1

Riemann-Hilbert problem

many different RH problems $\leftrightarrow$ all related

- Hilbert 21st problem:

  existence of linear differential equations with prescribed monodromy?

---

different questions:

1) a Fuchsian nth order linear ODE
2) an n-dim system first order w/ regular singularities
3) a Fuchsian 1st order system on $\mathbb{P}^1(\mathbb{C})$

(1) has negative answer [Poincaré]: number of free parameters in nth order linear Fuchsian eq smaller than dim of monodromy rep 1884 settled $\leftarrow$

(2) positive solution Plemelj 1908
(believed to also solve (3) until 1989

(3) negative solution Bolibrukh 1989

---

important aspect of the problem:
relation to Riemann-Hilbert boundary value problem

(also known as RH problem)

2

n Dirac system of linear ODEs

$$\frac{d\psi}{dz} = A(z)\,\psi \qquad \psi(z) \text{ matrix of solutions (fund matrix)}$$

$n\times n$ matrix

analytic in a disk $D_r(z_0) \subset \mathbb{C}$

- suppose $A(z)$ is analytic on $\mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_k\} = S$

(so can analytically continue local solutions along paths γ in)

Monodromy: pick a basept $z_0 \in S = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_k\}$ and consider $\pi_1(S, z_0)$ fundamental group

$\gamma \in \pi_1(S, z_0)$

↪ continuing a solution $\Psi(z)$ along γ gives another $\tilde{\Psi}(z)$ when returning to z_0

related by a linear transformation of the initial cond. data

$$\Psi(z) = \tilde{\Psi}(z)\, G_\gamma \qquad G_\gamma \in GL(n, \mathbb{C}) \text{ change of basis}$$

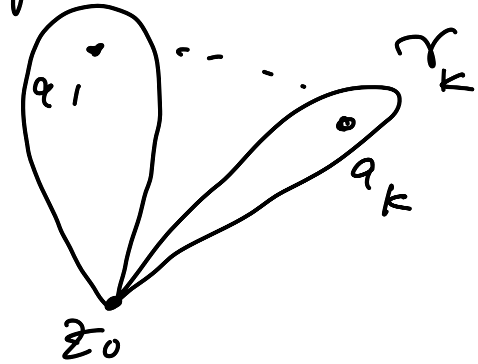
G_γ only depends on homotopy class of γ

$$\mu : \pi_1(S, z_0) \longrightarrow GL(n, \mathbb{C})$$

monodromy representation

[3] Collection of k fundamental loops γ_i

$\gamma_1, \dots, \gamma_k$



$$\underbrace{\gamma_1 \circ \cdots \circ \gamma_n}_{\substack{\text{(contractible} \\ \text{in } S = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_k\})}} = \underset{\substack{\uparrow \\ \text{trivial loop}}}{e} \in \pi_1(S, z_0)$$

$G_i := \mu(\gamma_i)$ monodromy matrices at the singular pts

$$G_1 \cdots G_k = id \in GL(n, \mathbb{C})$$

monodromy group $= \mu(\pi_1(S, z_0)) \subset GL(n, \mathbb{C})$ generated by $G_1, \dots, G_k$

if start from different fundam. solution $\tilde{\Phi}(z)$

then $\underset{\text{new}}{\tilde{G}_i} = H^{-1} G_i H$ for $\tilde{\Phi}(z) = \Phi(z) \cdot H$

for some $H \in GL(n, \mathbb{C})$

$\Rightarrow$ monodromy is element of

$$\mathcal{M} = \text{Hom}(\pi_1(S, z_0), GL(n, \mathbb{C})) \big/ GL(n, \mathbb{C})$$

action by conjugation

4

Fuchsian : system $\frac{d\Psi}{dz} = A(z)\Psi$

is Fuchsian if all sing. pts $a_1, \dots, a_k \in \mathbb{C}$ are poles of $A(z)$ of first order

