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Existence and uniqueness hypotheses:

$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ where $f(t,y)$ is continuous with $\frac{\partial f}{\partial y}$ continuous on a rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$ containing (t_0, y_0) $= \{\alpha < t < \beta, \gamma < y < \delta\}$

First observation: two aspects $\begin{cases} \text{existence (1)} \\ \text{uniqueness (2)} \end{cases}$

depend on two different types of mathematical results

(1) Fixed point theorem

(2) Implicit function theorem (~~related~~ similar to)

This means: to prove existence (when cannot explicitly write solution)

show that (the) a solution $\phi(t)$ with

$\begin{cases} \phi'(t) = f(t, \phi(t)) \\ \phi(t_0) = y_0 \end{cases}$ is (the) a fixed point

of a transformation acting on functions

$$T(\phi(t)) = \phi(t)$$

then show that the transformation T has the right properties to guarantee the existence of a fixed point

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Note: Knowing $\phi(t)$ is a fixed point of T usually gives also some useful information on how to find the solution: iterates of the transformation T applied to an initial choice ("arbitrary") of function converge to the ~~sol~~ fixed point

Start with a function $\phi(t)$ (not a solution), but an easy function (for example $\psi(t) = y_0$ for all t, a constant function)

and compute $T\psi$, then $T(T\psi)$, then $T^k(\psi)$, ... these are all new functions (not solutions) but $T^k(\psi) \underset{k\to\infty}{\longrightarrow} \phi$ converge to a function ϕ with $T(\varphi) = \phi$ (a fixed point) $\Rightarrow$ solution

* Need to explain what convergence $T^k(\psi) \to \phi$ of <u>functions</u> means

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What is the transformation T such that a solution $\phi(t)$ of $\begin{cases} \phi'(t) = f(t, \phi(t)) \\ \phi(t_0) = y_0 \end{cases}$ is a fixed point $T(\phi) = \phi$?

Key: A differential equation can also be written as an integral equation

$$\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases} \iff y = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds \quad (*)$$

(not a solution: still an equation because y appears also here, an integral equation for $y(t)$)

As the transformation T take that which maps a given function $\psi(t)$ to

$$T(\psi(t)) = y_0 + \int_{t_0}^{t} f(s, \psi(s))\, ds$$

Now if $\psi(t)$ is a solution of the integral equation $*$ then $\psi(t) = T(\psi(t))$, so it is a fixed point of T and conversely a fixed point of T is a solution of $*$ (and therefore a solution of the diff. equation)

So to know if solutions of $\begin{cases} y' = f(t,y) \\ y(t_0) = y_0 \end{cases}$ exist

need to know if fixed points

$T(\psi) = \psi$ exist

When does one know that fixed points exist?

There are different types of "fixed point theorems" available in mathematics

- Brower fixed point theorem (you cannot draw at the game of Hex)

  T a continuous function $T: X \to X$ mapping a compact convex set to itself then there exists at least one fixed point $T(x) = x$ in X.

- Banach fixed point theorem

  $T: X \to X$ continuous function mapping a complete metric space X to itself contracting distances

(a contraction)

$$\text{distance}(T(x), T(y)) \le k \cdot \text{distance}(x,y)$$

for some $0<k<1$

Then there is a unique fixed point $x \in X$.

These require more advanced mathematical tools (compactness, convexity, completeness, metric spaces, ...)

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but one can use them as black boxes

Note: the Banach fixed point theorem above also implies that starting from any $x_0 \in X$ and iterating T: $x_n = T(x_{n-1})$ ($x_n = T^n(x_0)$) gives $T^n(x_0) \xrightarrow[n\to\infty]{} x$ converges to the unique fixed point $T(x)=x$ in the sense that $\text{distance}(T^n(x_0), x) \xrightarrow[n\to\infty]{} 0$

Will use this fixed point theorem because it gives both existence and uniqueness at the same time.

