Logistic equation

$$\frac{dx}{dt} = x(\varepsilon_1 - \sigma_1 x)$$

$$\frac{dy}{dt} = y(\varepsilon_2 - \sigma_2 y)$$

these would be just two uncoupled growth phenomena without interaction

when growth of each influences other: $\boxed{\text{two cases}}$ 1) two species competing for same resources; 2) predator/prey models

Two competing species:

$$\begin{cases} \dfrac{dx}{dt} = x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) \\ \dfrac{dy}{dt} = y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) \end{cases}$$

the presence of species y population reduces the growth of species x

same for y: two species that compete for same resources

Nonlinear equation (system)

- Look for equilibrium solutions

$$\frac{dx}{dt} = 0 \ \& \ \frac{dy}{dt} = 0 \Rightarrow \begin{array}{l} x(\varepsilon_1 - \sigma_1 x - \alpha_1 y) = 0 \\ y(\varepsilon_2 - \sigma_2 y - \alpha_2 x) = 0 \end{array}$$

$$x = 0 < \begin{array}{l} y = 0 \\ \varepsilon_2 - \sigma_2 y = 0 \quad y = \frac{\varepsilon_2}{\sigma_2} \end{array}$$

one species (goes) extinct other either also extinct or reaches equilibrium

$$y = 0 < \begin{array}{l} x = 0 \\ \varepsilon_1 - \sigma_1 x = 0 \quad x = \frac{\varepsilon_1}{\sigma_1} \end{array}$$

Other equilibria

$x \neq 0$ & $y \neq 0$ but

$$\varepsilon_1 - \sigma_1 x - \alpha_1 y = 0 \ \& \ \varepsilon_2 - \sigma_2 y - \alpha_2 x = 0$$

$$\sigma_1 x + \alpha_1 y = \varepsilon_1$$
$$\alpha_2 x + \sigma_2 y = \varepsilon_2$$

$$\begin{pmatrix} \sigma_1 & \alpha_1 \\ \alpha_2 & \sigma_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

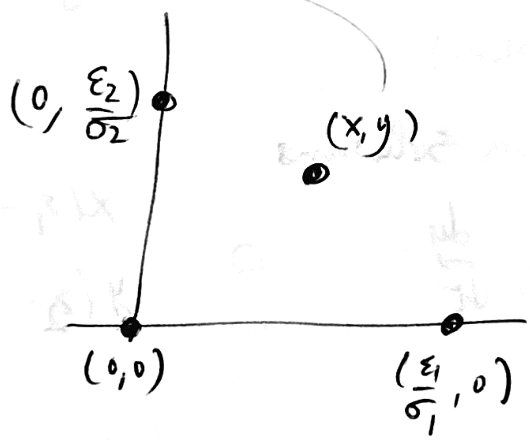
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \sigma_1 & \alpha_1 \\ \alpha_2 & \sigma_2 \end{pmatrix}^{-1} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$$

$$= \frac{1}{\sigma_1\sigma_2 - \alpha_1\alpha_2} \begin{pmatrix} \sigma_2 & -\alpha_1 \\ -\alpha_2 & \sigma_1 \end{pmatrix} \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix} = \frac{1}{\sigma_1\sigma_2 - \alpha_1\alpha_2} \begin{pmatrix} \sigma_2 \varepsilon_1 - \alpha_1 \varepsilon_2 \\ -\alpha_2 \varepsilon_1 + \sigma_1 \varepsilon_2 \end{pmatrix}$$

$$x = \frac{\sigma_2 \varepsilon_1 - \alpha_1 \varepsilon_2}{\sigma_2 \sigma_1 - \alpha_1 \alpha_2}$$

$$y = \frac{\sigma_1 \varepsilon_2 - \alpha_2 \varepsilon_1}{\sigma_1 \sigma_2 - \alpha_2 \alpha_1}$$

when both these $x > 0$ & $y > 0$ have one more equilibrium pt



- How the solutions behave near the equilibrium points:

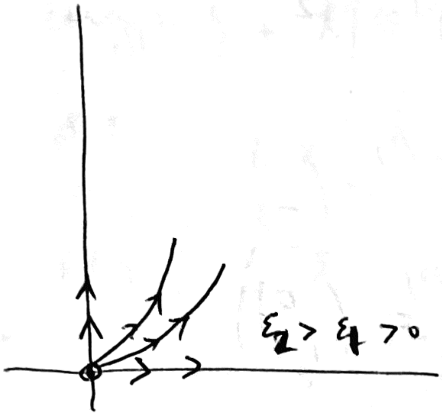
1) $x = 0$, $y = 0$ linearize the system near this solution

$$\begin{cases} \dfrac{dx}{dt} = \varepsilon_1 x \\ \dfrac{dy}{dt} = \varepsilon_2 y \end{cases}$$

- so locally solutions like those of lin. system

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \sim \begin{pmatrix} e^{\varepsilon_1 t} \\ e^{\varepsilon_2 t} \end{pmatrix}$$

if $\varepsilon_1 > 0$ & $\varepsilon_2 > 0$ repelling fixed pt.



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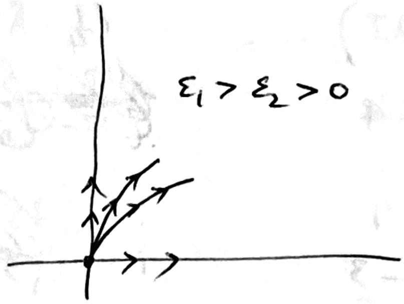


2) $x = 0 \quad y = \dfrac{\varepsilon_2}{\sigma_2}$

linearization: variable

$$v = y - \frac{\varepsilon_2}{\sigma_2} \qquad u = x$$

$$y = \frac{\varepsilon_2}{\sigma_2} + v$$

$$\begin{cases} \dfrac{du}{dt} = \varepsilon_1 u - \alpha_1 \dfrac{\varepsilon_2}{\sigma_2} u \\[2ex] \dfrac{dv}{dt} = \dfrac{\varepsilon_2}{\sigma_2}\left(-v\sigma_2 - \alpha_2 u\right) \end{cases}$$

$$\begin{pmatrix} \dfrac{du}{dt} \\[1ex] \dfrac{dv}{dt} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} \qquad A = \begin{pmatrix} \varepsilon_1 - \alpha_2 \dfrac{\varepsilon_2}{\sigma_2} & 0 \\[2ex] -\alpha_2 \dfrac{\varepsilon_2}{\sigma_2} & -\varepsilon_2 \end{pmatrix}$$