$\Rightarrow$ can write $A(z) = \sum_{i=1}^{k} \frac{B_i}{z - a_i}$

with $\operatorname{Res}_{z=\infty} A(z) = -\sum_{i=1}^{k} B_i = 0$

Riemann - Hilbert problem (case 3)

$$\mathcal{M}^* = \left\{ (B_1, \dots, B_k) : B_i \in M_{n\times n}(\mathbb{C}),\ \sum_{i=1}^{k} B_i = 0 \right\} \Big/ GL(n, \mathbb{C})$$

data of Fuchsian equation

$$\downarrow \mu$$

$$\mathcal{M} = \operatorname{Hom}(\pi_1(S, z_0), GL(n, \mathbb{C})) \Big/ GL(n, \mathbb{C})$$

monodromy map

(fixed singularities positions $a_1, \dots, a_k$)

is the map μ surjective?

5

Fuchsian versus regular-singular

$\frac{d\Psi}{dz} = A(z)\Psi$ with $A(z)$ analytic in a punctured disk $D_r(a)$

the sing. point a is regular-singular if any solution of the system has at most polynomial growth near z_0

- all singularities of a Fuchsian system are regular singular
- the converse is not true:

$$\frac{d\Psi}{dz} = \left(-\frac{3}{16z^2}\begin{pmatrix}0&1\\1&0\end{pmatrix}+\begin{pmatrix}0&1\\0&0\end{pmatrix}\right)\Psi$$

is not Fuchsian

but fundam. solution

$$\Psi(z) = \frac{1}{4}\begin{pmatrix}4z & 4z\\ 1 & 3\end{pmatrix} z^{-\frac{1}{4}\begin{pmatrix}3&0\\0&1\end{pmatrix}}$$

$z \in \mathbb{C} \setminus (-\infty, 0]$

$\Rightarrow z=0$ is a regular-singular point

---

difference between Fuchsian & regular singular

$\boxed{6}$ Plemelj proved:

any subgroup of $GL(n,\mathbb{C})$ generated by $G_1, \dots, G_k$ matrices satisfying $G_1 \cdots G_k = id$

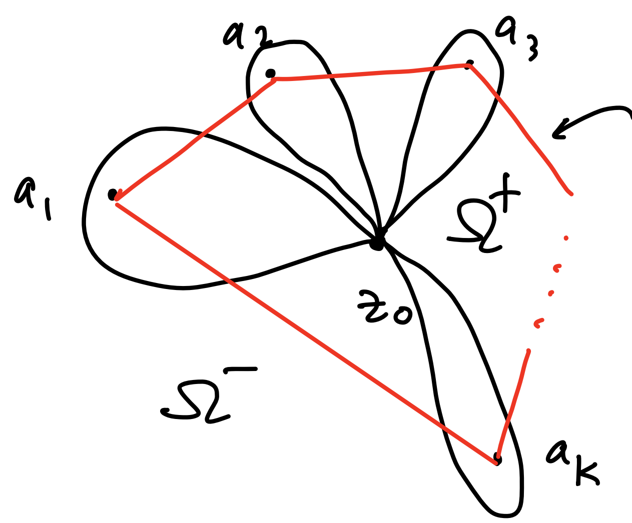
can be realized as monodromy of a system $\frac{d\Psi}{dz} = A(z)\Psi$ on $\mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_k\}$ with a_i regular singularities

$\rightsquigarrow$ can this be improved to realizability by the <u>more restrictive</u> class of Fuchsian systems?

Plemelj's original argument implied yes ... in fact not true!

