$\boxed{1}$ Nonlinear ODE's

example: $\begin{cases} x' = x + y^2 \\ y' = -y \end{cases}$

equilibria:
$x + y^2 = 0$
$y = 0$
$\Rightarrow (x=0, y=0)$ only equil. pt.

observe qualitative behavior:
when $y \ll 0$ then y^2 even smaller
so near $(0,0)$ can approximate by
linear equation $\begin{cases} x' = x \\ y' = -y \end{cases}$ $\Rightarrow$ saddle pt at $(0,0)$

in fact this nonlin system easy to solve explicitly
$y' = -y$ decouples $\Rightarrow y(t) = y_0 e^{-t}$
$x' = x + y_0^2 e^{-2t}$ first order non-autonomous

guess for form of solution ce^{-2t}
$-2ce^{-2t} = ce^{-2t} + y_0^2 e^{-2t}$ $\Rightarrow$ $c = -\frac{1}{3}y_0^2$

general solution of homogeneous equation can be added to particular solution
$x' = x \Rightarrow x_{homog}(t) = ce^{t}$

$x(t) = ce^{t} - \frac{1}{3}y_0 e^{-2t}$ is general solution

solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = X(t)$ w/ $X(0) = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$ is $\begin{cases} x(t) = (x_0 + \frac{2}{3}y_0^2)e^{t} - \frac{1}{3}y_0^2 e^{-2t} \\ y(t) = y_0 e^{-t} \end{cases}$

So can see explictly in this case the extent to which solutions of nonlinear eq. differ from linearization when moving away from the equilibrium pt. at (0,0) (see Figure)

- other observation in this simple example:

change of variables:

$$\begin{cases} u = x + \frac{1}{3}y^2 \\ v = y \end{cases} \qquad \begin{aligned} u' &= x' + \frac{2}{3}y y' \\ &= x + \frac{1}{3}y^2 = u \\ v' &= y' = -y = -v \end{aligned}$$

in these coordinates it becomes a linear equation

$$\begin{cases} u' = u \\ v' = -v \end{cases}$$

- in general not possible to change variables and linearize system in this way (because large discrepancies from linear behavior far from equilibrium pts)

Example:

$$\begin{cases} x' = \frac{1}{2}x - y - \frac{1}{2}(x^3 - y^2 x) \\ y' = x + \frac{1}{2}y - \frac{1}{2}(y^3 + x^2 y) \end{cases}$$

equilibrium again at $(x=0, y=0)$

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change to polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$x' = r'\cos\theta - r\sin\theta\,\theta' = \frac{1}{2}(r-r^3)\cos\theta - r\sin\theta$$

$$y' = r'\sin\theta + r\cos\theta\,\theta' = \frac{1}{2}(r-r^3)\sin\theta + r\cos\theta$$

equating coeff's of $\cos\theta$ & $\sin\theta$

$$\begin{cases} r' = r(1-r^2)/2 \\ \theta' = 1 \end{cases}$$ decoupled equations

$r = 1$ $\theta = \theta_0 + t$ circle counterclockwise constant speed angular movement

$r = 0$ equil. pt.
near equilibrium outward spiral (by linearization)

but don't spiral outward to ∞
because solutions cannot cross $r=1$ circle solution (uniqueness)

$r' > 0$ $0 < r < 1$ outward to circle $r=1$

$r' < 0$ $r > 1$ inward to circle $r=1$

(see Figure)

Example: $\begin{cases} x' = x^2 \\ y' = -y \end{cases}$ equilibrium at $(0,0)$

at all other pt. vectr field turns rightward $x' \geq 0$
and to $y=0$ ($y' < 0$ if $y > 0$ & $y' > 0$ if $y < 0$)

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(see figure of nonlin. solutions)