What kind of equilibrium this is depends on

$$\begin{cases} \det(A) = -\varepsilon_1 \varepsilon_2 + \alpha_2 \dfrac{\varepsilon_2^2}{\sigma_2} \quad \text{and} \\[2ex] Tr(A) = \varepsilon_1 - \left(1 + \dfrac{\alpha_2}{\sigma_2}\right)\varepsilon_2 \end{cases}$$

Example:

$$\begin{cases} \dfrac{dx}{dt} = x(1 - x - y) \\[2ex] \dfrac{dy}{dt} = y\left(\dfrac{3}{4} - y - \dfrac{1}{2}x\right) \end{cases} \qquad \frac{3}{8} \qquad x = 0, \quad y = \frac{3}{4}$$

$$\frac{du}{dt} = u\left(1 - \frac{3}{4}\right) = \frac{1}{4}u$$

$$\frac{dv}{dt} = \frac{3}{4}\left(-v - \frac{1}{2}u\right)$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ -\frac{3}{8} & -\frac{3}{4} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

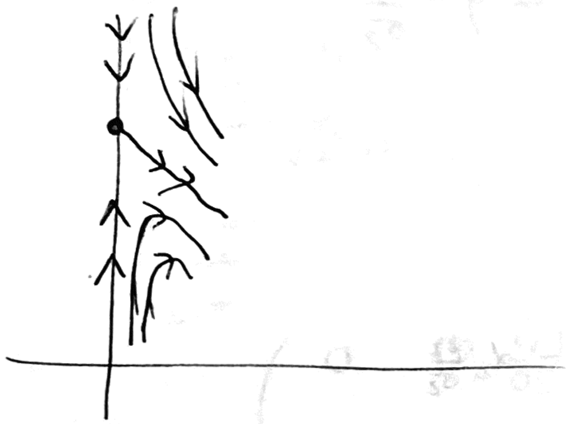
$\det = -\dfrac{3}{16} \qquad tr = -\dfrac{1}{2}$

$\det(A-\lambda I) = 0 = \left(\frac{1}{4}-\lambda\right)\left(-\frac{3}{4}-\lambda\right) = \lambda^2 + \frac{\lambda}{2} - \frac{3}{16}$

$\lambda_1 = \frac{1}{4}$ $\qquad \xi^{(1)} = \begin{pmatrix} 8 \\ -3 \end{pmatrix}$

$\lambda_2 = \frac{-3}{4}$ $\qquad \xi^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 8 \\ -3 \end{pmatrix} e^{\frac{1}{4}t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\frac{3}{4}t}$$



Other equilibrium pt: $\quad x=1, \; y=0$

$\frac{dx}{dt} = x(1-x-y)$

$\frac{dy}{dt} = y\left(\frac{3}{4} - y - \frac{1}{2}x\right)$

linearization

$u = x-1$

$v = y$

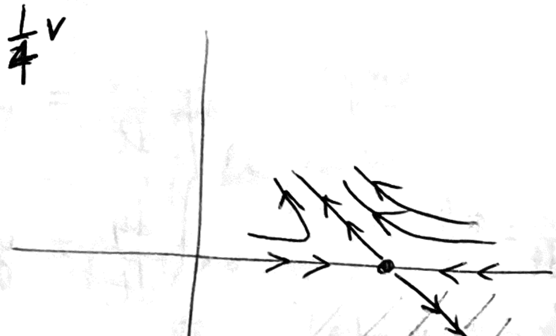
$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -1 & -1 \\ 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$$

$$\begin{cases} \frac{du}{dt} = -u - v \\ \frac{dv}{dt} = \frac{1}{4}v \end{cases}$$

$\det(A-\lambda I)$ $\qquad \lambda_1 = -1$ $\qquad \xi^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$\lambda_2 = \frac{1}{4}$

$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 4 \\ -5 \end{pmatrix} e^{\frac{1}{4}t} \qquad \xi^{(2)} = \begin{pmatrix} 4 \\ -5 \end{pmatrix}$$



The remaining equilibrium pt:

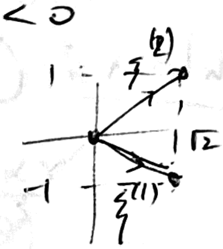
$$x = \frac{1}{2} \qquad y = \frac{1}{2}$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{4} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

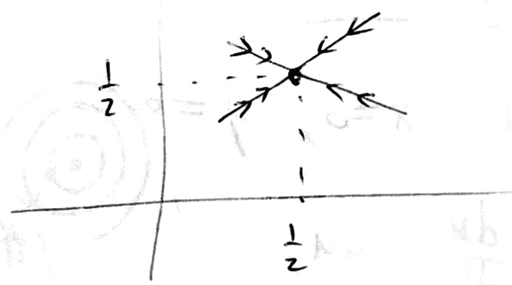
$\det(A - \lambda I) = 0$ gives $\lambda_1 = \frac{-2+\sqrt{2}}{4} < 0$

$\lambda_2 = \frac{-2-\sqrt{2}}{4} < 0$

$$\xi^{(1)} = \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} \qquad \xi^{(2)} = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$$

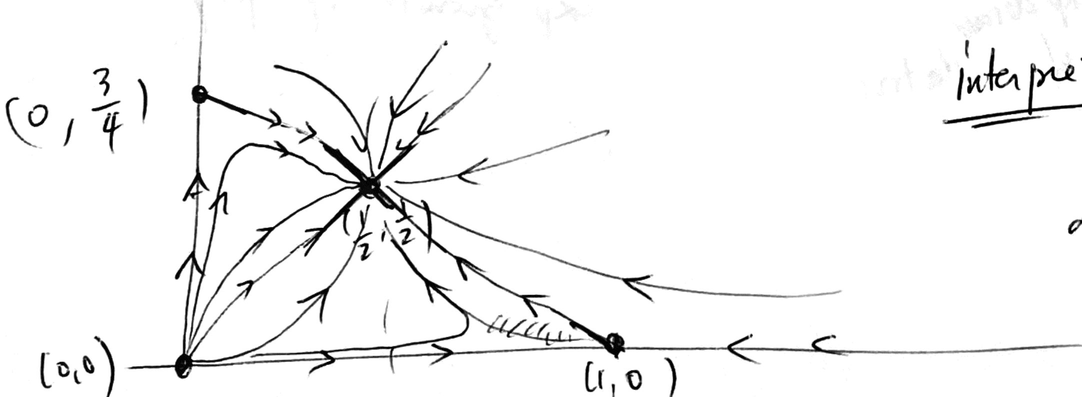


$$\begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} \sqrt{2} \\ -1 \end{pmatrix} e^{\frac{-2+\sqrt{2}}{4}t} + c_2 \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix} e^{\frac{-2-\sqrt{2}}{4}t}$$



So general picture for solutions (qualitative)