<u>Main idea of the regular-singular case:</u>

relation to Riemann-Hilbert boundary value problem



Γ closed contour

$\bigcup_i [a_i, a_{i+1})$ union of arcs

$(a_{k+1} := a_1)$

7

piecewise constant matrix-valued function

$$G : \Gamma \longrightarrow GL(n, \mathbb{C})$$

$$G(z) = (G_i G_{i-1} \cdots G_1)^{-1} \qquad z \in [a_i, a_{i+1}) \subset \Gamma$$

since $G_1 \cdots G_k = id$

$$G(z) = id \text{ for } z \in [a_k, a_1)$$

$$\mathbb{P}^1(\mathbb{C}) \setminus \Gamma = \Omega^+ \sqcup \Omega^-$$

RH boundary value problem:

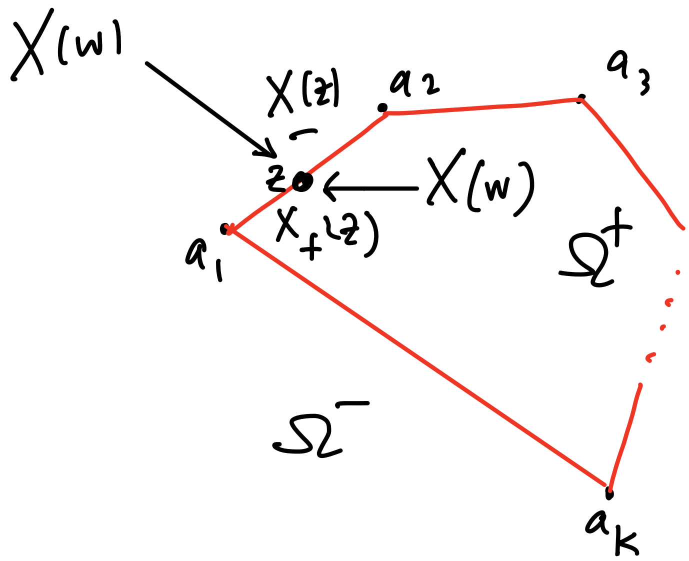
(**) find vector valued $X = X(z) \in \mathbb{C}^n \ \forall z$

a) $X(z)$ analytic on $\Omega^{\pm} \setminus \{\infty\}$
and extends continuously from either side of
$\Gamma \setminus \{a_1, .., a_k\}$

b) on each open segment (a_i, a_{i+1})

$$X_{\pm}(z) = \lim_{\substack{w \to z \\ w \in \Omega^{\pm}}} X(w)$$

satisfy the boundary condition $X_+(z) = X_-(z)\, G(z)$



c) $X(z)$ is of fin. deg. at ∞ i.e. $\exists$ vector of polynomials γ

$\gamma \in \mathbb{C}[z]^n$

s.t. $X(z) = \gamma(z) + O(z^{-1})$ as $z \to \infty$

and in small neighb. of pts $a_i \in \Gamma$

$$\|X(z)\| \leq \frac{C}{|z-a_i|^\alpha} \qquad 0 \leq \alpha < 1 \qquad C > 0$$

---

here requiring $X(z)$ extends continuously to $X_\pm(z)$ on $\Gamma \setminus \{a_1, \dots a_k\}$
can require solutions of this boundary RH problem
that extend to $X_\pm(z)$ on $\Gamma \setminus \{a_1, \dots a_k\}$

Hölder continuous $\quad |f(x) - f(y)| \leq c\,|x-y|^\alpha$

some $0 < \alpha \leq 1$

[9] for this assume the matrix $G(z)$ $z \in \Gamma$
("jump matrix")

is Hölder continuous on all of Γ

(not only on each (a_i, a_{i+1}))

[look first at this continuous everywhere on Γ case]

so problem of finding $X_\pm$ w/ jump G on Γ and $\gamma(z)$ at ∞

(behavior at a_i not required if

$\Leftrightarrow$ problem of finding a funct. $X_-(\lambda)$ (Hölder) cont on $\Gamma \ni \lambda$ s.t.

$$\frac{1}{2\pi i}\int_\Gamma \frac{X_-(\lambda)G(\lambda)}{z-\lambda}\,d\lambda = 0 \quad \forall z \in \Omega^-$$

no poles in Ω^+

(in interior Ω^+ region no poles)

$$-\frac{1}{2\pi i}\int_\Gamma \frac{X_-(\lambda)}{z-\lambda}\,d\lambda + \gamma(z) = 0 \quad \forall z \in \Omega^+$$

no poles in Ω^-

(in exterior Ω^- region once pole at ∞ subtracted no other pole)