Then the key property is the "distance contracting property" of the transformation T

(also the completeness of the metric space but that requires more advanced methods from "functional analysis")

One observation about the hypothesis that

$\frac{\partial f}{\partial y}(t,y)$ is continuous on $R=(\alpha,\beta)\times(\gamma,\delta)$
(and not just $f(t,y)$ continuous)

this property implies that

$$|f(t,y_1)-f(t,y_2)| \leq K_\varepsilon |y_1-y_2|$$

for all $t \in [\alpha+\varepsilon, \beta-\varepsilon]$ with some $K_\varepsilon > 0$ independent of t.

the reason why this is true: use the mean value theorem in the variable y (for a fixed t)

$\exists\ y_t \quad y_1 \leq y_t \leq y_2$ such that

$$\frac{\partial f(t,y_t)}{\partial y} = \frac{f(t,y_2)-f(t,y_1)}{y_2-y_1}$$

so $$|f(t,y_1)-f(t,y_2)| \leq \left|\frac{\partial f(t,y_t)}{\partial y}\right| |y_1-y_2| \qquad (*)$$

Since $\frac{\partial f}{\partial y}$ is a continuous function, it has a maximum on every compact set $[\alpha+\varepsilon, \beta-\varepsilon]\times[\gamma+\varepsilon, \delta-\varepsilon] = \bar{R}_\varepsilon$

let $$K_\varepsilon = \max_{(t,y)\in \bar{R}_\varepsilon} \left|\frac{\partial f}{\partial y}(t,y)\right|$$

then the right hand side of $(*)$ is also

$$\leq K_\varepsilon |y_1-y_2|$$

This property (fix one ε so write $K = K_\varepsilon$)

$$|f(t, y_1) - f(t, y_2)| \leq K\,|y_1 - y_2|$$

"Looks" like the Kind of contraction property

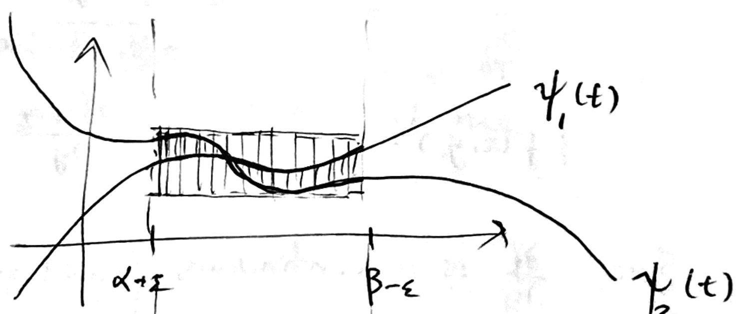
$\text{dist}(T\psi_1, T\psi_2) \leq c\ \text{dist}(\psi_1, \psi_2)$ $(c<1)$ that want to obtain to use the fixed point theorem

How to go from one to the other ?

first answer question: what is the distance between two functions ?

$$\text{distance}(\psi_1, \psi_2) = \sup_{\alpha+\varepsilon \leq t \leq \beta-\varepsilon} |\psi_1(t) - \psi_2(t)|$$

(Suppose ψ_1, ψ_2 functions $\psi(t)$ with $t \in [\alpha+\varepsilon, \beta-\varepsilon]$)



max distance of graphs of these two functions inside the interval $[\alpha+\varepsilon, \beta-\varepsilon]$

<u>Note</u>: one sees already a reason here why one may have to restrict the interval from (α, β) to (t_0-h, t_0+h) smaller

other reasons too ...