Note: certain solutions will separate areas w/ different behavior of solutions



interpret different solutions in population dynamics terms

Predator / Prey model:

$$\begin{cases} \dfrac{dx}{dt} = ax - \alpha xy \\[2ex] \dfrac{dy}{dt} = -cy + \gamma xy \end{cases} \qquad a, c, \alpha, \gamma > 0$$

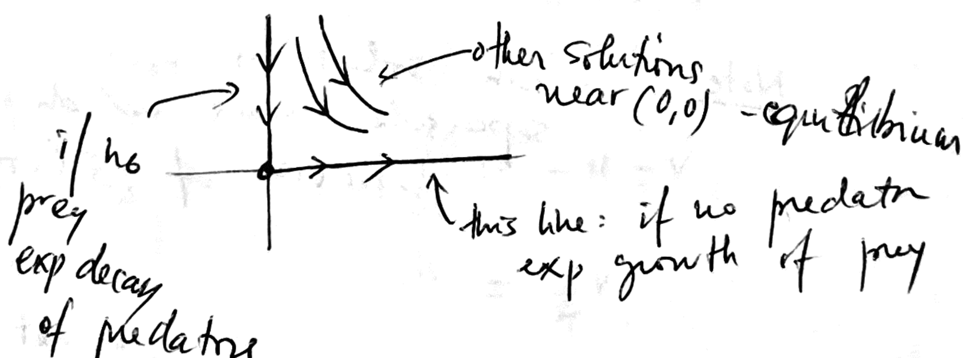
<u>Equilibria</u>: (1) $x = 0$ & $y = 0$

(2) $y = \dfrac{a}{\alpha}$ & $x = \dfrac{c}{\gamma}$

<u>Linearizations</u>:

at $x=0$, $y=0$

$$\begin{cases} \dfrac{du}{dt} = au \\[2ex] \dfrac{dv}{dt} = -cv \end{cases} \qquad \begin{pmatrix} u \\ v \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{at} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-ct}$$



near $\begin{cases} x = \frac{c}{\gamma} \\ y = \frac{a}{\alpha} \end{cases}$ equilibrium

$u = x - \frac{c}{\gamma}$

$v = y - \frac{a}{\alpha}$

$$\frac{du}{dt} = -\frac{c\alpha}{\gamma} v$$

$x(a-\alpha y) \sim (\frac{c}{\gamma} + u) \cdot \alpha v \sim \frac{c\alpha}{\gamma} v$

$$\frac{dv}{dt} = +\frac{a\gamma}{\alpha} u$$

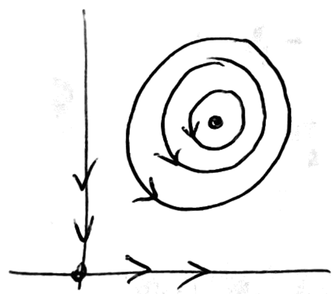
$$A = \begin{pmatrix} 0 & -\frac{c\alpha}{\gamma} \\ \frac{a\gamma}{\alpha} & 0 \end{pmatrix}$$

$$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & -\frac{c\alpha}{\gamma} \\ \frac{a\gamma}{\alpha} & -\lambda \end{pmatrix} = \lambda^2 + \frac{c\cancel{\alpha}\alpha\cancel{\gamma}}{\cancel{\alpha}\cancel{\gamma}}$$

$$= \lambda^2 + c\alpha = 0 \qquad \lambda = \pm\sqrt{c\alpha}\, i$$

purely imaginary

Solutions are circles



Nonlinear systems with periodic solutions and limiting cycles

1

$\underline{x}'(t) = \underline{F}(\underline{x}(t))$ non-lin. system

A periodic solution is $\underline{x}(t)$ s.t. $\underline{x}(t+T) = \underline{x}(t)$

---

e.g. a linear system with $Tr(A) = 0$ $\det(A) > 0$ has center : orbits around it are periodic

- Also in example of Lotka-Volterra closed curves around equilibrium point (period depends on curve)

- Non linear pendulum

Periodic solutions can appear as "limiting cycles"

$$\begin{cases} \dfrac{dx}{dt} = x + y - x(x^2+y^2) \\ \dfrac{dy}{dt} = -x + y - y(x^2+y^2) \end{cases}$$

Equilibria (critical points):

$$\begin{cases} x+y-x(x^2+y^2) = 0 \\ -x+y-y(x^2+y^2) = 0 \end{cases}$$

$$x+y = x(x^2+y^2)$$
$$\Downarrow$$
$$-x+y-\frac{y}{x}(x+y) = 0$$
$$-x^2 + \cancel{xy} - \cancel{xy} - y^2 = 0$$
$$x^2+y^2 = 0 \Rightarrow$$
$$x = 0 \;\&\; y = 0$$

So the only equilibrium point is $(x,y) = (0,0)$

② linearization at $(0,0)$ $\quad \begin{cases} u = x \\ v = y \end{cases}$

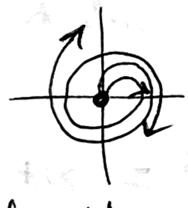
$$\begin{cases} \dfrac{du}{dt} = u + v \\[2ex] \dfrac{dv}{dt} = -u + v \end{cases} \qquad \frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix}$$

with $A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ $\qquad \lambda = 1 \pm i$

$$\det\begin{pmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{pmatrix} = (1-\lambda)^2 + 1 = \lambda^2 + 1 - 2\lambda + 1 = \lambda^2 - 2\lambda + 2$$

$$\lambda = +1 \pm \sqrt{1-2} = 1 \pm i$$

So spiral point : solutions spiral outward :



All there is? No

even if no other equilibria

behavior of solutions <u>change</u> when farther from $(0,0)$

To see better rewrite in polar coordinates

$$r^2 = x^2 + y^2 \qquad x = r\cos\theta \quad y = r\sin\theta$$

$$\Downarrow$$

$$r\frac{dr}{dt} = \underbrace{x\frac{dx}{dt} + y\frac{dy}{dt}}$$

from equation $x(x+y) - x^2(x^2+y^2)$
$+ y(-x+y) - y^2(x^2+y^2)$

$$= (x^2+y^2) - (x^2+y^2)^2$$

So equation

$$r\frac{dr}{dt} = r^2(1-r^2)$$

critical points

$r=0$, $\underline{\underline{r=1}}$ $\uparrow$

behavior of r completely decoupled from behavior of θ-variable

behavior of θ: from equations also have

$$y\frac{dx}{dt} - x\frac{dy}{dt} = y(\cancel{x}+y) - \cancel{xy(x^2+y^2)}$$
$$-x(-x+\cancel{y}) + \cancel{xy(x^2+y^2)} = x^2+y^2 = r^2$$