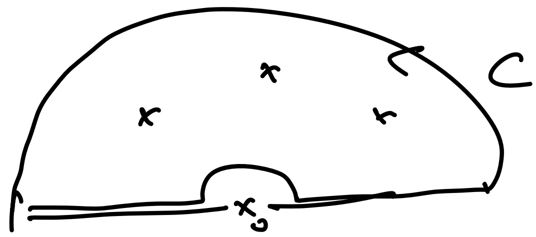
$$f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}\,d\zeta$$

Cauchy integral formula

$\boxed{10}$ $$PV\int_a^b \frac{f(x)}{x-x_0}\,dx = \lim_{\delta\to 0}\left(\int_a^{x_0-\delta}\frac{f(x)}{x-x_0}\,dx+\int_{x_0+\delta}^{b}\frac{f(x)\,dx}{x-x_0}\right)$$

$$PV\left(\int_C f(z)\,dz\right) = 2\pi i\left(\frac{1}{2}\,\mathrm{Res}_{x=x_0}(f) + \sum_{\substack{z\in \mathrm{int}(D)\\ \partial D = C}} \mathrm{Res}_z(f)\right)$$

$$\int \frac{f(x)\,dx}{x-x_0\pm i\varepsilon} = PV\int\frac{f(x)}{x-x_0}\,dx \mp i\pi f(x_0)$$



$$PV\int_\Gamma \frac{X_-(\lambda)}{\lambda-z}\,d\lambda = \lim_{\delta\to 0}\sum_i \int_{a_i+\delta}^{a_{i+1}-\delta}\frac{X_-(\lambda)}{\lambda-z}\,d\lambda$$

$$PV\int_\Gamma \frac{X_-(\lambda)}{\lambda-z}\,d\lambda = \underbrace{\int_\Gamma\frac{X_-(\lambda)}{\lambda-z}\,d\lambda}_{=-2\pi i\gamma(z)} - i\pi X_-(z)$$

for $z\in\Gamma$

(***)
$$-\frac{1}{\pi i}PV\int_\Gamma \frac{X_-(\lambda)}{\lambda-z}\,d\lambda + 2\gamma(z) = X_-(z)$$

& similarly $z\in\Gamma$
$$X_-(z)G(z) = \frac{1}{\pi i}PV\int_\Gamma\frac{X_-(\lambda)G(\lambda)}{\lambda-z}\,d\lambda$$

$\to G(z)^{-1}$

write as an integral equation

$$X_-(z) - \frac{1}{\pi i} \mathrm{PV} \int_\Gamma X_-(\lambda) \frac{K(z,\lambda)}{\lambda - z} \, d\lambda = \gamma(z)$$

(*)

with Kernel

$$K(z,\lambda) := \frac{1}{2} \left(G(\lambda) - G(z) \right) G(z)^{-1}$$

$(z,\lambda) \in \Gamma \times \Gamma$

Case of a class of general problems of this kind (singular Fredholm equations)

$$(K\phi)(z) := \phi(z) - \frac{1}{\pi i} \mathrm{PV} \int_\Gamma \phi(\lambda) \frac{K(z,\lambda)}{\lambda - z} \, d\lambda$$

Q: when is equation (*) solvable in space of continuous functions on Γ

Q: how to go from sol'n of (*) to solution of original problem (**) (w/ Hölder contin. hypothesis)

- an accompanying problem
- if accomp. probl has no nontrivial sol'ns then sol'ns of (*) give sol'n of (**) (w/ cond.)

