1 Painlevé equations

Main properties of Painlevé (PI):

$$w'' = 6w^2 + z$$

1) All local solutions : initial condition $\begin{cases} w(z_0) = w_0 \neq \infty \\ w'(z_0) = w_0' \neq \infty \end{cases}$
admit an analytic continuation as single-valued meromorphic functions in $\mathbb{C}$

2) expansion at poles : all poles are double
and $w'(z) = -2\,w(z)^{3/2} - \frac{1}{2} z\, w(z)^{-1/2} - \frac{1}{2} w(z)^{-1} + \cdots$

3) however these solutions are not algebraic

Note : $\mathcal{A}(\mathbb{C}(z))$ = merom functions satisfying an algebraic diff eq. of first order :

$$(w')^m + \sum_{j=1}^{m} P_j(z,w)(w')^{m-j} = 0 \qquad P_j(z,w) \in \mathbb{C}(z)[w]$$

(and $A(\mathbb{C}(z))$ alg. diff eq. of any order) look at corresp. alg. equation

$$\underset{z,w}{P}(X) = X^m + \sum_j P_j(z,w)\, X^{m-j} = 0$$ polyn. in X, w coeffs in $\mathbb{C}(z)$

defining a surface in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$

branches of this algebraic function

$X = g_0(z,w), \dots, g_{m-1}(z,w)$: $\exists$ loops γ_j in $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ s.t. an. cont. of g_0 along γ_j gives g_j

$$P_1(z,w) = -\sum_{j=0}^{m-1} g_j(z,w) \qquad \text{by } \prod_i (X - g_i) = P(X)$$

suppose a solution of (P1) is of this form

So sol'n of (P1) that also satisfies

$$\underline{w' = g_j(z,w)} \quad \Leftarrow \quad (w')^m + \sum_j P_j (w')^{m-j} = 0$$

by (P1): and $P_1(z,w) = -\sum_{j=0}^{m-1} g_j(z,w)$ get

that this should satisfy

expansion for w' at poles of w

but $P_1(z,w)$ is <u>polynomial</u> in w So contradiction

$\boxed{3}$ Painlevé (P2) : $w'' = 2w^3 + zw + \alpha$ $\quad \alpha \in \mathbb{C}$ param.

1) again all solutions analytically continue to merom. functions in $\mathbb{C}$ and can be represented non-uniquely as quotient of entire functions $w(z) = \dfrac{v(z)}{u(z)}$

Then (P2) becomes equations

$$\begin{cases} uu'' = (u')^2 - v^2 \\ v''u^2 + v(u')^2 - 2u'v'u \\ \qquad = v^3 + z\,vu^2 + \alpha u^3 \end{cases}$$

problems equivalent up to e^{az+b} factor:

$$w(z_0) = \frac{v_0}{u_0}$$

$$w'(z_0) = \frac{u_0 v_0' - v_0 u_0'}{u_0^2}$$

from eq:

$$w^2 = -\left(\frac{u'}{u}\right)' = \left(\frac{v}{u}\right)^2$$

so two sol'ns (u, v) $(\tilde{u}, \tilde{v})$ $\Rightarrow$ same w

iff $-\left(\frac{u'}{u}\right)' = \left(\frac{v}{u}\right)^2 = \left(\frac{\tilde{v}}{\tilde{u}}\right)^2 = -\left(\frac{\tilde{u}'}{\tilde{u}}\right)'$

$$\frac{u'}{u} + a = \frac{\tilde{u}'}{\tilde{u}} \qquad \tilde{u} = u \cdot e^{az+b}$$

$$\log u + az + b = \log \tilde{u}$$

Bäcklund transform and solutions of (P2)

dependence on parameter $\alpha \in \mathbb{C}$

$$\mathcal{P}_2(\alpha) = \text{set of solutions of (P2) with param. } \alpha$$
(arb. initial cond)

$$\mathcal{P} = \bigcup_{\alpha \in \mathbb{C}} \mathcal{P}_2(\alpha)$$

construct a transformation $S : \mathcal{P} \to \mathcal{P}$

given solution constructs new solutions (different parameter)

$$w_\alpha \in \mathcal{P}_2(\alpha)$$

$S(w_\alpha) = -w_\alpha$ maps $\mathcal{P}_2(\alpha)$ to $\mathcal{P}_2(-\alpha) \ni -w_\alpha$

and $S \circ S = id$

another transformation

$$w \in \mathcal{P}_2(\alpha - 1) \longmapsto Tw \in \mathcal{P}_2(\alpha)$$

$$Tw := \begin{cases} -w & \text{if } \alpha = \frac{1}{2} \text{ and } w' = -w^2 - \frac{z}{2} \\ -w - \dfrac{\alpha - 1/2}{w' + w^2 + z/2} & \text{otherwise} \end{cases}$$