Note that if

$x = r\cos\theta \qquad y = r\sin\theta$

$$\frac{dx}{dt} = \frac{dr}{dt}\cos\theta - r\sin\theta\frac{d\theta}{dt} \qquad \frac{dy}{dt} = \frac{dr}{dt}\sin\theta + r\cos\theta\frac{d\theta}{dt}$$

$$y\frac{dx}{dt} - x\frac{dy}{dt} = \cancel{r\frac{dr}{dt}\cos\theta\sin\theta} - r^2\sin^2\theta\frac{d\theta}{dt}$$
$$\cancel{-r\frac{dr}{dt}\sin\theta\cos\theta} - r^2\cos^2\theta\frac{d\theta}{dt}$$
$$= -r^2\frac{d\theta}{dt}$$

$$\Rightarrow \quad -r^2\frac{d\theta}{dt} = r^2 \qquad \text{i.e. } \frac{d\theta}{dt} = 1$$

constant angular velocity

So at $r=1$ periodic solution

$\theta = t_0 - t$ going around unit circle

④ While other solutions

$$\frac{dr}{dt} = r(1-r^2) \qquad \frac{dr}{r(1-r^2)} = dt$$

$$\frac{dr}{dt} = r(1-r^2)$$

$$\frac{dr}{r(1-r^2)} = dt \qquad\qquad \frac{1}{r(1-r^2)} = \frac{1}{r} + \frac{r}{1-r^2}$$

$$\int \left(\frac{1}{r} + \frac{r}{1-r^2}\right) dr = \int dt$$

$$\log r - \frac{1}{2}\log|1-r^2| = t + t_0$$

$$\log\left(r|1-r^2|^{-1/2}\right) = t + t_0$$

$$r^2(1-r^2)^{-1} = \pm e^{2(t+t_0)}$$

$$\frac{r^2}{1-r^2} = C e^{2t} \qquad r^2 = Ce^{2t} - Ce^{2t} r^2$$

$$r^2(1+Ce^{2t}) = Ce^{2t}$$

if $\rho > 1$
$C_0 > 0$
$r \to 1_+$
as $t \to \infty$

$$r^2 = \frac{Ce^{2t}}{1+Ce^{2t}} = \frac{1}{C^{-1}e^{-2t}+1}$$

$$r = \frac{1}{\sqrt{1+C_0 e^{-2t}}} \qquad C_0 = C^{-1}$$

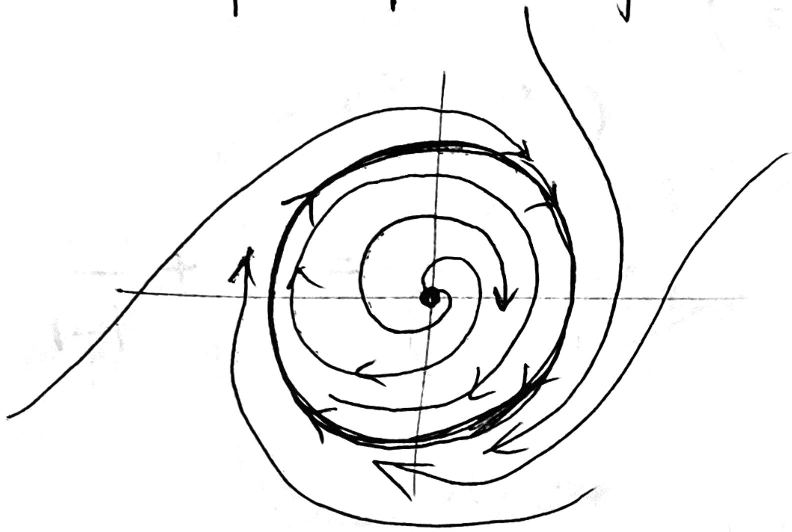
C depends on initial $r(0) = \rho$ by

$$\frac{\rho^2}{1-\rho^2} = C$$

$$C_0 = \left(\frac{1}{\rho^2} - 1\right)$$

if $\rho < 1$
$C_0 < 0$
$r \to 1_-$ as $t \to \infty$

(6) So final picture for solutions



In general cannot solve the equations explicitly : so how to know if there are limiting cycles ?

System
$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

1 Suppose that F, G have continuous derivatives $\dfrac{\partial F}{\partial x}, \dfrac{\partial F}{\partial y}, \dfrac{\partial G}{\partial x}, \dfrac{\partial G}{\partial y}$ in some region $\mathcal{D}$ of the plane

then if there is a limiting cycle in this region it must enclose a critical point (in $\mathcal{D}$ if $\mathcal{D}$ simply conn.) & the critical point cannot be a saddle

---

2 Same hypothesis with $\mathcal{D}$ simply connected if $F_x + G_y$ does not change sign in $\mathcal{D}$ then no closed trajectory in $\mathcal{D}$

---

3 same hypothesis as in 1

& no critical pt in R

$\mathcal{D}_1$ $\mathcal{D}$ $R = \overline{\mathcal{D}_1}$

Then if $\exists$ t_0 s.t. $\begin{cases} x(t) = \phi(t) \\ y(t) = \psi(t) \end{cases}$ solution that stays in R $\forall t \geq t_0$ $\Rightarrow$ either periodic or spirals towards a periodic solution as $t \to \infty$

($\Rightarrow \exists$ periodic solution) [R cannot be simply conn. by 1]

② **Bendixson criterion :**

② $\dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} \neq 0$ anywhere in $\mathcal{D}$

suppose there is a closed trajectory in $\mathcal{D}$

apply Green's theorem (normal / flux form)

$$\oint_C (F\,\underline{e}_1 + G\,\underline{e}_2)\cdot \underline{n}\; ds$$

$\underline{n}$ = normal vector

$$= \boxed{\oint_C (F\,dy - G\,dx) = \iint_\Omega \left(\frac{\partial F}{\partial x} + \frac{\partial G}{\partial y}\right) dx\,dy}$$

$\partial\Omega = C$
$\Omega \subset \mathcal{D}$

$\neq 0$ by hypothesis

but this must be zero because C closed trajectory
i.e. tangent vector to C is the velocity field

if (F, G) = tangent vector then $(F,G)\cdot\underline{n} = 0$

Green's thm :

$$\oint_C (A\,dx + B\,dy) = \iint_\Omega \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx\,dy$$

$\partial\Omega = C$

$\omega = A\,dx + B\,dy \qquad d\omega = -\dfrac{\partial A}{\partial y}\, dx\wedge dy + \dfrac{\partial B}{\partial x}\, dx\wedge dy$

unit normal $\quad$ if (dx, dy) tg. direction along C then outward normal is $90°$ to right
$\Rightarrow (dy, -dx) = \underline{n}\cdot ds$