12 accompanying problem:

$$\hat{X}(z) := \begin{cases} \frac{1}{2\pi i} \int_\Gamma \frac{X_-(\lambda)}{\lambda - z}\, d\lambda - \gamma(z) & z \in \Omega^+ \\ \frac{1}{2\pi i} \int_\Gamma \frac{X(\lambda)\, G(\lambda)}{\lambda - z}\, d\lambda & z \in \Omega^- \end{cases}$$

if $\underset{z \in \Omega^\pm}{\hat{X}(z) \equiv 0}$ then $\underset{\lambda \in \Gamma}{X_-(\lambda)}$ solves (***)

$\Longrightarrow$ determines a sol'n of (**) w/ contin. hypothesis

Problem for $\hat{X}$:

find $\hat{X}$ vector valued in $\mathbb{C}^n$

- $\hat{X}(z)$ analytic in $\Omega^\pm$ extend H. cont. on Γ from either side $\hat{X}_\pm$

- ptwise limits

$$\hat{X}_\pm(z) = \lim_{\substack{w \to z \\ w \in \Omega^\pm}} \hat{X}(w) \qquad z \in \Gamma$$

satisfy jump condition

$$\hat{X}_+(z) = \hat{X}_-(z) \cdot G(z)^{-1} \qquad \forall z \in \Gamma$$

- $\hat{X}(z) = \mathcal{O}(z^{-1})$ as $z \to \infty$ vanishing at ∞

if $\hat{X}$ defined by $\boxed{\circledast}$ is <u>not</u> $\equiv 0$
then it is a non-trivial sol'n of this problem

- if the accompanying problem only has trivial solution $\hat{X} \equiv 0$

then solutions of $\circledast$ equiv to sol'ns of $\boxed{\circledast}$
hence sol'ns of $\circledast\circledast$ w/ continuity condition

---

So reduced to the integral equation form w/ Kernel $\frac{K(z,\lambda)}{\lambda - z}$ <u>up to</u> the issue of the continuity hypothesis in the original problem (for $G(z)$ on Γ)

Want to eliminate this continuity hypothesis and replace with <u>piecewise constant</u>

$$G(a_j \mp 0) = \lim_{\lambda \to a_j \mp} G(\lambda)$$

limits from two sides of a_j on Γ

a_σ where $G(a_\sigma - 0) \neq G(a_\sigma + 0)$ pts of discontinuity

$$G_\sigma := G(a_\sigma - 0)\, G(a_\sigma + 0)^{-1} \qquad \in GL(n, \mathbb{C})$$

$\lambda_1^\sigma, \dots, \lambda_n^\sigma$ eigenvalues of G_σ

14

- write then eigenvalues as

$$\lambda_j^\sigma = e^{2\pi i \gamma_j^\sigma} \qquad \gamma_j^\sigma \in \mathbb{C}$$

$$-\tfrac{1}{2} < \mathrm{Re}\,(\gamma_j^\sigma) \leq \tfrac{1}{2}$$

- fix $z_0 \in \Omega^+$ $\qquad L^\sigma$ = line segment between z_0 and a_σ

$\rightsquigarrow$ extend to half line to ∞

$\Rightarrow$ every branch

$$d_j^\sigma(z) = (z - z_0)^{\gamma_j^\sigma} \qquad z \in \mathbb{C} \setminus L^\sigma \quad j = 1, \dots, n$$

is single valued and

$\forall z \in \Gamma \setminus \{a_\sigma\}$

$$\lim_{\substack{z \to w \\ z \in \Omega^+}} d_j^\sigma(z) = \lim_{\substack{z \to w \\ z \in \Omega^-}} d_j^\sigma(z)$$

and

$$\frac{d_j^\sigma(a_\sigma - 0)}{d_j^\sigma(a_\sigma + 0)} = e^{2\pi i \gamma_j^\sigma}$$