5

Note: can write (P2) as a system

$$* \begin{cases} w' = \varepsilon w^2 + \frac{1}{2}\varepsilon z + v \\ v' = \alpha - \frac{\varepsilon}{2} - 2\varepsilon w v \end{cases} \qquad (\varepsilon = \pm 1)$$

if (w,v) sol'n of $*$ then w soln of (P2)
and if w sol'n of P2 then v defined by first eq of $*$ gives (w,v) sol'n of $*$

then see that if take $\varepsilon = +1$ and $v = w' - w^2 - \frac{z}{2}$

then

$$v = -(y' + y^2 + \tfrac{z}{2}) \quad \text{for } y = T^{-1}w$$

this used to show
that differentiating & using $*$
get soln of (P2) for $\alpha - 1$ if w sh fn α

<u>Note</u>: equivalently $w_{\alpha+1} = T(w_\alpha)$

$$w_{\alpha+1} = -w_\alpha - \frac{\alpha + \frac{1}{2}}{w_\alpha' + w_\alpha^2 + z/2} \qquad (\alpha \neq \pm 1/2)$$

Then two trans of S, T generate a group that acts on $\mathcal{P}$

[6] and to determine solutions for arbitrary α sufficient to determine for $0 \leq \mathrm{Re}(\alpha) \leq \frac{1}{2}$ and then act with this group

---

Rational solutions of (P2) ? $\iff \exists$ solutions (u, v) u, v polynomials $(v \not\equiv 0)$

this happens iff $\alpha \in \mathbb{Z}$

if $\alpha = 0$ $w \equiv 0$ is a solution then apply T repeatedly:

for $\alpha = 1$: $\frac{1}{z}$ and following appl. of T still rational funct's for other $\alpha \in \mathbb{N}$

and by S for all $\alpha \in \mathbb{Z}$

$w \equiv 0$ is only rational solution for $\alpha = 0$

$$w'' = 2w^3 + zw$$

$w = \frac{P}{Q}$ polyn.

$$\frac{w''}{w} = \frac{P''}{Q} - \frac{Q''}{Q} - 2\frac{Q'}{Q}\frac{P'}{P} + 2\left(\frac{Q'}{Q}\right)^2$$

$\Rightarrow \frac{w''}{w} \to 0$ when $z \to \infty$ by degrees of polyn's

but $\frac{w''}{w} = 2w^2 + z$ so $\frac{w''}{w} \to \infty$ as $z \to \infty$

so cannot have nonzero rat sol'n at $\alpha = 0$

but then if had other rat slns at other $\alpha \in \mathbb{Z}$ appl. of same S, T's would give a sol'n at 0 different from 0

so there are ONLY rat. sol'ns : only one for each $\alpha \in \mathbb{Z}$

7 Airy solutions of P_2

algebraic solutions (first order alg. diff eq.)

$$(w')^m + \sum_{j=1}^{m} P_j(z,w)\,(w')^{m-j} = 0$$

an earlier result (Erëmenko)
reduces $P_j(z,w)$ to form

$$P_j(z,w) = \sum_{i=0}^{n_j} q_{ij}(z)\,w^i$$

where $q_{ij}(z) \in \mathbb{C}(z)$ rational functions

Riccati equations:

$$w' = \varepsilon w^2 + \tfrac{1}{2}\varepsilon z \qquad \varepsilon = \pm 1$$

solutions are also solutions
of (P_2) with $\alpha = \frac{\varepsilon}{2}$

acting w/ Backlund transform grp
$\rightsquigarrow$ solutions for $\alpha = \frac{2m+1}{2}$ $m \in \mathbb{Z}$
satisfying a first order eq.

Example: $w' = w^2 + \frac{z}{2}$ $\leftarrow$ $w_{1/2}(z)$ a solution

$$y = w_{3/2} = -w_{1/2} - \frac{1}{w_{1/2}' + w_{1/2}^2 + \frac{z}{2}}$$

and inverse transform:

$$w_{1/2} = \underbrace{-y + \frac{1}{y' - y^2 - z/2}}$$

get from in Riccati equation the algebraic diff. equation satisfied by y:

$$(y')^3 - \left(y^2 + \frac{z}{2}\right)(y')^2 - \left(y^4 + zy^2 + 4y + \frac{z^2}{4}\right) y' + y^6 + \frac{3}{2} z y^4 + 4y^3 + \frac{3}{4} z^2 y^2 + 2yz + 2 + \frac{z^3}{8} = 0$$

Again an argument like for rational funct's showing these are the <u>only</u> Airy soln's

---

Painlevé (P3) & (P5) have similar properties while (P1) (P2) & (P4) have in common all soln's are meromorphic

(P4) has 2 parameters $\rightsquigarrow$ more complicated phenomena in param. space & Backlund transform

Painlevé (P4):

$$w'' = \frac{(w')^2}{2w} + \frac{3}{2} w^3 + 4zw^2 + 2(z^2 - \alpha) w + \frac{\beta}{w}$$

$$\alpha, \beta \in \mathbb{C}$$

$\boxed{9}$ first as for (P2) write (P4) in terms of Riccati equations : <u>pair</u> of Riccati

$$\begin{cases} w' = q + 2\mu z w + \mu w^2 + 2\mu w u \\ u' = p - 2\mu z u - \mu u^2 - 2\mu w u \end{cases}$$

$$q^2 = -2\beta \qquad p = -1 - \alpha\mu - \tfrac{1}{2} q \qquad \mu^2 = 1$$

then take

$$\tilde{\alpha} := \tfrac{1}{4}(2\mu - 2\alpha + 3\mu q) \qquad \tilde{\beta} := \tfrac{1}{2}(1 + \alpha\mu + \tfrac{1}{2} q)^2$$