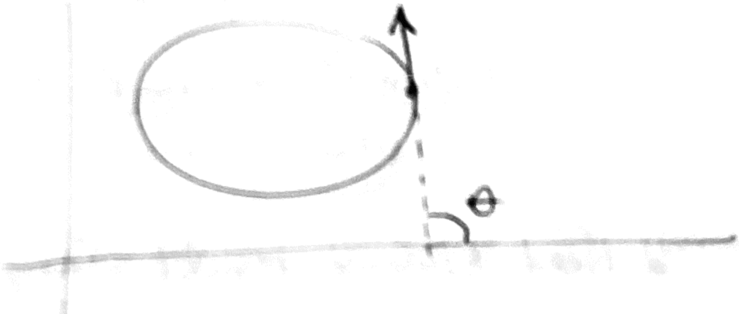
Given vector (F, G) then $(F,G)\cdot\underline{n} = F\,dy - G\,dx$

$ds = \sqrt{dx^2 + dy^2}$ length

① Suppose $\mathcal{D}$ contains Ω : $\partial\Omega = C$
and no critical point in Ω

then $F^2 + G^2 \neq 0$ everywhere in Ω

Let θ = angle between tangent vector to C and x-axis



$$\oint_C d\theta = 2\pi$$

$$\tan\theta = \frac{y'}{x'} = \frac{G}{F}$$

So get
$$d\theta = \frac{F\,dG - G\,dF}{F^2+G^2}$$
or equivalently
$$\frac{d\theta}{dt} = \frac{F\frac{dG}{dt} - G\frac{dF}{dt}}{F^2+G^2}$$

then
$$\oint_C d\theta = \oint_C \frac{F\,dG - G\,dF}{F^2+G^2} = \iint_\Omega \left(\frac{\partial}{\partial F}\left(\frac{F}{F^2+G^2}\right) + \frac{\partial}{\partial G}\left(\frac{G}{F^2+G^2}\right)\right) dF\,dG$$

this would say that the 1-form $d\theta$ is exact in $\mathcal{D}$

but know it is not because $\oint d\theta \neq 0$
(would be for an exact form)

So there must be some point in Ω where $F^2+G^2=0$

④

③ A trajectory that stays within ~~R~~ R $\forall t \geq t_0$ is either

(i) approaching a critical pt in R as $t \to \infty$

(ii) approaching a closed orbit in R

(iii) is a closed orbit (or a critical pt) in R

~~(iv) approaching~~

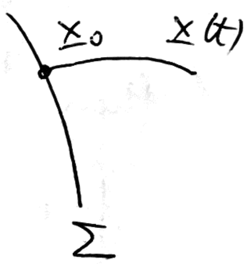
if no critical pt. Then only (ii) (iii) options

Poincaré – Bendixson theorem

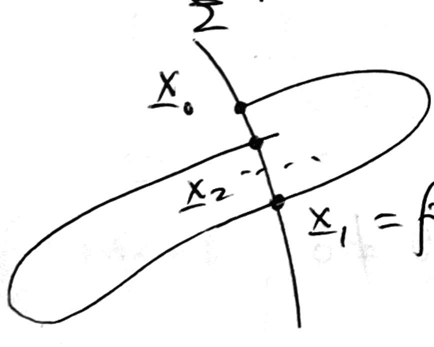
• Need topology result: any simple closed curve (Jordan curve) in the plane separates out two components

Jordan curve theorem:

Start with $\underline{x}_0$ in R and take an arc Σ that is transversal to the flow $\underline{x}(t)$ (solution w/ $\underline{x}(t_0) = \underline{x}_0$)



x_n = points of subsequent intersection of flow w/ Σ



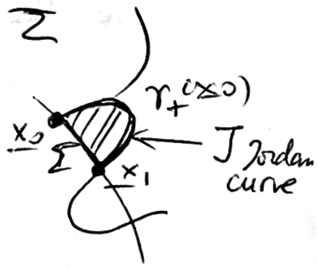
$\underline{x}_1$ = first return to Σ etc.

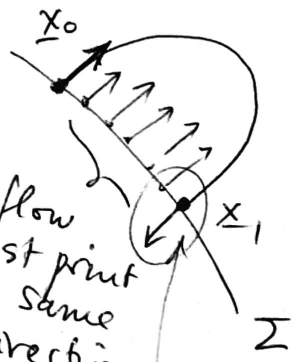
if $x_1 = x_0$ is periodic orbit (done)

$\{x_n\}$ (this set may be empty: don't know yet it returns to Σ

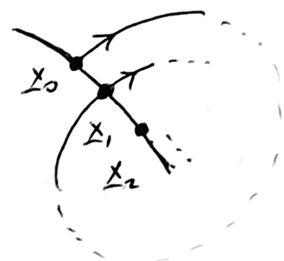
(i) first thing to prove: x_n's must follow monotonically same order on Σ

or





So the first return must be in same direction:



flow must point in same direction otherwise Σ somewhere not transversal

contradicts this by continuity

So sequence $\underline{x}_n$ monotonically along segment Σ (bounded

if infinite $\Rightarrow$ has limit point $\underline{x}_n \to \underline{y} \in \Sigma$

(if finite $\Rightarrow$ periodic solution)

$\underline{x}_n = \gamma(\underline{x}_{n-1})$

first return map

if $\underline{y}$ not a fixed pt of the flow

Apply first return map to $\underline{y}$

$$\gamma(\underline{y}) = \lim_n \gamma(\underline{x}_n) = \lim_n \underline{x}_{n+1} = \underline{y}$$

So $\underline{y}$ returns to itself

$\Rightarrow$ is on a periodic orbit

$\exists$ periodic orbit

open neighborhood of $\underline{x}_0$

$L : U(\underline{x}_0) \longrightarrow \mathbb{R}$ $\qquad \underline{x}_0 = (x_0, y_0)$ critical pt.

Lyapunov function $\qquad$ continuous $\qquad U(\underline{x}) > 0$ for $\underline{x} \neq \underline{x}_0$

and $L(\underline{x}(t_0)) \geq L(\underline{x}(t_1))$

if $t_0 < t_1$ $\qquad$ if $\underline{x}(t_i) \in U(\underline{x}_0) \setminus \{\underline{x}_0\}$

for any $\underline{x}(t)$ solution

strict Lyapunov function if is strict $>$ inequality

(Note: if strict Lyapunov function then $U(\underline{x}_0) \setminus \{\underline{x}_0\}$ contains no periodic orbits)

Sublevel sets of L

$$S_\delta = \{ \underline{x} \in U(\underline{x}_0) \mid L(\underline{x}) \leq \delta \}$$

S_δ^0 = connected component of S_δ that contains $\underline{x}_0$

- if $S_\delta^0 \subset U(\underline{x}_0)$ is closed (no boundary in common w/ $\partial U(\underline{x}_0)$) then invariant under forward time evolution (positively invariant)

Pf: suppose $\underline{x}(t)$ leaves S_δ^0 at some time t_0

$\underline{y} = \underline{x}(t_0)$ $\qquad$ because S_δ closed $\Rightarrow \underline{y} \in S_\delta^0$ (all previous pts in orbit are in S_δ $\Rightarrow$ limit also)

② but $\underline{x}(t)$ leaves S_δ at t_0 i.e. $\exists$ ball

$B_r(\underline{y}) \subseteq U(\underline{x}_0)$ s.t.