Note very different behavior from linearization at $(0,0)$ where

$$\begin{cases} x' = 0 \\ y' = -y \end{cases}$$ all pts $y=0$ are equilibria

linear behavior coming from a $\lambda = 0$ eigenvalue

$\Rightarrow$ in nonlin. system significant qualitative change of behavior

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<u>Bifurcations</u> family of nonlinear ODEs depending on parameters

$$X' = F_a(X) \quad (\text{dependence on } a \text{ is } C^\infty)$$

bifurcation: values of parameters a where behavior of solutions changes suddenly

e.g. $a < a_0$ and $a > a_0$ (for $a \in \mathbb{R}$) have ~~different #~~ of equilibria and/or different type of equilibria

$x' = x^2 + a$

$a = 0$ single equil. at $x = 0$

$a > 0$ no equilibrium

$a < 0$ a pair of equilibrium pts $x = \pm |a|^{1/2}$

Saddle-node bifurcation

in general this happens when

$$x' = f_a(x) \quad \text{with}$$

$$f_{a_0}(x_0) = 0 \qquad \& \quad \frac{\partial f_{a_0}}{\partial a}(x_0) \neq 0$$

$$f'_{a_0}(x_0) = 0$$

$$f''_{a_0}(x_0) \neq 0$$

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for $G(x,a) = f_a(x)$ $\qquad G(x_0,a_0)=0$

$\frac{\partial G}{\partial a}(x_0,a_0) \neq 0$

implicit function thm

$\exists\ a = a(x)$ smooth $\quad G(x, a(x)) = 0$

$$a'(x) = \frac{-\frac{\partial G}{\partial x}}{\frac{\partial G}{\partial a}} \qquad f_{a_0}'(x_0) = 0$$

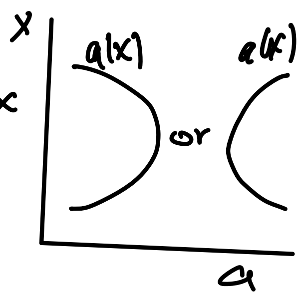
$\Rightarrow a'(x_0) = 0$

$$a''(x) = \frac{-\frac{\partial^2 G}{\partial x^2}\frac{\partial G}{\partial a} + \frac{\partial G}{\partial x}\frac{\partial^2 G}{\partial x \partial a}}{\left(\frac{\partial G}{\partial a}\right)^2}$$

$$a''(x_0) = \frac{-\frac{\partial^2 G}{\partial x^2}(x_0,a_0)}{\frac{\partial G}{\partial a}(x_0,a_0)} \neq 0 \qquad f_{a_0}''(x_0) \neq 0$$

graph of $a(x)$ concave or convex

so two equilibria one side no equil. other side



- saddle-node is generic case of bifurcation:
  (need $f_{a_0}(x_0)=0$ $f_{a_0}'(x_0)=0$ and then take the general case $f_{a_0}''(x_0)\neq 0$, $\frac{\partial f_a}{\partial a}(x_0,a_0)\neq 0$)

  (see figure)

- pitchfork bifurcation

  example: $\quad x' = x^3 - ax$

  3 equilibria $x=0$ $\ x=\pm\sqrt{a}$ when $a>0$

  and $x=0$ only for $a<0$

  (see figure)

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- example for 2-dim systems in plane of saddle-node bifurcation

$$\begin{cases} x' = x^2 + a \\ y' = -y \end{cases}$$

bifurcation for $a<0$, $a=0$, $a>0$

(see figure)

- another planar example in polar coord's

$$\begin{cases} r' = r - r^3 \\ \theta' = \sin^2(\theta) + a \end{cases}$$

$r=0$ origin is equilibrium

$a>0$ no other equilibrium

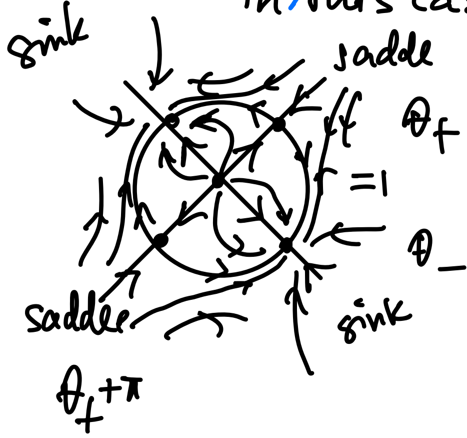
$a=0$ $r=1$
$\theta = 0, \pi \pmod{2\pi}$