Then have a Backlund transform:

$$T : w_{\alpha,\beta} \longmapsto w_{\tilde{\alpha},\tilde{\beta}}$$

$$w_{\tilde{\alpha},\tilde{\beta}}(z) = \frac{1}{2\mu w_{\alpha\beta}} \left(w'_{\alpha\beta} - \mu w^2_{\alpha\beta} - 2\mu z w_{\alpha\beta} - q \right)$$

inverse T^{-1}

$$w_{\alpha,\beta}(z) = -\frac{1}{2\mu w_{\tilde{\alpha},\tilde{\beta}}} \left(w'_{\tilde{\alpha},\tilde{\beta}} - \tilde{q} + 2\mu z w_{\tilde{\alpha},\tilde{\beta}} + \mu w^2_{\tilde{\alpha},\tilde{\beta}} \right)$$

10

$\tilde{q}^2 = -2\tilde{\beta}$ $\quad \alpha = (-2\mu - 2\tilde{\alpha} - 3\mu\tilde{q})/4$

$q = \tilde{\alpha}\mu - 1 - \tilde{q}/2$ $\quad \beta = -q^2/2$

For (and only for) (α, β) with

$$\beta = -2(\alpha\mu + 2n + 1)^2 \qquad n \in \mathbb{N}, \quad \mu^2 = 1$$

or

$$\beta = -2n^2 \qquad n \in \mathbb{N}$$

$\exists$ a one-parameter family of solutions of (P4) that satisfy a 1st order algebraic equation

$P(z, w, w') = 0$

all obtained from Riccati equation

$$w' = \mu w^2 + 2\mu z w - 2(1 + \alpha\mu)$$

and $\beta = -2(1+\alpha\mu)^2$ by

repeated application of Backlund transform and

$$\tilde{W}(z) = \lambda^{-1} w(\lambda z) \qquad \lambda^2 = -1$$

Also sufficient to know sol'ns for param's in

$$0 \le \mathrm{Re}(\alpha) \le 1 \qquad \mathrm{Re}\sqrt{-2\beta} \ge 0$$

$$\mathrm{Re}\left(\sqrt{-2\beta} + 2\alpha\right) \le 2$$

and rest obtained by Backlund transform

11

Hermite polynomials

$$H_n(\tau) = (-1)^n \exp(\tau^2) \frac{d^n}{d\tau^n} \exp(-\tau^2)$$

$W(z) = \dfrac{u(z)}{v(z)}$ for (P4) gives equations

$$(**)\quad \begin{cases} u u'' - (u')^2 + v^2 + 2 z u v = 0 \\ 2u^2 v v'' + v^2 (u')^2 - 2uv\, u'v' - u^2 (v')^2 \\ \quad = v^4 + 4 z u v^3 + 4(z^2-\alpha) u^2 v^2 + 2\beta u^4 \end{cases}$$

taking $\begin{cases} v(z) = P(z) \exp(g(z)) \\ u(z) = Q(z) \exp(g(z)) \end{cases}$ (***)

P, Q polynomials (deg n, m resp.)
$g(z)$ some entire funct.

then
$$(*)\quad \begin{cases} QQ'' - (Q')^2 + g'' Q^2 + 2zPQ + P^2 = 0 \\ 2PQ^2 P'' + 2P^2 Q^2 g'' + P^2 (Q')^2 - 2PQP'Q' - (P')^2 Q^2 \\ \quad = P^4 + 4z P^3 Q + 4(z^2-\alpha) P^2 Q^2 + 2\beta Q^4 \end{cases}$$

then use this like in case of (P2):

12 (P4) has rational solution iff $\circledast$ has (P, Q) polyn. solution for some $g(z)$ that is compatible w/ $(\ast\ast)$ & $(\ast\ast\ast)$

---

from this then get:

(P4) has rational solution $w(z)$ iff (α, β) are of the form

$$\alpha = n_1 \qquad \beta = -\frac{2}{9}(6n_2 - 3n_1 + 1)^2 \qquad n_1, n_2 \in \mathbb{Z}$$

or

$$\alpha = n_1 \qquad \beta = -2(1 + 2n_2 - n_1)^2 \qquad n_1, n_2 \in \mathbb{Z}$$

These rational solutions obtained by Backlund: e.g. for start w/ $\underset{0,-2}{w(z)} = -2z$

then get solutions ... these include

$$w(z) = -\sqrt{\mu}\,\frac{H_n'(t)}{H_n(t)} \qquad \alpha = -\mu(1+n) \quad \beta = -2n^2$$
$$z = \sqrt{\mu}\, t \qquad \mu^2 = 1$$

$$w(z) = -2z + i\sqrt{\mu}\,\frac{H_n'(t)}{H_n(t)} \qquad \alpha = n\mu,\ \beta = -2(1+n)^2$$
$$z = i\sqrt{\mu}\, t \qquad \mu^2 = 1$$