$\underline{x}(t_0+\varepsilon) \in B_r(\underline{y}) \setminus S^o_\delta$ for small $\varepsilon > 0$

$\Rightarrow$ outside of sublevel set S^o_δ so $L(\underline{x}(t_0+\varepsilon)) > \delta = L(\underline{y})$

but this contradicts the Lyapunov property of L

(should have $L(\underline{x}(t_0+\varepsilon)) \leq L(\underline{x}(t_0))$)

- $\forall \varepsilon > 0 \quad \exists \delta > 0$ s.t.

$$\underbrace{S^o_\delta \subseteq B_\varepsilon(\underline{x}_0)}_{} \quad \text{and} \quad B_\delta(\underline{x}_0) \subseteq S^o_\varepsilon$$

Pf: (1) if is false then $\forall n \in \mathbb{N}$

$\exists\ \underline{x}_n \in S^o_{1/n}$ with $|\underline{x}_n - \underline{x}_0| \geq \varepsilon$

but $S^o_{1/n}$ is connected there are in fact $\underline{y}_n$ with $\underbrace{|\underline{y}_n - \underline{x}_0| = \varepsilon}_{\text{on sphere of radius } \varepsilon \text{ around } \underline{x}_0}$ in $S^o_{1/n}$

$\exists$ convergent subsequence

$\underline{y}_{n_k} \longrightarrow \underline{y}$

L continuous $\Rightarrow L(\underline{y}_{n_k}) \longrightarrow L(\underline{y})$

but $\underline{y}_{n_k} \in S^o_{1/n}$ so $L(\underline{y}_{n_k}) \longrightarrow 0$ (it is $\in S^o_{1/n_k}$ i.e. $\leq \frac{1}{n_k}$)

$\Rightarrow L(\underline{y}) = 0$ so $\underline{y} = \underline{x}_0$ contradicts $|\underline{y}_{n_k} - \underline{x}_0| = \varepsilon$ (stay away from $\underline{x}_0$)

(2) if second property $B_\delta(\underline{x}_0) \subseteq S^0_\varepsilon$ is false
then $\exists\ \underline{x}_n$ with

$$|\underline{x}_n - \underline{x}_0| \le \frac{1}{n} \quad \text{but with } L(\underline{x}_n) \geq \varepsilon$$

but then L continuous so $\underline{x}_n \to \underline{x}_0$

$\Rightarrow L(\underline{x}_n) \to L(\underline{x}_0) = 0$ so cannot be $L(\underline{x}_n) \geq \varepsilon$

Thus given any neighborhood $V(\underline{x}_0)$
can find a δ s.t. $S^0_\delta \subseteq V(\underline{x}_0)$
is positively invariant

$\Rightarrow$ the critical pt $\boxed{\underline{x}_0 \text{ is stable}}$

i.e. once trajectories enter a $V(\underline{x}_0)$ they stay inside it

<u>Conclusion</u>: $\underline{x}_0$ critical pt.
if $\exists$ Lyapunov function L on some $U(\underline{x}_0)$
$\Rightarrow \underline{x}_0$ stable

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the trajectories in $U(\underline{x}_0)$ could still be
closed or approaching closed trajectories
rather than going to $\underline{x}_0$

- Suppose $\underline{x}_0$ crit. pt. & L Lyapunov function on $U(\underline{x}_0)$

if L not <u>constant</u> on any orbit entirely contained in $U(\underline{x}_0) \setminus \{\underline{x}_0\}$
then $\underline{x}_0$ <u>asymptotically stable</u>
every orbit entering $U(\underline{x}_0)$ converges to $\underline{x}_0$

(4) • this is the case when L = strict Lyapunov function

<u>Note</u>: if Lyapunov function is a differentiable function then $L(\underline{x}(t_0)) \geq L(\underline{x}(t_1))$ when $t_0 < t_1$ holds iff

$$\frac{d}{dt} L(\underline{x}(t)) \leq 0$$

$$\parallel$$

$$(\nabla L)(\underline{x}(t)) \cdot \underset{\underset{\frac{d\underline{x}}{dt}}{\parallel}}{(F, G)}$$

where $\frac{d\underline{x}}{dt} = \begin{pmatrix} F(\underline{x}) \\ G(\underline{x}) \end{pmatrix}$ equation

So the Lyapunov condition becomes

$$(\nabla L) \cdot (F, G) \leq 0$$

Lie derivative of L along vector field (F, G).

<u>Example</u>:

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}$$

$$L(x,y) = x^2 + y^2$$

$$\nabla L = (2x, 2y) \qquad \nabla L \cdot (y, -x) = 2xy - 2yx = 0$$

L is a Lyapunov function (not strict)

in fact it is constant along trajectories

$\Rightarrow$ $(0,0)$ stable but not asympt. stable $L(x,y) = \delta$ are closed traj.

- $L(x,y) = x^2+y^2$ Still a Lyapunov function for modified example

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\eta y - x \end{cases} \quad \text{with } \eta \geq 0$$

$\nabla L = (2x, 2y)$ $\qquad$ $\nabla L \cdot (F,G) = \cancel{2xy} - \cancel{2yx} - 2\eta y^2$

in this case $(0,0)$ asympt. stable $\qquad = -2\eta y^2 \leq 0$

Liapunov stability

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y) \end{cases}$$

suppose equilibrium pt at $(0,0)$

i.e. $F(0,0) = G(0,0) = 0$

Suppose there is some domain $D \ni (0,0)$
and a function (potential energy) $V(x,y)$ on D

say pos. def. on D if $V(x,y) > 0$ on D
pos. semi-def if $V(x,y) \geq 0$; neg. similarly

define $$\dot{V}(x,y) = \frac{\partial V}{\partial x}(x,y)\, F(x,y) + \frac{\partial V}{\partial y}(x,y)\, G(x,y)$$

then $$\frac{dV(x(t), y(t))}{dt} = V_x(x(t), y(t)) \frac{dx(t)}{dt} + V_y(x(t), y(t)) \frac{dy(t)}{dt}$$

$$= V_x F + V_y G = \dot{V}(x(t), y(t))$$

how energy changes along trajectories of the system

Thm: If $\exists V$ continuous w/ continuous V_x, V_y which is pos. def ($V(0,0)=0$ & $V>0$ $(x,y)\neq(0,0)$) on $D \ni (0,0)$ and s.t. $\dot{V}$ neg def. $\dot{V}<0$ $(x,y)\neq(0,0)$, $\dot{V}(0,0)=0$ on $D \ni (0,0)$

$\Rightarrow$ $(0,0)$ is an asymptotically stable critical point

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Thm: if $\exists V$ contin. w. V_x, V_y contin.