$-1 < a < 0$ on circle $r=1$

$\theta =$ solution of $\sin^2(\theta) = -a$

$\theta_\pm$, $\theta_\pm + \pi$ $0 < \theta_+ < \frac{\pi}{2}$

in this case $-\frac{\pi}{2} < \theta_- < 0$

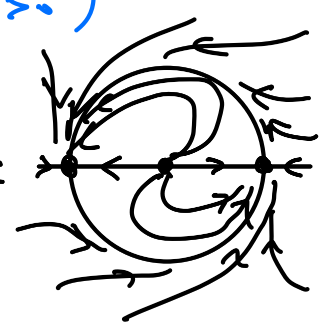


$\theta_- < \theta < \theta_+$ have $\theta' < 0$

$\theta_+ < \theta < \theta_- + \pi$ have $\theta' > 0$

(see figure for other cases $a=0$, $a>0$)

$a=0$ x-axis invariant
no other trajectory crosses it



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Hopf bifurcation

in this case number of equilibria stays same but type stable/unstable changes and a periodic orbit appears

equilibria don't capture all behavior of nonlinear systems

Example:

$$\begin{cases} x' = ax - y - x(x^2+y^2) \\ y' = x + ay - y(x^2+y^2) \end{cases}$$

equil. at $(0,0)$

linearization $X' = \begin{pmatrix} a & -1 \\ 1 & a \end{pmatrix} X$

polar coord's $\begin{cases} r' = ar - r^3 \\ \theta' = 1 \end{cases}$ (no equilibria at $r \neq 0$)

$a < 0$ origin is sink ($r' < 0$ everywhere)

$a > 0$ $r' = 0$ on $r = \sqrt{a}$ periodic solution period 2π

$r' > 0$ $r < \sqrt{a}$

$r' < 0$ $r > \sqrt{a}$

solutions spiral out of origin to circle periodic orbit

and from outside spiral around/to circle of periodic orbit (see Figure)

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other nonlinear phenomena

Null clines: nonlinear system $X' = F(X)$ $\begin{cases} x_1' = f_1(x_1, \dots, x_n) \\ \vdots \\ x_n' = f_n(x_1, \dots, x_n) \end{cases}$

j-th nullcline $N_j = \{ X = (x_1, \dots, x_n) \in \mathbb{R}^n \mid f_j(x_1, \dots x_n) = 0 \}$

$N_j \subset \mathbb{R}^n$ is a hypersurface (a real codimension one locus)

(not nec. alg. unless f_j is a polynomial)

$\mathbb{R}^n \setminus N_j = \bigsqcup_a \mathcal{U}_{j,a}$ open sets

in which $\mathcal{U}_{j,a}$ the j-th component of vector field $F(X)$ is nonzero, (>0 or <0 on all component $\mathcal{U}_{j,a}$)

for all j: $\mathcal{U}_{\underline{a}}$ "

get intersections $\mathcal{U}_{1,a_1} \cap \cdots \cap \mathcal{U}_{n,a_n}$ of open sets of each nullcline

where $\underline{a} = (a_1, \ldots, a_n)$ $\underbrace{(\pm, \pm, \ldots \pm)}_{\text{sign of each component of vector field}} = \varepsilon_{\underline{a}}$

get in each of these regions $\mathcal{U}_{\underline{a}}$ the qualitative behavior of solution w/ $\varepsilon_{\underline{a}}$ describing how vector field points in that region