13

Painlevé (P3) & Painlevé (P5)

Painlevé (P3) : $\alpha\ \beta\ \gamma\ \delta \in \mathbb{C}$

$$w'' = \frac{(w')^2}{w} - \frac{w'}{z} + \frac{\alpha w^2 + \beta}{z} + \gamma w^3 + \frac{\delta}{w}$$

- different behavior w/resp to merom solutions

e.g. $\beta = \gamma = 0$ and $\alpha \neq 0$ $\delta \neq 0$

$$w(z) = C z^{1/3} \quad \text{for } C^3 = -\delta/\alpha$$

is a non-meromorphic solution

but : changing variables

$$W(\xi) = z w(z) = e^{\xi/2} w(e^{\xi/2}) \qquad z = e^{\xi/2}$$

gives meromorphic solutions :

replace $\alpha\ \beta\ \gamma\ \delta \rightsquigarrow 4\alpha\ \ 4\beta\ \ 4\gamma\ \ 4\delta$ get eq.

$$W'' = \frac{(W')^2}{W} + \alpha W^2 + \gamma W^3 + \beta e^{\xi} + \frac{\delta e^{2\xi}}{W}$$

(Modified P3) for this eq:

All local solutions admit a single-valued merom continuation to $\mathbb{C}$

(Hinkkanen-Laine thm)

$\boxed{14}$

- Some special values of param's for which (P3) can be integrated in terms of known functions

again change var. to $z = \exp(\xi)$

$$W'' = \frac{(W')^2}{W} + e^{\xi}(\alpha W^2 + \beta) + e^{2\xi}\left(\gamma W^3 + \frac{\delta}{W}\right)$$

then solutions would satisfy identities

$$(\cdot) \quad \frac{d}{dz}\left(\left(\frac{W'}{W}\right)^2 + \left(\frac{\delta}{W^2} - \gamma W^2\right)e^{2\xi} + 2\left(\frac{\beta}{W} - \alpha W\right)e^{\xi}\right)$$
$$= 2\left(\frac{\delta}{W^2} - \gamma W^2\right)e^{2\xi} + 2\left(\frac{\beta}{W} - \alpha W\right)e^{\xi}$$

and

$$(\cdot\cdot) \quad \underbrace{\left(\frac{W'}{W}\right)'}_{} = \left(\frac{\delta}{W^2} - \gamma W^2\right)e^{2\xi} + \left(\frac{\beta}{W} + \alpha W\right)e^{\xi}$$

$2(\cdot\cdot) + (\cdot) \Rightarrow \quad \frac{d}{dz}\left(\left(\frac{W'}{W}\right)^2 + \underbrace{2\frac{W'}{W}}_{} + G(\xi, W)\right) = 4(\delta e^{2\xi} + \beta e^{\xi} W)\frac{1}{W^2}$

for $\underline{G(\xi, W)} = \left(\frac{\delta}{W^2} - W^2\right)e^{2\xi} + 2\left(\frac{\beta}{W} - \alpha W\right)e^{\xi}$

$$\Rightarrow \quad \frac{d}{dz}\left(\left(\frac{W'}{W}\right)^2 - \underbrace{2\frac{W'}{W}}_{} + G(z, W)\right) = -4(\gamma e^{2\xi} W^2 + \alpha e^{\xi} W)$$

in case with $\alpha = \beta = \gamma = \delta = 0$

$\Rightarrow$ ~~from this all solutions~~ will have to be $W(\xi) = C_1 e^{C_2 \xi}$

so $w(z) = C_1 z^{C_2}$ power functions

then for special cases:

for $\underline{\beta = \delta = 0}$ from $\boxed{\odot}$

$$\left(\frac{W'}{W}\right)^2 + 2\,\frac{W'}{W} - 2\alpha e^{\xi} W - \gamma e^{2\xi} W^2 = c_1$$

or for $\underline{\alpha = \gamma = 0}$ from $\boxed{ae}$

$$\left(\frac{W'}{W}\right)^2 - 2\,\frac{W'}{W} + \delta e^{2\xi}\frac{1}{W^2} + 2\beta e^{\xi}\frac{1}{W} = C_2$$

$$\frac{1}{V} = e^{\xi} W$$

$$(V')^2 = 2\alpha V + \gamma + (1 + C_1) V^2$$

$$(V')^2 = -2\beta V - \delta + (1 + C_2) V^2$$

general solution (for $\underline{1 + c_1 \neq 0}$)

$$V(\xi) = -\frac{\alpha}{1 + C_1} + \frac{(1 + C_1)\, e^{2(1+C_1)^{1/2}(c + \mu\xi)} + 4(\alpha^2 - \gamma - \gamma C_1)}{4(1 + C_1)^{3/2}\, e^{(1+C_1)^{1/2}(c + \mu\xi)}}$$