Now look at functions $\psi(t)$

continuous on $[\alpha+\varepsilon, \beta-\varepsilon]$ and such that

$\psi(t)$ has values in $[\gamma+\varepsilon, \delta-\varepsilon]$

that's where expect to find a solution

Want to compute then $T(\psi) = y_0 + \int_{t_0}^{t} f(s,\psi(s))\,ds$

(and iterates $T^k(\psi)$)

OK OK

but is $T(\psi)$ still a continuous function on $[\alpha+\varepsilon, \beta-\varepsilon]$

with values in $[\gamma+\varepsilon, \delta-\varepsilon]$

↑ here is where the problem may be

$$\sup_{t_0 \le t \le \beta-\varepsilon} \left| T(\psi)(t) - y_0 \right| = \sup \left| \int_{t_0}^{t} f(s,\psi(s))\,ds \right|$$

$$\le \sup_{t\in[t_0,\beta-\varepsilon]} \int_{t_0}^{t} \left| f(s,\psi(s)) \right| ds$$

$$\le (t-t_0) \underbrace{\sup_{t\in[t_0,\beta-\varepsilon]} \left| f(t,\psi(t)) \right|}_{C_{t_0,\varepsilon}}$$

$$\le C_{t_0,\varepsilon} \cdot (\beta-\varepsilon-t_0)$$

↑ length of interval

(or $\beta-\alpha-2\varepsilon$ if both sides of t_0)

this condition holds when

$$C_{t_0,\varepsilon} \cdot (\beta-\varepsilon-t_0) \le (\delta-\gamma-2\varepsilon)$$

This condition may not be satisfied but can be satisfied by restricting the interval $(\alpha+\varepsilon, \beta-\varepsilon)$ to a smaller (t_0-h, t_0+h)

for which $C_{t_0,\varepsilon} \cdot 2h \leq (\delta-\gamma-2\varepsilon)$

* this is the main reason why end up with a smaller interval of existence

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Then have $T : \{\text{functions } \psi : (t_0-h, t_0+h) \to (\gamma+\varepsilon, \delta-\varepsilon)\}$

maps this set to itself

and can check if it is a contraction of distances

$$\underbrace{\sup_{t\in[t_0,t_0+\eta]} \left| \int_{t_0}^{t} f(s,\psi_1(s))\,ds - \int_{t_0}^{t} f(s,\psi_2(s))\,ds \right|}_{\text{distance}(T(\psi_1),\, T(\psi_2))}$$

$$\leq \int_{t_0}^{t} \sup_{t\in[t_0,t_0+\eta]} |f(s,\psi_1(s)) - f(s,\psi_2(s))| \, ds$$

$$\leq K_\varepsilon \sup_{t\in[t_0,t_0+\eta]} |\psi_1(t) - \psi_2(t)| \int_{t_0}^{t} ds$$

$$= K_\varepsilon \cdot \eta \; \text{dist}(\psi_1, \psi_2)$$

for small η $\quad K_\varepsilon \cdot \eta < 1$

move in small steps of lenght η from t_0: each step solution on $(t_0, t_0+\eta)$ ... $[t_0+k\eta, t_0+(k+1)\eta]$ until reach t_0+h limit of interval...

Contraction $\Rightarrow \exists!\; T(\psi) = \psi$

Use of fixed pt. theorem method to find solutions to non linear equations ①

(exist & unique ok but practical ?)

$$(*)\ \begin{cases} y' = 2t(1+y) \\ y(0) = 0 \end{cases} \iff y(t) = \int_0^t 2s\,(1+y(s))\,ds$$

diff. eq. integral eq.

Now this is the transf. $\phi(t) \mapsto T(\phi(t)) = \int_0^t 2s(1+\phi(s))\,ds$ for which we seek a fixed pt.

$\phi(t) = T(\phi(t))$ which is a solution of eq. diff. (*)

Know that $T^k(\psi) \xrightarrow[k\to\infty]{} \phi = T(\phi)$ iterates converge to fixed points

So start with any function ψ e.g. $\psi(t) \equiv 0$

and compute $T(\psi) = \int_0^t 2s\,ds = t^2$

then compute $T(T(\psi)) = \int_0^t 2s(1+s^2)\,ds$

$$= t^2 + \frac{t^4}{2}$$

then continue:

$$T(T(T(\psi))) = T^3(\psi) = \int_0^t 2s\left(1+s^2+\frac{s^4}{2}\right)ds$$

$$= t^2 + \frac{t^4}{2} + \frac{t^6}{2\cdot 3}$$

Have a guess ?