Example: $\begin{cases} x' = y - x^2 \\ y' = x - 2 \end{cases}$

x-nullcline $y = x^2$

y-nullcline $x = 2$

where nullclines all meet equilibrium [$y = x^2$ and $x = 2$ meet at $(2,4)$]

(see Figure) qualitative behavior on the basis of nullclines

one/other side of parabola point right/left ; one/other side of line point down/up

9 Nullclines and bifurcations

$$\begin{cases} x' = x^2 - 1 \\ y' = -xy + a(x^2-1) \end{cases}$$

x-nullclines
$x = \pm 1$

y-nullclines
$xy = a(x^2 - 1)$

equilibrium pts
$(\pm 1, 0)$

$a = 0$ $\quad x' = x^2 - 1$
$\quad\quad\quad y' = -xy$

y-nullclines $xy = 0$: axes $x = 0, y = 0$

$y' = 0$ on axes $x = 0$ & $y = 0$
vect field tg to x axis ($y = 0$)

$x' > 0$ for $|x| > 1$
$x' < 0$ for $|x| < 1$

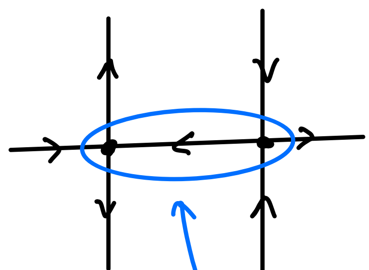
each equilibrium a saddle
(one dir. contracts one expands)

(see Figure nullclines and trajectories)

$(-1, 0)$: $y=0$ axis contracts, $x = -1$ line expands

$(+1, 0)$: $y = 0$ axis expands, $x = +1$ line contracts

(attractive repelling along $y = 0$ line)



heteroclinic connection (between saddle pts)

for $a \neq 0$ same x-nullclines $x = \pm 1$
but no longer flow tangent to $y = 0$ axis
between the two equilibrium pts $(-1,0)$ $(+1,0)$

A = region between $x = \pm 1$ lines : vector field
and between branches of curve now points in this region
$y = a(x^2-1)/x$

bifurcation at $a = 0$, $a > 0$
the heteroclinic connection is broken when $a > 0$

(see Figure)

Example Nonlinear pendulum

$$m l \frac{d^2\theta}{dt^2} = -bl \frac{d\theta}{dt} - mg \sin\theta$$

m = mass l = length (radius of circle pendulum moves on)

friction (proportional to velocity) $b > 0$ prop. factor

take units w/ $m = l = g = 1$

$$\begin{cases} \theta' = v \\ v' = -bv - \sin\theta \end{cases}$$

equilibria: $\theta = 0, v = 0$ stable
$\theta = \pi, v = 0$ unstable

at $(0,0)$ linearization $X' = \begin{pmatrix} 0 & 1 \\ -1 & -b \end{pmatrix} X$

eigenvalues: $b = 0$ purely imaginary: circles
$b > 0$ neg. real part: inward spirals to crit. pt.

What else can one say?

Energy functional

$$E(\theta, v) = \frac{1}{2} v^2 + 1 - \cos\theta$$

$$\frac{d}{dt} E = v v' + \sin\theta \, \theta' = -bv^2 \leq 0$$

conserved quantity when $b = 0$
then level sets of E contain trajectories (see Figure: ideal pendulum)

in the case where $b > 0$

have $\frac{d}{dt} E = -bv^2 \leq 0$

$= 0$ at $v = 0$ (hence at equilibrium pts)

$E(0,0) = 0$ $\quad E(\theta, v) > 0$ in a neighborhood $\mathcal{U}$ of $(0,0)$

these properties
Lyapunov function

$\alpha =$ min value of E on $\partial \mathcal{U}$ > 0

$X(t) = \begin{pmatrix} \theta(t) \\ v(t) \end{pmatrix}$ solution starting in $\mathcal{U} \setminus (0,0)$

if $X(t_n) \longrightarrow X_0$ $\quad t_n \to \infty$

$E(X_0) \leq E(X(t_n))$ because nonincreasing along trajectories

(and $<$ strict for $v \neq 0$ along trajectory) $\leftarrow$ decreasing (unless at equil. $v=0$ on all trajectory)

if $\tilde{X}(t)$ trajectory starting at X_0
should have $E(\tilde{X}(t)) \leq E(X_0)$ and $E(\tilde{X}(t)) < E(X_0)$ if not equil.