$\mu^2 = 1$, c = arbitrary constant

for $1 + C_1 = 0$, $\alpha \neq 0$

$$V(\xi) = \frac{1}{2}\left(\alpha c^2 - \frac{\gamma}{\alpha} - 2\mu\alpha c\xi + \alpha\xi^2\right)$$

& $1 + C_1 = 0$, $\alpha = 0$: $\quad V(\xi) = c + \mu\sqrt{\gamma}\,\xi$

- as for (P2) (P4) values of param for which solution $w(z)$ of (P3) that also satisfies a Riccati equation:

$$\circledast \quad w' = a(z)w^2 + b(z)w + c(z) \qquad a(z) \not\equiv 0$$

meromorphic coeff's

→ substitute into (P3) to get

$$a^2 = \gamma \neq 0 \qquad b = \frac{\alpha - a}{az} \qquad c^2 = -\delta \qquad a\beta + c(\alpha - 2a) = 0$$

$\Rightarrow$ $\circledast$ is of free form

$$w' = aw^2 + \frac{\alpha - a}{az} w + c$$

linear equation

$$\frac{d^2u(\tau)}{d\tau^2} + \frac{a - \alpha}{a\tau} \frac{du(\tau)}{d\tau} + u(\tau) = 0$$

reduces to by $w(z) = -\frac{1}{a\,u(z)} \frac{du(z)}{dz}$ $\qquad z = \lambda\tau$

$\lambda^2 = (ac)^{-1}$

$(c \neq 0)$

→ this eq. admits a sol'n by classical special functions (SLP) Bessel functions $J_\nu(\tau)$

$$u(\tau) = \tau^{\alpha/2a} \left(C_1 J_{\alpha/2a}(\tau) + C_2 J_{-\alpha/2a}(\tau) \right)$$

for $\frac{\alpha}{2a} \notin \mathbb{Z}$ or for $\frac{\alpha}{2a} \in \mathbb{Z}$: $Y_{\frac{\alpha}{2a}}$ second kind Bessel if $c=0$ $a\neq 0$ also explicit solution

$$w(z) = z^{\frac{\alpha}{a}-1} \cdot \left(C - \frac{a^2 z^{\alpha/a}}{\alpha}\right)^{-1}$$

- another example of solution by knows special functions:

$$\alpha = a(1+2n) \quad n \in \mathbb{N}_{\geq 0}$$

then special case of Bessel functions

$$u(\tau) = \tau^{1+2n}\left(C_1 \frac{d^n}{(\tau d\tau)^n} \frac{\cos\tau}{\tau} + C_2 \frac{d^n}{(\tau d\tau)^n} \frac{\sin\tau}{\tau}\right)$$

---

outside of the integrable cases

$$\alpha=\beta=\gamma=\delta=0 \quad \text{or} \quad \alpha=\gamma=0 \quad \text{or} \quad \beta=\delta=0$$

in other cases can still generate new solutions from given solutions by a transformation

$w(z) = \phi(z, \alpha, \beta, \gamma, \delta)$ solution of (P3)

$$T: w \longmapsto \sigma_1 w^{k_1}$$
$$z \longmapsto \sigma_2 z^{k_2} \qquad \sigma_j, k_j \neq 0$$

then

$$T_1(\sigma_1,\sigma_2) : \phi \longmapsto \underbrace{\tilde{\phi}(z, \alpha\sigma_1\sigma_2, \beta\sigma_1^{-1}\sigma_2, \gamma\sigma_1^2\sigma_2^2, \delta\sigma_1^{-2}\sigma_2^2)}_{\shortparallel}$$

$$\sigma_1^{-1}\phi(\sigma_2 z, \alpha, \beta, \gamma, \delta)$$

18

$T_2 : \phi \mapsto \tilde{\phi}(z, -\beta, -\alpha, -\delta, -\gamma) := [\phi(z, \alpha, \beta, \gamma, \delta)]^{-1}$

$T_3 : \phi(z, \alpha, \beta, 0, 0) \mapsto \tilde{\phi}(z, 0, 0, 2\alpha, 2\beta) := \phi(z^2, \alpha, \beta, 0, 0)^{1/2}$

generate the solutions of (P3)

Consider then cases: (non integrable cases)

(i) $\gamma\delta \neq 0$

(ii) $\gamma = 0, \alpha\delta \neq 0$

(iii) $\delta = 0 \quad \beta\gamma \neq 0$

(iv) $\delta = \gamma = 0 \quad \alpha\beta \neq 0$

under T_3, T_2 (iii) & (iv) $\rightsquigarrow$ (i) & (ii)

so (i) & (ii) only

can reduce to $\gamma = 1 \quad \delta = -1$ for (i)

$\alpha = 1 \quad \delta = -1$ for (ii)

Rational solutions of (P3): similar discussion of algebraic & rational solns

$w = \dfrac{P(\tau)}{Q(\tau)}$ w/ variable $z = \tau^3$ (since know ~~that~~ $z^{1/3}$ behavior)