hypothesis: $T^k(\psi) = t^2 + \frac{t^4}{2!} + \frac{t^6}{3!} + \cdots + \frac{t^k}{k!}$

check if correct : check if it satisfied at $k+1$

$$T^{k+1}(\psi) = T(T^k(\psi)) = \int_0^t 2s\left(1 + \cdots + \frac{s^k}{k!}\right) ds$$

$$= t^2 + \frac{t^4}{2} + \cdots + \frac{t^{k+1}}{(k+1)!} \qquad \text{ok}$$

Then $T^k(\psi) \xrightarrow[k\to\infty]{} \sum_{r=1}^{\infty} \frac{t^{2r}}{r!} = \exp(t^2) - 1$

check indeed $\phi(t) = \exp(t^2) - 1$ is a fixed pt

$$T(\phi(t)) = \int_0^t 2s\left(1 - \exp(s^2) + 1\right) ds = \int_0^t 2s\, e^{s^2} ds$$

$$= e^{t^2} - 1 = \phi(t)$$

Solution (hence of dif. eq.)

Difference equations

a "discrete" version of differential equations

also in general solved using fixed point arguments

$\to \exists!$ $\quad I(\psi) = \psi$ $\qquad$ until 'reach' to th limit of iterated

A metric on a space

X set (will be some subset of $\mathbb{R}$ or of $\mathbb{R}^N$)

$d : X \times X \longrightarrow \mathbb{R}_+ = (0, \infty)$

$d(x,y) = d(y,x)$ symmetric

$d(x,z) \leq d(x,y) + d(y,z)$ triangle inequality

$d(x,y) = 0$ iff $x = y$

example: $\underline{x} = (x_1, \dots, x_n)$ $\quad y = (y_1, \dots, y_n)$ in $\mathbb{R}^n$

$$d(\underline{x}, \underline{y}) = \left(\sum_i |x_i - y_i|^2\right)^{1/2} = \|x - y\|_2$$

or in $\mathbb{R}$

$|x - y| = d(x,y)$

$f : X \to \cancel{X}$ $\quad$ Lipschitz map if

$d_X \quad d_Y$

$$d_Y(f(x_1), f(x_2)) \leq C \, d_X(x_1, x_2)$$

(if $C < 1$, f is a contraction)

Complete metric space: (X, d)

every $\{x_n\}$ Cauchy sequence is convergent

$\forall \varepsilon > 0 \ \exists N = N(\varepsilon)$ s.t. $\forall m, n > N$ $\quad d(x_n, x_m) < \varepsilon$

e.g. $(\mathbb{R}, |x-y|)$ $\quad (\mathbb{R}^N, \|x-y\|_2)$

The contraction fixed point theorem

(X, d) complete metric space

$f : X \to X$ contraction $\quad (d(f(x), f(y)) \leq C d(x,y))$
with $C<1$, $C>0$

then f has a unique fixed point:

$\exists !\ x \in X$ s.t. $f(x) = x$

Proof: start with any point $x_0 \in X$

define a sequence $\quad x_n = f^n(x_0) = \underbrace{f(f(f \cdots (f(x_0))}_{n\text{-th iteration}}$

so that $\quad x_{n+1} = f(x_n) \quad x_n = f(x_{n-1})$ etc.

then $\quad d(x_{n+1}, x_n) \leq C d(x_n, x_{n-1}) \leq \cdots \leq c^n d(x_1, x_0)$

so have

$$d(x_m, x_n) \leq \sum_{j=n}^{m-1} d(x_{j+1}, x_j) \leq (C^n + C^{n+1} + \cdots + C^{m-1}) d(x_1, x_0)$$

triangle inequality

$$\leq \frac{C^n d(x_1, x_0)}{1-C}$$

then $\{x_n\}$ is Cauchy : $\forall \varepsilon\ \exists N$ s.t. $\frac{C^n}{1-C} < \varepsilon$ for $n > N$

Since (X,d) complete $\Rightarrow \{x_n\}$ convergent

$\exists\ x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n(x_0)$ $\qquad f(x) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x$

so fixed pt.