so any trajectory starting very near X_0 will also have
$E(\tilde{X}(t)) < E(X_0)$ but $E(X(t_n + \varepsilon))$ starting at $E(X(t_n))$ also should

contradicts

so only possible limit point of trajectory $X(t)$ in $\mathcal{U}$ is equilibrium $X_0 = (0,0)$

Gradient flows

$$X' = -\nabla V(X)$$

$V: \mathbb{R}^n \to \mathbb{R}$ $\quad \mathcal{C}^\infty$- function

$$\nabla V = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n} \right)$$

$$\dot{V}(X) = \frac{d}{dt}\Big|_{t=0} V(X(t))$$

V is strictly decreasing along trajectories $X(t)$ and $\dot{V}(X) = 0$ iff X equilibrium points

$(\dot{V}(X) = 0$ iff $\nabla V(X) = 0)$

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in fact here

$$\frac{d}{dt} V(X(t)) = \nabla V(X(t)) \cdot X'(t)$$
$$= -\nabla V(X(t)) \cdot \nabla V(X(t))$$
$$= -\|\nabla V(X(t))\|^2 \leq 0$$

and $= 0$ iff $\nabla V(X(t)) = 0$

So level sets of potential $V(X)$

$V^{-1}(c)$ $\quad c \in \mathbb{R}$ $\quad$ hypersurfaces (by implicit funct. thm) locally graph of some function

some $\frac{\partial V}{\partial x_j} \neq 0$ if $\underline{c \text{ regular value}}$

$\Rightarrow V(\hat{x}, g_j(\hat{x})) = c$ $\quad g_j: \mathbb{R}^{n-1} \to \mathbb{R}$ $\quad \mathcal{C}^\infty$- funct.

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and ∇V vectr is transverse to level sets with $\nabla V(x) \cdot W = 0$ W tangent to level set

at all regular pts of V

critical pts of V $\nabla V = 0$ are equilibrium pts

if critial pt & isolated minimum of V
then also attractive equilibrium
(asymptotically stable)
$\lim_{t \to \infty} X(t) = X_0$ for traj. starting in a neighb.

Hamiltonian systems   classical mechanics for 2D case

$$\begin{cases} x' = \dfrac{\partial H}{\partial y}(x,y) \\ y' = -\dfrac{\partial H}{\partial x}(x,y) \end{cases}$$

$H : \mathbb{R}^2 \to \mathbb{R}$
$\mathcal{C}^\infty$ - function
Hamiltonian

Example: ideal pendulum

$$\begin{cases} \theta' = v \\ v' = -\sin\theta \end{cases}$$

energy
$E(\theta, v) = \frac{1}{2} v^2 + 1 - \cos\theta$

is Hamiltonian

If system is Hamiltonian then

$$\dot{H} = \frac{\partial H}{\partial x} x' + \frac{\partial H}{\partial y} y' = \underbrace{\frac{\partial H}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial H}{\partial y}\frac{\partial H}{\partial x}}_{=0}$$

H is constant along trajectories

So can study solutions (in 2D) by level sets of H <u>and</u> direction of vector field: complete behavior of solutions

(In higher dim solutions are curves on the hypersurfaces given by level sets of H

$$\begin{cases} q_i' = \frac{\partial H}{\partial p_i} \\ p_i' = -\frac{\partial H}{\partial q_i} \end{cases}$$