& $\alpha = 1 \quad \delta = -1$ normalize

$$w'' = \frac{(w')^2}{w} - \frac{w'}{\tau} + 9\tau\left(w^2 + \beta - \frac{\tau^3}{w}\right)$$

existence of rational solns $\exists$ iff $\beta = 2k \quad k \in \mathbb{Z}$

- for $\beta = 0$ have $w = \lambda \tau$ $\lambda^3 = 1$

then use a Backlund transform (for $\alpha = -\delta = 1$ case)

$$\tilde{w}(z, \hat{\beta}) := w^{-2} \cdot \left((\varepsilon - \beta) w + z - \varepsilon z w' \right)$$

$\hat{\beta} = \beta - 2\varepsilon$ $\varepsilon^2 = 1$

for τ in the form

$$\tilde{w}(\tau, \beta - 2\varepsilon) = \frac{3(\varepsilon - \beta) w + 3\tau^3 - \varepsilon \tau w_\tau'}{3 w^2}$$

to get rational solutions for all $\beta = 2k$ $k \in \mathbb{Z}$

e.g. -2ε: $\quad w = \dfrac{2\varepsilon\lambda + 3\tau^2}{3\lambda^2 \tau}$

-4ε: $\quad w = \dfrac{9\lambda\tau^5 + 24\varepsilon\lambda^2\tau^3 + 20\tau}{(2\varepsilon\lambda + 3\tau^2)^2}$

etc

So sufficient condition; for necessity use estimates of solution behavior at 0 & ∞

Relations between Painlevé (P3) & (P5)

take case $\delta\gamma \neq 0$ transform between solutions of these two eqns

- if $w(z)$ solution of (P3) with

$$R := w' - \varepsilon w^2 + (\alpha\varepsilon - 1)\frac{w}{z} + 1 \neq 0$$

then $u(\tau) = 1 - \dfrac{2}{R(\sqrt{2\tau})}$ is a solution of (P5) w/

$a = \frac{1}{32}(\beta - \alpha\varepsilon + 2)^2$ $\quad b = -\frac{1}{32}(\beta + \alpha\varepsilon - 2)^2$ $\quad c = -\varepsilon$ $\quad d = 0$

- (P5) has a symmetry:

if $u = u(t, a, b, c, d)$ solution of (P5) then also

$$u_1(t) = 1/u(t, -b, -a, -c, d)$$ is also (P5) solution

if $w(z)$ solution of (P3) with

$$R_1(z) := w' - \varepsilon w^2 - (\alpha\varepsilon - 1)\frac{w}{z} - 1 \neq 0$$

then

$$u_1(t, a, b, c, d) = 1 + \frac{2}{R_1(\sqrt{2}t)}$$

is a solution of (P5) with

$$a = \frac{1}{32}(\beta + \alpha\varepsilon - 2)^2 \quad b = \frac{-1}{32}(\beta - \alpha\varepsilon + 2)^2 \quad c = \varepsilon \quad d = 0 \qquad \varepsilon^2 = 1$$

---

other direction: if $u(\tau)$ is (P5) solution with a, b, $c^2 = 1$ and $d = 0$ then

$$\phi(\tau) := \tau u'(\tau) - \sqrt{2a}\, u(\tau)^2 + (\sqrt{2a} + \sqrt{-2b})\, u(\tau) - \sqrt{-2b} \neq 0$$

then $$w(z) = \frac{\sqrt{2\tau}\, u(\tau)}{\phi(\tau)} \quad \text{for } z^2 = 2\tau$$

is a solution of (P3) with parameters

$$\alpha = 2c(\sqrt{2a} - \sqrt{-2b} - 1) \qquad \beta = 2\sqrt{2a} + 2\sqrt{-2b}$$
$$\gamma = 1 \qquad \delta = -1$$

---

solns of (P3) w/ $\gamma\delta \neq 0$ $\leftrightarrow$ solns of (P5) w/ $c \neq 0$ $d = 0$

[21] (P5) $w'' = \left(\frac{1}{2w} + \frac{1}{w-1}\right)(w')^2 - \frac{w'}{z} + \frac{(w-1)^2}{z^2}\left(\alpha w + \frac{\beta}{w}\right)$
$+ \frac{\gamma w}{z} + \frac{\delta w(w+1)}{w-1}$

$\alpha\, \beta\, \gamma\, \delta$ param $\in \mathbb{C}$

again has non merom sol'ns

but again change of var $z = \exp(\xi)$
$\rightsquigarrow$ modified (P5)
for which all solutions are meromorphic