So there is a fixed pt. $x = f(x)$

it is unique: suppose another one $y \neq x$

$$f(y) = y$$

then $d(f(x), f(y)) \le C\, d(x,y)$ with $C < 1$

but $f(x) = x$ & $f(y) = y$ so get $d(x,y) \le C\, d(x,y)$

which contradicts $C < 1$.

Existence theorem for solutions of differential equations

$$F : I \times X \longrightarrow Y \quad \text{continuous}$$

$$\frac{dx}{dt} = F(t, x) \quad \text{equation} \qquad x(t) = \phi(t)$$

Solution is $\phi : J \to X$, $J \subseteq I$, s.t. $\phi'(t) = F(t, \phi(t))$

$\frac{\partial f}{\partial y}$ continuous

<u>Assume</u> F is Lipschitz in the second variable $x \in X$

$$d_Y(F(t,x_1), F(t,x_2)) = \| F(t,x_1) - F(t,x_2) \| \le C \, \| x_1 - x_2 \| = C\, d_X(x_1,x_2)$$

for some $0 < C < 1$

$\forall x_1, x_2 \in X$

<u>then</u> $\forall (t_0, x_0) \in I \times X$ $\exists$ a neighborhood U of x_0 (ball around x_0 of some radius)

s.t. for any $J \subseteq I$ suff. small

$\exists$ $\phi : J \to X$ with $\phi(t_0) = x_0$ solving the equation

Idea of the proof:

if $x = f(t)$ is a solution of the equation then

$$f(t) - f(t_0) = \int_{t_0}^{t} F(s, f(s))\,ds \qquad \text{given } f(t_0) = x_0 \text{ initial condition}$$

$$\phi(t) - \phi(t_0) \quad f(s, \phi(s)), \qquad \phi(t_0) = x_0$$

this is
$$f(t) = x_0 + \int_{t_0}^{t} F(s, f(s))\,ds$$

$$\phi(t) \quad f(s, \phi(s))$$

(integral equation)

fix small interval J around t_0 in I

denote by $\mathbb{B}(J) = \{ h : J \to X \text{ bounded continuous functions} \}$ (ψ)

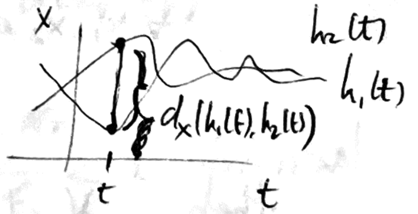
$$K : \mathbb{B}(J) \longrightarrow \mathbb{B}(J)$$

$$K(h)(t) = x_0 + \int_{t_0}^{t} F(s, h(s))\,ds$$

$$\psi \quad f(s, \psi(s))$$

Want to show that the map K is a contraction

for this want a metric on $\mathbb{B}(J)$



$$d_{\mathbb{B}(J)}(h_1, h_2) = \sup_{t \in J} \| h_1(t) - h_2(t) \| \qquad (\psi_1, \psi_2)$$

$$= \sup_{t \in J} d_X(h_1(t), h_2(t)) \qquad (\psi_1, \psi_2)$$

then let δ = length of J, have

$T(\psi_1)$ $T(\psi_2)$ $f(s,\psi_1(s))$ $f(s,\psi_2(s))$

$$\| K(h_1)(t) - K(h_2)(t) \| = \left\| \int_{t_0}^{t} (F(s,h_1(s)) - F(s,h_2(s)))\,ds \right\|$$

distance in X ~~for~~ a fixed $t \in J$

$f(s,\psi_1(s))$ $f(s,\psi_2(s))$

$$\leq \int_{t_0}^{t} \underbrace{\| F(s,h_1(s)) - F(s,h_2(s)) \|}_{\text{assuming that } F \text{ Lipschitz in second variable}}\,ds$$

f $\frac{\partial f}{\partial y}$ cont.

$$\leq C \int_{t_0}^{t} \| h_1(s) - h_2(s) \|\,ds$$

ψ_1 ψ_2

$$\leq C \delta\, d_{B(J)}(h_1, h_2)$$

ψ_1 ψ_2

Since this inequality true for all t in l.h.s. get by passing to $\sup_t$ in l.h.s.