15

take a branch of $(z-a_\sigma)^{\gamma_j^\sigma}$ in $\mathbb{C}$ with cut along $[a_\sigma, \infty) \subset L_\sigma$

and $\left(\frac{z-a_\sigma}{z-z_0}\right)^{\gamma_j^\sigma}$ uniquely def. on $\mathbb{C}$ w/ cut $[z_0, a_\sigma] \subset L_\sigma$

such that $\left(\frac{z-a_\sigma}{z-z_0}\right)^{\gamma_j^\sigma} \to 1$ when $z \to \infty$

diagonal matrix

$$C_\sigma(z) = \bigoplus_{j=1}^{n} d_j^\sigma(z) \leftarrow \text{jumps at } a_\sigma$$

assume $G(a_\sigma - 0)\, G(a_\sigma + 0)^{-1}$ is diagonalizable

then $\exists N_\sigma$

$$N_\sigma G(a_\sigma - 0)\, G(a_\sigma + 0)^{-1} N_\sigma^{-1} = \bigoplus_{j=1}^{n} \lambda_j^\sigma$$

and take

$$M_\sigma = C_\sigma(a_\sigma + 0)^{-1} N_\sigma G(a_\sigma + 0)$$

(if not diag. similar arg w/ Jordan form)

also satisfies

$$M_\sigma = C_\sigma(a_\sigma - 0)^{-1} N_\sigma G(a_\sigma - 0)$$

so matrix

$$C_\sigma(z)^{-1} N_\sigma G(z) - M_\sigma$$ is H. continuous (Lipschitz cont.) on $(a_{\sigma-1}, a_{\sigma+1}) \subset \Gamma$

$$A_\sigma(z) = \bigoplus_{j=1}^{n} \left(\frac{z-a_\sigma}{z-z_0}\right)^{\gamma_j^\sigma}$$ ← analytic in Ω^- (as → 1 at ∞)

$$B(z) = \bigoplus_{j=1}^{n} (z-a_\sigma)^{\gamma_j^\sigma}$$ ← analytic in Ω^+

(✱) $$T(z) := X(z) \cdot \begin{cases} M_\sigma^{-1} B_\sigma(z)^{-1} & \text{for } z \in \Omega^+ \\ N_\sigma^{-1} A_\sigma(z)^{-1} & \text{for } z \in \Omega^- \end{cases}$$

if $X(z)$ solves original problem (✱✱)
then $T(z)$ solves probl. w/ H. continuity hypoth.
on $(a_{\sigma-1}, a_{\sigma+1})$

iterate for each a_σ discontinuity

---

also these singularities are regular singular

$A(z) := \frac{d\Psi}{dz} \cdot \Psi(z)^{-1}$ where $\Psi(z)$ solution of boundary RH problem

is single valued analytic
on $\mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_k\}$

and behavior at a_i of soln's of (✱)
⟹ regular singular

<u>Plemelj's thm</u> (1908) any subgr $GL(n,\mathbb{C})$ gen. by $G_1, \dots, G_k$ w/ $\prod_{i=1}^{k} G_i = id$ is realized as monodr. of a system $\frac{d\Psi}{dz} = A(z)\Psi$ w/ regular sing.

The Bolibrukh counterexample (1989)

$$\frac{d\psi}{dz} = A(z)\psi \qquad A(z) = \begin{pmatrix} 0 & a_{12}(z) & a_{13}(z) \\ 0 & & \\ 0 & \multicolumn{2}{c}{B(z)} \end{pmatrix}$$

$$a_{12}(z) = \frac{1}{z^2} + \frac{1}{z+1} - \frac{1}{z-\frac{1}{2}}$$

$$a_{13}(z) = \frac{1}{z-1} - \frac{1}{z-\frac{1}{2}}$$

$$B(z) = \frac{1}{z}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{6(z+1)}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix} + \frac{1}{2(z-1)}\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} + \frac{1}{3(z-\frac{1}{2})}\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

- has Fuchsian singularities $a_1 = -1$ $a_2 = \frac{1}{2}$ $a_3 = 1$ and one non-Fuchsian singularity $a_4 = 0$
- all singularities are regular-singular pts
- this system has non-trivial monodromy
- this monodromy rep cannot be realized by a Fuchsian system

closely related to the fact that the monodromy rep is reducible

small perturb. of this system $\rightsquigarrow$ irred rep $\rightsquigarrow$ has Fuchsian realization

Kostov-Bolibrukh: if irreducible rep of monodr. $\Rightarrow$ can be realized by Fuchsian

From Riemann-Hilbert problem to Riemann-Hilbert correspondence:

generalization of linear systems

$\frac{d\Psi}{dz} = A(z)\Psi$ w/ reg sing.