---

possibility of rational solutions ?
in a variable $z = \tau^s$ then (P5) becomes:

$$6s^2w^4\alpha - 2s^2w^5\alpha + 2s^2\beta + s^2w^3(-6\alpha - 2\beta + \tau^s(-2\gamma + \tau^s))$$
$$\odot \quad + \tau^2(w')^2 + w^2(s^2(2\alpha + 6\beta + \tau^s(2\gamma + \tau^s)) + 2\tau(w' + \tau w''))$$
$$- w(6s^2\beta + \tau(w'(2+3\tau w') + 2\tau w'')) = 0$$

search for solutions $w(\tau) = \frac{P(\tau)}{Q(\tau)}$ deg p & q

$\rightsquigarrow$ $p = q$ & s arbitrary or $q = p + s$
(case $q = p - s$ avoided due to symmetry of (P5)
$T_2 : w(z, \alpha, \beta, \gamma, \delta) \mapsto \frac{1}{w}(z, -\beta, -\alpha, -\gamma, \delta)$)

behavior at ∞ of $\odot$ for $\tau = \frac{1}{t}$ $\quad$ $p = q$ arbitrary s

solution w/ $w(t) = \sum_{k=0}^{\infty} b_k t^k$ $\quad b_0 \neq 0$

and equation gives $b_0^2(b_0+1)s^2 = 0 \Rightarrow b_0 = -1$
(b_k determined by recursion)

then by analyzing asymptotic behavior

gives another $\phi(z) = \sum_{j=0}^{\infty} a_j z^{-j}$ w/ $a_0 = -1$ a_i uniquely defined

requirement of "algebraic solution"

and $w(t) = \phi(t^{-s})$ must coincide w/

$\Rightarrow b_k = 0$ for $k \notin s\mathbb{Z}$

similar for $q = p+s$ same rel' $b_k = 0$ for $k \notin s\mathbb{Z}$

$\longleftrightarrow$ for $\delta \neq 0$ rational solutions are the only alg. solutions

- special case for $\delta = 0$ $\gamma \neq 0$ rewrite P5 as

$$(*) \quad u'' = \frac{3u-1}{2u(u-1)}(u')^2 - \frac{u'}{z} + \frac{(u-1)^2}{z^2}\left(au + \frac{b}{u}\right) + \frac{c}{z}u$$

can be reduced to (P3) $\gamma\delta \neq 0$ case

then rational solutions iff

$$a = \frac{(2n-1)^2}{8} \quad n \in \mathbb{N}$$

or $a \neq 0$ $b = -\frac{(2n-1)^2}{8}$

from (P3) case

Backlund transform for (P5):

$(*)$ thm

$w = w(z, \alpha, \beta, \gamma, \delta)$ $\delta \neq 0$ solution of (P5):

with

$$F_1(z) := zw' - \varepsilon_1 c w^2 + (\varepsilon_1 c - \varepsilon_2 a + \varepsilon_3 kz) w + \varepsilon_2 a \neq 0$$

and $c^2 = 2\alpha$ $a^2 = -2\beta$ $k^2 = -2\delta$ then

$$T_{\varepsilon_1, \varepsilon_2, \varepsilon_3} : w_1(z, \alpha_1, \beta_1, \gamma_1, \delta_1) = 1 - \frac{2k\varepsilon_3 z w}{F_1(z)}$$

claim
$$\begin{cases} \alpha_1 = -\frac{1}{16\delta}\left(\gamma + \varepsilon_3 k\left(1 - \varepsilon_2 a - \varepsilon_1 c\right)\right)^2 \\ \beta_1 = \frac{1}{16\delta}\left(\gamma - \varepsilon_3 k\left(1 - \varepsilon_2 a - \varepsilon_1 c\right)\right)^2 \\ \gamma_1 = \varepsilon_3 k\left(\varepsilon_2 a - \varepsilon_1 c\right) \\ \delta_1 = \delta \end{cases}$$

$\varepsilon_i^2 = 1 \quad i = 1,2,3$

is also a solution of (P5)

The "nonlinear superposition formula" that links solutions of (P5) via repeated application of $T_{\varepsilon_1,\varepsilon_2,\varepsilon_3}$ transformations

get "auto Backlund transform" that links solutions w/ <u>same</u> parameter values

$S_0 : w(z, \alpha, \beta, 0, \delta) \longmapsto w(-z, \alpha, \beta, 0, \delta)$

$S_1 : w(z, \alpha, -\alpha, 0, \delta) \longmapsto \frac{1}{w(z, \alpha, -\alpha, 0, \delta)}$

$S_2 : w(z, \alpha, -\alpha, \gamma, \delta) \longmapsto 1/w(-z, \alpha, -\alpha, \gamma, \delta)$

$S_3 : w(z, \alpha, \beta, 0, 0) \longmapsto w(\lambda z, \alpha, \beta, 0, 0)$

then can apply compositions

$$T^{-k} \circ S_j \circ T^{k}$$

to get more auto-Backlund transforms

- for $\delta = -1/2$ and $j = 0, 2$

are auto BT for • $\gamma = n$ $n \in \mathbb{Z}$ & $j=0$

or • $-2\beta = (\sqrt{2\alpha} + n)^2$ $n \in \mathbb{N} \cup \{0\}$ & $j=2$

also auto BT for w/ $j=1$

• $\gamma = m$ $-2\beta = (\sqrt{2\alpha} + k)^2$ $m \in \mathbb{Z}$ $k \in \mathbb{N} \cup \{0\}$ $k+m \in 2\mathbb{Z}$

or • $\gamma \notin \mathbb{Z}$ $8\alpha = (2k-1)^2$ $-8\beta = (2m-1)^2$ $k, m \in \mathbb{Z}$