<u>Example in 2D:</u>

$$\begin{cases} x' = y \\ y' = -x^3 + x \end{cases}$$

Hamiltonian

$$H(x,y) = \frac{x^4}{4} - \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{4}$$

only chosen so that min $= 0$ (at $(\pm 1, 0)$)

equilibria: $(\pm 1, 0)$
$(0,0)$

linearization at $(0,0)$ $\quad X' = AX \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

eigenvalues ± 1 $\Rightarrow$ saddle

at $(\pm 1, 0)$ $\quad X' = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix} X$

$-(x^2-1)x$

$-(x-1)(x+1)x$

$x=1$: $-2(x-1)$

$x=-1$: $-2(x+1)$

eigenvalues
$\pm\sqrt{2}\, i$
centers for linearized system

(see Figure trajectories of nonlinear system)

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Closed orbits and limit sets

limiting behavior of solutions of nonlinear ODEs

Example: planar system $\begin{cases} r' = \frac{1}{2}(r - r^3) \\ \theta' = 1 \end{cases}$

As seen before
all solutions except equilibrium $(0,0)=(x,y)$
tend to periodic orbit $r=1$ (see Figure)
↑ limit set

Example: $x' = \sin x \left(-\frac{1}{10}\cos x - \cos y\right)$
$y' = \sin y \left(\cos x - \frac{1}{10}\cos y\right)$

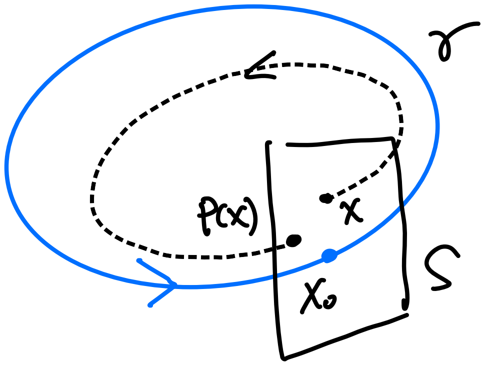
- $(0,0)$, $(0,\pi)$, (π,π), $(\pi,0)$ are equilibria all saddles
- heteroclinic solutions between them (sides of square)
- spiral source equilibrium at $\left(\frac{\pi}{2},\frac{\pi}{2}\right)$ center of square
- all solutions accumulate at sides of square (heteroclinic solutions)
↗ limit set

[see Figure of trajectories]

Poincaré return map

periodic orbits can be stable (neighborhood from which trajectories do not leave)
asymptotically stable (attracting)
or unstable (repelling in at least some directions)

to study behavior instead of linearization
use Poincaré return map
hypersurface transversal to periodic orbit
when pts on hypersurface return to it following trajectories



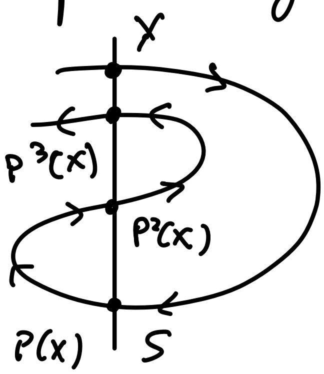
periodic orbit

$P(x)$ first return to S of point $x \in S$

$P(x_0) = x_0$

implicit function theorem: P defined and smooth near x_0

case of planar system: S is a line



$P: \mathbb{R} \to \mathbb{R}$

$P(0) = 0$ set: $(0 = x_0)$

if $|P'(0)| < 1$

expansion at 0: $P(x) = ax + \cdots$ higher order

$|a| < 1$

$\Rightarrow$ $P(x)$ closer to 0 than x

$\Rightarrow$ solution moves closer to per. orbit at each passage through S

$\Rightarrow$ γ per. orbit is asymptotically stable

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monotone sequences of pts

following a solution return map to S (2D case)

x_0, $P(x_0) = x_1$

$P(x_1) = x_2$, ..., $P(x_{n-1}) = x_n$ ...