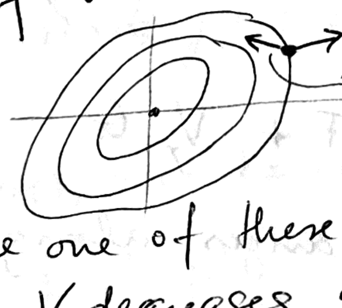
$V(0,0)=0$ and (1) $\forall D \ni (0,0)$ $\exists (x,y)\in D$ $V(x,y)>0$

(2) and $\exists \tilde{D} \ni (0,0)$ $\dot{V}(x,y)>0$ $\forall (x,y)\in\tilde{D}$ $(x,y)\neq(0,0)$ $\rightarrow$ the trajectory starting at this (x,y) will move away from origin

$\Rightarrow$ $(0,0)$ unstable

Note: do not need for this to know solutions (it helps checking if info from linearization ok with NL system)
but don't know in general how to construct such $\underline{\underline{V}}$

Idea: (1) level sets of V curves around origin



tg. to solution curve points always inward for each level set

trajectories inside one of these curves remain inside because V decreases along trajectories

solutions keep crossing level sets inward until hit origin as $t\to\infty$

van der Pol equation

$$u'' - \mu(1-u^2)u' + u = 0 \qquad \mu \geq 0$$

(triode oscillator)

$\mu = 0$ linear $u'' + u = 0$ solutions $\sin t$ & $\cos t$

write as system

$$\begin{cases} x' = y \\ y' = -x + \mu(1-x^2)y \end{cases}$$

only critical point at $(0,0)$

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$ linearization at $(0,0)$

eigenvalues $\dfrac{\mu \pm \sqrt{\mu^2 - 4}}{2}$

unstable ~~spiral point~~:

$1 = \det(A) > 0 \quad tr(A) = \mu > 0$

$\det(A) - \frac{tr(A)^2}{4}$

spiral pt. $0 < \mu < 2$

node $\mu \geq 2$

Periodic solutions

How can say if there are closed cycles?

when cannot solve explicitly equations

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + \mu(1-x^2)y \end{cases}$$

written as a system

$u'' - \mu(1-u^2)u' + u = 0$

van der Pol equation for triode oscillations

$\mu > 0$

Critical pts : $y=0$ & $x=0$ $\quad (0,0)$

Linearization :
$$\begin{cases} \dfrac{dx}{dt} = -y \\ \dfrac{dy}{dt} = -x + \mu y \end{cases}$$

$$A = \begin{pmatrix} 0 & 1 \\ -1 & \mu \end{pmatrix} \qquad \lambda = \frac{\mu \pm \sqrt{\mu^2 - 4}}{2}$$

$$\det \begin{pmatrix} -\lambda & 1 \\ -1 & \mu-\lambda \end{pmatrix} = -\lambda(\mu-\lambda)+1 = \lambda^2 - \lambda\mu + 1 = 0$$

- When $0 < \mu < 2$ unstable spiral
- when $\mu \geq 2$ unstable node

Periodic solutions : $\dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} = \mu(1-x^2)$

$\Rightarrow$ no closed trajectories in $|x|<1$ where $\dfrac{\partial F}{\partial x} + \dfrac{\partial G}{\partial y} > 0$

What else ?

try again polar coordinates

$$x\frac{dx}{dt} + y\frac{dy}{dt} = r\frac{dr}{dt}$$

(2) $\quad x\dfrac{dx}{dt} = xy$

$$y\frac{dy}{dt} = -xy + \mu(1-x^2)y^2$$

$$r\frac{dr}{dt} = x\frac{dx}{dt} + y\frac{dy}{dt} = \mu(1-x^2)y^2 = \mu(1-r^2\cos^2\theta)\,r^2\sin^2\theta$$

$$\Rightarrow \quad \frac{dr}{dt} = \mu(1-r^2\cos^2\theta)\,r\sin^2\theta$$

no longer decoupled from θ

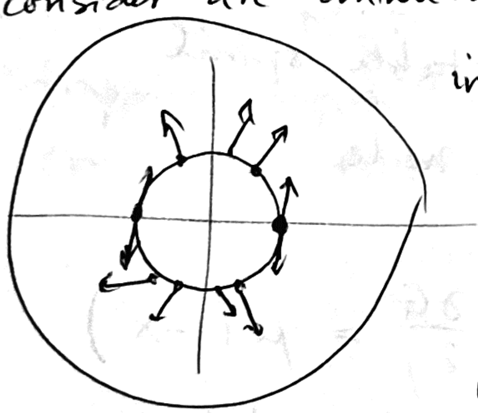
__Note__: when r small dominant term in

is linear term in r

$\sim \mu r \sin^2\theta \qquad$ so $\dfrac{dr}{dt} > 0$

except where $\sin\theta = 0$
(x-axis) where $\dfrac{dr}{dt} = 0$

So consider an annular region R



inner radius r_1 is __small__

then trajectories starting at pts. of inner circle enter region R
(except at x-axis ~~only~~ where tangent to R)

Suppose r_2 outer radius is large

then dominant term becomes

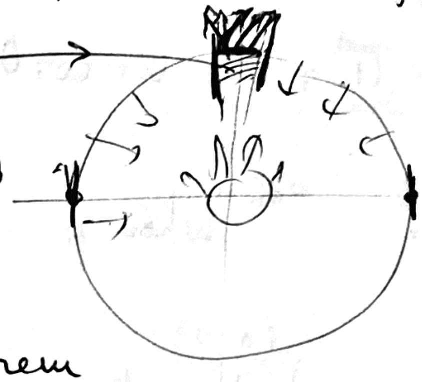
$$-\mu r^3\cos^2\theta\sin^2\theta$$

$1 - r_2^2\cos^2\theta > 0$

and $\dfrac{dr}{dt} < 0 \quad$ (except where $\sin\theta = 0$ or when it is $\dfrac{dr}{dt} = 0$) and very near y axis

