$\xi_1 = t^2 - x^2 - y^2 \qquad \xi_2 = Ct - Bx - Ay$

$h = C^2 - A^2 - B^2 \neq 0$

then $u(\xi) = i \log w(\xi)$ solution of
if $w(\xi)$ solution of Painlevé (P3)
w/ parameters
$\alpha = -\beta - \nu/8 \qquad \gamma = -\delta = -\mu/8$

Nonlinear Schrödinger eqn

$$i u_t + u_{xx} + \gamma u^2 \bar{u} = 0$$

(optical waves in nonlinear crystals, plasma waves in solids, nonlinear optics, laser optics)

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$$u(x,t) = (2t)^{-1/2}\, v(\tau)\, \exp(i\,Q(\tau))$$

$$\tau = \frac{x\,t^{-1/2}}{2}$$

real functions

this gives:

$$Q''v + 2v'Q' - 2\tau v' - 2v = 0$$

$$v'' - (Q')^2 v + 2\tau Q' v + 2\gamma v^3 = 0$$

setting $Q'(\tau) = \tau + g/g_\tau \qquad v^2 = 2g_\tau$

and integrating

int. param.

$$(g'')^2 + 4(\tau g' - g)^2 + 8\gamma (g')^3 + 2\gamma \mu^2 g' = 0$$

for $\gamma = -1$ taking

$$g = \frac{w(w-\tau)^2}{2} + (w_\tau^2 - 2w_\tau + 1 - \mu^2)/8w$$

$\Rightarrow$ Painlevé (P4)

$$w w'' = \frac{1}{2}(w')^2 - 6w^4 + 8\tau w^3 - 2x^2 w^2 - (\mu-1)^2/2$$

Other forms of NL Schrödinger equation:

$$i u_t + u_{xx} + \underset{\pm 1}{\varepsilon}\, i (u^2 \bar{u})_x + \underset{\text{param.}}{\mu} / (t^2 u \bar{u}^2) = 0$$

$\rightsquigarrow$ (P4) Painlevé for w where:

for param's
$\alpha = 2\gamma\varepsilon\lambda^2$
$\beta = 8(\gamma^2 - \mu)$
w/ $\theta(\eta)$ satisfying
$\sigma^2\theta' = \sigma^2\eta/4 - 3\varepsilon\sigma^4/4 + \gamma$

$\underline{\mu \neq 0}$:
$w(z) = \sigma^2/\lambda$
$z = \eta/k$
$\lambda^4 = -1$
$k = -2\varepsilon\lambda$

$\underline{\mu = 0}$:
$u = t^{-1/4} f(\eta)$
$f = \sigma e^{i\theta}$
$w(\eta) = \sigma^2$
$\eta = x t^{-1/2}$

Kawamoto form of NL Schrödinger:

$$i u_t + u_{xx} + (-2\alpha x + 2|u|^2) u = 0$$

$u(x,t) = f(x) \exp(ikt)$
modif. version of Painlevé (P2)

$$f'' - (2\alpha x + k) f + 2|f|^2 f = 0$$

Equations of Einstein type: coming from axially symmetric vacuum solutions of GR

complex Einstein eq:

(*) $\nabla^2 u = 2\bar{u} (\nabla u)^2 / (u\bar{u} - 1)$

$u = u(\rho, z)$, $\nabla^2 u = \frac{\partial^2 u}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial u}{\partial \rho} + \frac{\partial^2 u}{\partial z^2}$, $(\nabla u)^2 = \left(\frac{\partial u}{\partial \rho}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2$

[21]

if $w(\rho)$ solution of Painlevé (P5) with
$\alpha = -\beta = -\frac{1}{2}b^2 \quad \gamma = 0 \quad \delta = -2a^2$
$a, b \in \mathbb{R}$ so that $w > 0$ some domain $E \subset \mathbb{R}$

$u = \exp(iaz + is)\, w(\rho)^{1/2}$
with s solution of

$$\frac{ds}{d\rho} = b(1-w^2)/2\rho w$$

gives solution of ⊛

Toda equation

(Toda lattice model nearest neighb interaction of particles on a line)

$$q_n'' = \exp(q_{n-1} - q_n) - \exp(q_n - q_{n+1})$$

$q_n(t)$ = distance of n-th particle from equil. pt.

$$q_n - q_{n-1} = -\log p_n$$

$\Rightarrow p_n$ satisfies

$$(**) \quad \frac{d^2(\log p_n)}{dt^2} = p_{n+1} - 2p_n + p_{n-1}$$

equivalently written as a system

$$\begin{cases} q_n' = p_{n-1} - p_n \\ p_n' = p_n(q_n - q_{n+1}) \end{cases}$$

plug into (**)

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Consider $H = H_2(q, p, k) = \frac{1}{2}p^2 - (q^2 + \frac{t}{2})p - kq$

Hamiltonian for

$$q'' = 2q^3 + tq + k - \frac{1}{2}$$

Painlevé (P2)

$k \in \mathbb{C}$ parameter:

$p(k), q(k)$
$H(k)$

$$\begin{cases} \frac{\partial H}{\partial q} = -p' \\ \frac{\partial H}{\partial p} = q' \end{cases} \qquad \begin{cases} \frac{\partial H}{\partial q} = -2qp - k \\ \frac{\partial H}{\partial p} = p - (q^2 + \frac{t}{2}) \end{cases}$$

$$q' = p - (q^2 + \frac{t}{2})$$

$$\begin{aligned} q'' &= p' - 2qq' - \frac{1}{2} \\ &= 2qp + k - 2q(p - (q^2 + \frac{t}{2})) - \frac{1}{2} \\ &= 2q^3 + tq + k - \frac{1}{2} \end{aligned}$$

have.

$$H(k-1) = H(k) + q(k)$$

and correspondingly from

$$q(k-1) = -q(k) + \frac{k-1}{p(k) - 2q(k)^2 - t}$$

$$p(k-1) = -p(k) + 2q(k)^2 + t$$

transf. that produces new soln's w/ change of parameter

τ - function : defined by

$$\frac{d}{dt} \log \tau(k) = H(k) \quad \text{satisfies}$$

$$\frac{d^2}{dt^2}(\log \tau(k)) = \frac{\tau(k-1)\tau(k+1)}{\tau(k)^2}$$

Toda equation