Painlevé (P6) $\alpha \beta \gamma \delta \in \mathbb{C}$

$$w'' = \frac{1}{2}\left(\frac{1}{w} + \frac{1}{w-1} + \frac{1}{w-z}\right)(w')^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{w-z}\right)w'$$

$$+ \frac{w(w-1)(w-z)}{z^2(z-1)^2}\left(\alpha + \frac{\beta z}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + \frac{\delta z(z-1)}{(w-z)^2}\right)$$

sing pts $z = 0, 1, \infty$

All other (P_i) can be obtained from (P6) by coalescence of sing. of (P6)

<u>e.g.</u> $z \mapsto 1 + \varepsilon z$ $\delta \mapsto \delta \varepsilon^{-2}$ $\gamma \mapsto \gamma \varepsilon^{-1} - \delta \varepsilon^{-2}$ in (P6) gives (P5) when $\varepsilon \to 0$

<u>Painlevé (P6) and the Weierstrass $\wp$-function:</u>

(25) Fuchs presentation of (P6):

$$\mathcal{L}_z \int_\infty^w \frac{dt}{\sqrt{t(t-1)(t-z)}} =$$

$$\sqrt{w(w-1)(w-z)}\left(\alpha + \frac{\beta}{w^2} + \frac{\gamma(z-1)}{(w-1)^2} + (\delta - \tfrac{1}{2})\frac{z(z-1)}{(w-z)^2}\right)$$

$\mathcal{L}_z$ = Picard-Fuchs linear diff. operator

$$\mathcal{L}_z = z(1-z)\frac{d^2}{dz^2} + (1-2z)\frac{d}{dz} - \frac{1}{4}$$

$$u = u(w,z) = \int_\infty^w \frac{dt}{\sqrt{t(t-1)(t-z)}} \quad \text{elliptic int}$$

with

$$w = w(u) = \wp(u; z) \quad \text{Weierstrass}$$

$\Lambda_z = \mathbb{Z} + \mathbb{Z}z$

- special case of $\alpha = \beta = \gamma = 0$ $\delta = \frac{1}{2}$:
  u satisfies Legendre equation

$$4z(z-1)\frac{d^2u}{dz^2} + 4(2z-1)\frac{du}{dz} + u = 0$$

with solutions

$$K(z) = \int_0^1 \frac{ds}{\sqrt{(1-s^2)(1-\mu^2 s^2)}}$$

$$K'(z) = \int_0^{1/\mu} \frac{ds}{\sqrt{(s^2-1)(1-\mu^2 s^2)}} \qquad \mu^2 = z$$

$4K(z)$, $2iK'(z)$ } periods of $\wp$ for E_z ell. curve

26

special case of hypergeom. eq

$\Rightarrow K(z) = \frac{\pi}{2} F(1/2, 1/2, 1, z)$ hypergeom function

$K'(z) = \frac{\pi}{2} F(1/2, 1/2, 1, 1-z)$

so this special case has general solution

$$w(z) = \wp\Big(\underbrace{C_1 F(1/2,1/2,1,z) + C_2 F(1/2,1/2,1,1-z)}_{u}\,;\, z\Big)$$

More general relation of (P6) to Weierstrass $\wp$:
(Manin) : change of variable

$(w, z) \longmapsto (q, \tau)$

$$w := \frac{\wp(q;\tau) - e_1}{e_2 - e_1} \qquad z := \frac{e_3 - e_1}{e_2 - e_1}$$

$e_j = \wp(\omega_j;\tau)$ ω_j half-period points
$\omega_1 = 1/2 \quad \omega_2 = \frac{\tau}{2} \quad \omega_3 = \frac{1+\tau}{2}$

then in these new variables (P6) becomes

$$(2\pi i)^2 \frac{d^2 q}{d\tau^2} = \sum_{n=0}^{3} \alpha_n \wp'(q + \omega_n;\tau)$$

$\alpha_0 = \alpha \quad \alpha_1 = -\beta \quad \alpha_2 = \gamma \quad \alpha_3 = -\delta + \frac{1}{2}$

$\omega_0 = 0 \qquad \wp'(q;\tau)$ deriv. in q

$$\wp(q;\tau) = \frac{1}{q^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(q+m+n\tau)^2} - \frac{1}{(m+n\tau)^2} \right)$$

Backlund transform for (P6)

$W(z) = w(z, \alpha, \beta, \gamma, \delta)$ soln of (P6)

$T_j : w \longmapsto w_j \qquad j = 1, 2, 3$

$$w_1(z, -\beta, -\alpha, \gamma, \delta) = 1/w(1/z)$$

$$w_2(z, -\beta, -\gamma, \alpha, \delta) = 1 - 1/w(1/(1-z))$$

$$w_3(z, -\beta, -\alpha, -\delta + \tfrac{1}{2}, -\gamma + \tfrac{1}{2}) = z/w(z)$$

repeated compositions of T_j's gives a finite group of order 24

- other constructions of Backlund transforms for (P6)
- some auto-Backlund transforms from compositions of T_j's (6 elements in this group form a subgp. of auto-BT)