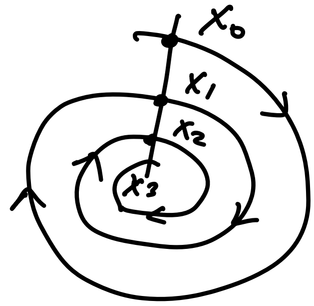
monotone along orbit:

flow along solution $x(t)$ $\longrightarrow$ $\phi_{t_n}(x_0) = x_n$ $\quad 0 \le t_1 < t_2 \cdots < t_n < \cdots$

$x(0) = x_0$ at time t $\quad \phi_t(x_0) = x(t)$

monotone along line S:

here monotone on both solution and line



S is local section of flow if $X(t)$ flow is not tangent to S anywhere

Note: if local section then monotone on trajectory $\Rightarrow$ monotone on S

i.e. case (*) above does not arise

↳ (would have to flip direction without being ever along S)

Some properties of Poincaré return map

Poincaré - Bendixson thm (2D systems)

$\Omega \neq \phi$ Ω closed & bounded $\Omega \subset \mathbb{R}^2$

limit set of trajectories of some system $X' = F(X)$

and Ω contains no equilibrium pt

$\Rightarrow$ Ω is a closed orbit

Pf. $Y \in \Omega$ lim set of traj. starting at Y is $\subset \Omega$

pt Z in this lim set and loc. section S at Z

$\{(s,u) \mid |s| < \sigma, (a,b) \in S\}$

$\parallel$

V = flow box assoc. to section S

V s.t. if $X \in V$ then $\phi_t(X) \in S$ for unique $t \in (-\sigma, \sigma)$

solution starting at Y meets S in one pt

and $\exists\, t_n$ $\phi_{t_n}(Y) \to Z$ so all in V eventually

can find pts on the traj. from these that cross S $\exists\, r, s \in \mathbb{R}$ $r > s$ $\phi_r(Y), \phi_s(Y) \in S$

but then $\phi_r(Y) \neq \phi_s(Y)$ $\Rightarrow$ $\phi_{r-s}(Y) = Y$ but no fixed pt superorbit suppose meet at two pts (or more)

y_1, y_2 on $X(t)$ flow w/ $X(0) = y$

and $y_1, y_2 \in S$

if y in limit set then y_1, y_2 also

V_1, V_2 flow boxes (intervals $J_1, J_2 \subset S$)

disjoint (see Figure)

solution $X(t)$ enters V_1 & V_2 ∞-many times

(as accumulation at y_1, y_2)

$\Rightarrow \exists$ sequence a_i, b_j : $a_1 b_1 a_2 b_2, \ldots$

monotone along solution w/ $a_i \in J_1$ $b_j \in J_2$

but then not monotone on section S

contradicts previous result

Consequences of Poincaré-Bendixon:

- closed bounded set $K \subset \mathbb{R}^2$
  invariant (forward or backward flow)
  is either limit cycle or equilibrium pt

(because if $x \in K$ then limit pts of flow from x also in K $\to$ PB thm this limit set $\nearrow$ either equi pt or periodic orbit (lim cycle)

- γ closed orbit $\gamma = \partial \mathcal{U}$ $\mathcal{U}$ open set in $\mathbb{R}^2$

then $\mathcal{U}$ contains equil. pt or lim cycle

(if neither then trajectories in $\mathcal{U}$ flow to γ

pt $z \in \gamma$ $\exists$ $\phi_{t_n}(x)$, $\phi_{s_n}(x)$ $\to z$ $t_n \to +\infty$ $s_n \to -\infty$

contradicts monotone seq's $\in S$

• γ closed orbit $\gamma = \partial U$
then U contains critical pt

(by previous U contains per orbit or
crit. pt. if only per orbits (fin. many)
choose smallest one ⇒ by previous
must contain
other per orbit (excluded)
or crit. pt

(if ∞-many per. orbits γ_n)
$x_n \in \gamma_n$ if $x_n \to x$ in U
x must be fixed pt
as x would be on a smallest per orbit
otherwise and same as