Thus, no matter how large r_2 radius of R
there are trajectories in R that leave R

region
where
$1 - r_2^2\cos^2\theta > 0$



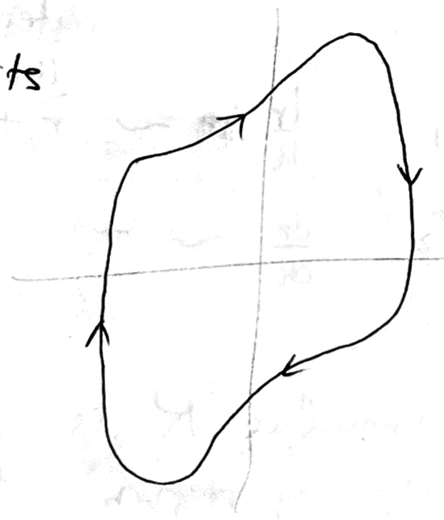
So cannot use this annuli
for Poincaré-Bendixson theorem

(would have to know that there are solutions that stay inside region)

but in fact a unique limiting cycle exists

looks
something
like this;

(more or less
squashed
depending on μ)



$$\frac{dx}{dt} = F(x,y)$$

$$\frac{dy}{dt} = G(x,y)$$

F, G continuous & differentiable w/ contin. first derivatives

then if there is a closed trajectory it must encircle a critical point

(so if no critical pts in a region no closed trajectories there)

Also if simply connected domain and $F_x + G_y$ does not change sign

No closed trajectories in domain

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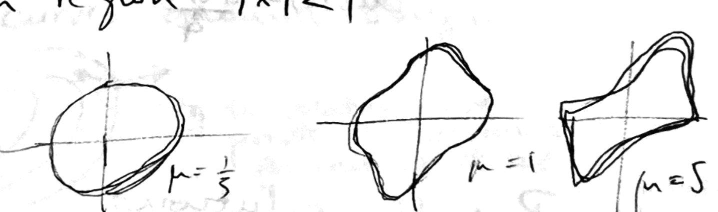
in example

These say: if closed traj : around origin

$$F_x + G_y = \mu(1 - x^2)$$

so no closed trajectories in region $|x| < 1$

More information?



In this case van der Pol equation does have one closed trajectory

Stability / instability of critical points

- Look at linearization
- But when linearization has a center as crit. pt. this sensitive to small perturbations and nonlinear terms can change to spiral

Example: damped pendulum

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -\omega^2 \sin x - \gamma y \end{cases}$$

Equilibrium positions

$$\begin{cases} y = 0 \text{ and} \\ \sin x = 0 \end{cases}$$

$x = \pm n\pi \quad n \in \mathbb{N}$

linearization near $(2n\pi, 0)$ $\quad n \in \mathbb{Z}$

$$\begin{cases} u = x - 2n\pi \\ v = y \end{cases}$$

$$\sin(x) = \sin(2n\pi + u)$$
$$= \cancel{\sin(2n\pi)\cos u} + \overset{1}{\cos(2n\pi)} \sin(u)$$

$$\begin{cases} \dfrac{du}{dt} = v \\ \dfrac{dv}{dt} = -\omega^2 u - \gamma v \end{cases}$$

linear term of $\sin(u)$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$Tr(A) = -\gamma$
$\det(A) = \omega^2$

<u>Note</u>: if $\gamma = 0$ no damping effect

have a center, but for even very small γ get stable spiralling pt.

at $((2n+1)\pi, 0)$ points linearization is

$$\frac{du}{dt} = v$$

$$\frac{dv}{dt} = +\omega^2 u - \gamma v$$

$$\frac{d}{dt}\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \omega^2 & -\gamma \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix}$$

$$\det(A) < 0$$

because $\sin(x) = \sin((2n+1)\pi + u)$

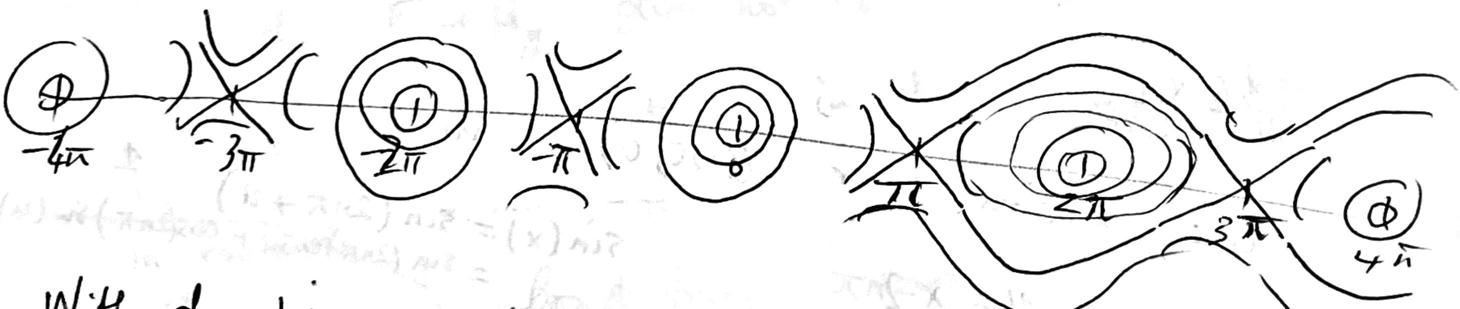
$$= \cancel{\sin((2n+1)\pi)\cos u} + \cos((2n+1)\pi)\sin u$$

$$= -\sin(u)$$

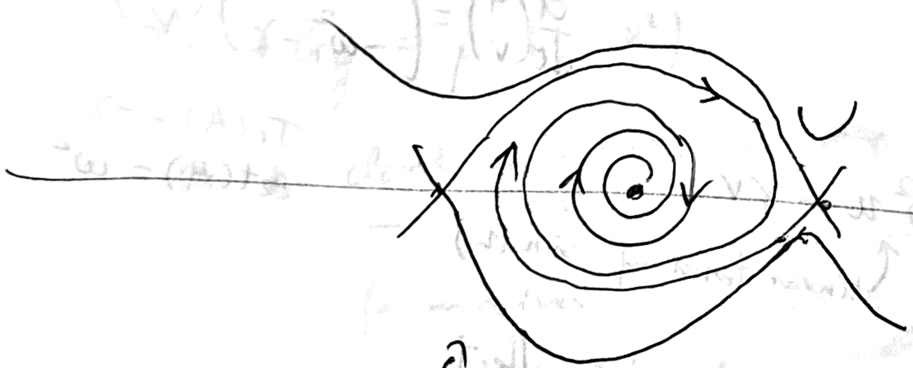
$$-\sin(u) \sim -u$$

So these are all saddles

No damping $\gamma = 0$:



With damping $\gamma \neq 0$



destroys the orbits between sad pts.