$T(\psi_1)$ $T(\psi_2)$ ψ_1 ψ_2

$$d_{B(J)}(K(h_1), K(h_2)) \leq C \cdot \delta\, d_{B(J)}(h_1, h_2)$$

If J is sufficiently small that $\delta \cdot C < 1$ (δ suff. small)

then K (T) is a contraction

Also need to know that the metric space $(B(J), d_{B(J)})$ is complete (which we don't prove here)

$T(\phi) = \phi$

$\Rightarrow \exists!$ fixed point $h \in B(J)$ (ϕ) s.t. $K(h) = h$

$\phi(t)$ $T(\phi)(t)$ $f(s,\phi(s))$

which means $h(t) = K(h)(t) = x_0 + \int_{t_0}^{t} F(s, h(s))\,ds$

i.e. h solution of equation

<u>Example</u> :

$$\begin{cases} y' = y^{1/3} \\ y(0) = 0 \end{cases}$$

$$y^{-1/3}\,dy = dt \qquad \frac{3}{2} y^{2/3} = t + c \qquad y = \left(\frac{2}{3}(t+c)\right)^{3/2}$$

initial condition gives $c = 0$

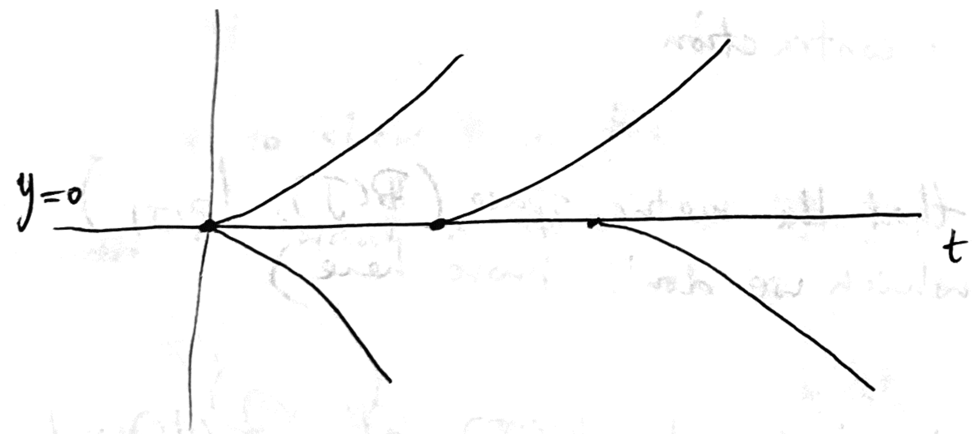
$\Rightarrow \; y(t) = \left(\frac{2}{3} t\right)^{3/2}$ is a solution

<u>but</u> $\quad y(t) = -\left(\frac{2}{3} t\right)^{3/2}$ is also a solution with the same initial value

and <u>also</u>

$$y(t) = \begin{cases} 0 & 0 \le t \le t_0 \\ \pm\left(\frac{2}{3}(t - t_0)\right)^{3/2} & t \ge t_0 \end{cases}$$

is also a solution with same initial value



and $y(t) \equiv 0$ for all t also a solution

<u>WHY</u> not uniqueness? the function $F(y) = y^{1/3}$ is <u>not</u> lipschitz (when near $y=0$ derivative blows up)

(A Lipschitz funct. $f: \mathbb{R} \to \mathbb{R}$ is abs. cont. hence a.e. differentiable and $|f'|$ bounded by K Lipschitz constant)