$\Downarrow$

connections on locally free sheaves

$(\mathcal{F}, \nabla)$

RH correspondence

generalization of monodromy representation

$\mu: \pi_1(S, z_0) \longrightarrow GL(n, \mathbb{C})$

$\Downarrow$

local systems:

Category of finite dim representations of $\pi_1(X)$

$\Updownarrow$

Category of local systems (sheaves locally isomorphic to $\mathbb{C}^n$)

X complex mfld

$\begin{matrix} V \\ \downarrow \\ X \end{matrix}$ holom. vector bdle over X $\qquad$ $\mathcal{F}$ = loc. free sheaf of holom. sections of V

$\Omega^{\bullet} = \bigoplus_i \Omega^i \quad \Omega^i = \Lambda^i \Omega^1$ sheaf of 1-forms on X

$\Omega^i(\mathcal{F}) = \Omega^i \otimes_{\mathcal{O}} \mathcal{F}$ ← i-forms w/ coefficients in $\mathcal{F}$

$\mathcal{O}$ ← ring of functs on X

<u>connection</u>: $\nabla: \mathcal{F} \longrightarrow \Omega(\mathcal{F}) = \Omega^1 \otimes \mathcal{F}$

$\nabla(\varphi f) = d\varphi \otimes f + \varphi \nabla f$ (Leibnitz rule)

$\varphi \in \mathcal{O}$ $\quad f \in \mathcal{F}$ (multipl. of a section by a funct.)

(19) connection is <u>flat</u> if

$$\underbrace{\nabla^2 = \nabla \circ \nabla}_{\text{curvature of the connection}} : \mathcal{F} \to \Omega^2(\mathcal{F}) \quad \underline{\text{vanishes}}$$

where $\nabla : \Omega^p(\mathcal{F}) \to \Omega^{p+1}(\mathcal{F})$ w/ Leibniz rule

$$\nabla(\omega \otimes f) = d\omega \otimes f + (-1)^p \omega \wedge \nabla f \quad \omega \in \Omega^p \quad f \in \mathcal{F}$$

determined by $\nabla : \mathcal{F} \to \Omega^1(\mathcal{F})$

---

given $(\mathcal{F}, \nabla)$ loc. free sheaf w/ flat connection ∇

$\mathcal{F}^\nabla = \{ f \in \mathcal{F} : \nabla f = 0 \}$ sheaf of "horizontal sections"

map $(\mathcal{F}, \nabla) \longmapsto \mathcal{F}^\nabla$

equivalence of categories between
$\{\text{flat connections}\} \longleftrightarrow \{\text{local systems}\}$

<u>Note</u>: if V local system on X

$\rightsquigarrow \mathcal{F} = \mathcal{O} \otimes_{\mathbb{C}} V$ w/ flat connection

$$\nabla : \mathcal{F} \to \Omega(\mathcal{F})$$
$$\nabla(f \otimes v) = df \otimes v$$

$\boxed{20}$. Riemann–Hilbert problem and the Painlevé transcendents

2×2 system w/ 4 reg sing pts

$$\frac{d\Psi}{d\lambda} = \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + \frac{A_2}{\lambda - x}\right)\Psi$$

$\left(\begin{array}{l}\text{w/ } \mathrm{Tr}(A_i) = 0 \\ -\sum_i A_i = \begin{pmatrix}\delta & 0\\ 0 & -\delta\end{pmatrix} =: A_\infty\end{array}\right)$

eigenvalues $\pm\alpha, \pm\beta, \pm\gamma$

sing pts at $\{0, 1, \infty, x\}$ $\leftarrow$ up to an autom. of $\mathbb{P}^1(\mathbb{C})$ always put in this form

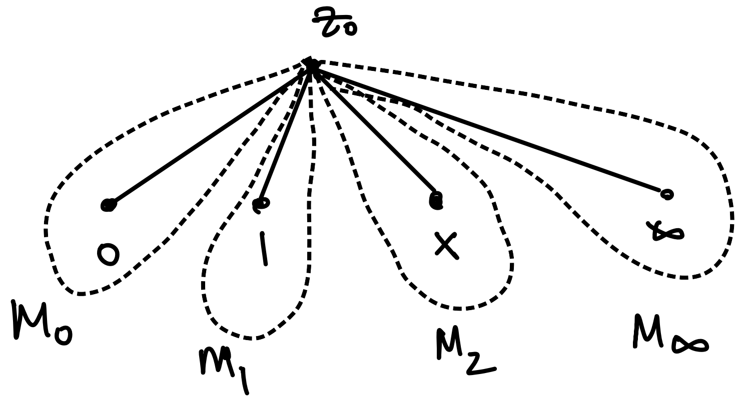
then consider dependence on position of fourth point x

$\rightsquigarrow$ diff system (nonlinear) describing deformations in the variable x

assume $2\alpha, 2\beta, 2\gamma \notin \mathbb{Z}$

monodromy representation

$M_\nu \quad \nu = 0, 1, 2, \infty$



$$\begin{cases} M_\infty M_2 M_1 M_0 = \mathrm{id} \\ M_\infty = \begin{pmatrix} e^{2\pi i\delta} & 0 \\ 0 & e^{-2\pi i \delta}\end{pmatrix} \\ \det M_\nu = 1 \quad \text{all } \nu \end{cases}$$

$e^{\pm 2\pi i\alpha}$, $e^{\pm 2\pi i \beta}$, $e^{\pm 2\pi i\gamma}$ eigenvalues of M_0, M_1, M_2

$0 \le \mathrm{Re}(\alpha, \beta, \gamma) < \frac{1}{2}$

(these fix the conj. class of the monodr. rep.)

isomonodromic deformations

$$\frac{d\psi}{d\lambda} = A(\lambda)\psi$$

reg. sing

dependent on an additional parameter

$A(\lambda,t)$ so that

monodromy rep. is constant in t

$$X = \mathbb{P}^1(\mathbb{C}) \setminus \{a_1, \dots, a_k\}$$

$$d\psi \cdot \psi^{-1} = \Theta(\lambda,t)$$

well def on univ. cover $\tilde{X}$ of X

$$d\psi = \sum_j \frac{\partial\psi}{\partial t_j} dt_j$$

$t = (t_j)$

Isomonodromy

since $\frac{\partial}{\partial t_j} M_\gamma = 0$ then

$\Theta(\lambda,t)$ single valued on X:

$$d(\psi(\lambda)M_\gamma)(\psi(\lambda)M_\gamma)^{-1} = d\psi(\lambda)\cdot\psi(\lambda)^{-1}$$

So system of equations

$$\frac{\partial\psi}{\partial\lambda} = A(\lambda,t)\psi$$

$$d\psi = \Theta(\lambda,t)\psi \iff \frac{\partial\psi}{\partial t_j} = \Theta_j(\lambda,t)\psi$$

with $\Theta(\lambda,t) = \sum_j \Theta(\lambda,t_j)\, dt_j$

overdetermined system $\begin{cases} \frac{\partial\psi}{\partial\lambda} = A\psi \\ \frac{\partial\psi}{\partial t_j} = \Theta_j\psi \end{cases}$

& compatibility conditions
(from cross-differentiating and matching terms)

get $\dfrac{\partial A}{\partial t_j} = \dfrac{\partial \Theta_j}{\partial \lambda} + [\Theta_j, A]$ holds for all λ

isomonodromy deformation equation
system on nonlinear ODEs
(also overdetermined if $t = (t_j)_{j=1}^{q}$ w/ $q > 1$)

Case of the form (Fuchsian)

$$\frac{d\psi}{d\lambda} = A(\lambda)\psi \qquad A(\lambda) = \sum_{\nu=1}^{k} \frac{A_\nu}{\lambda - a_\nu}$$

with $\left(\sum_\nu A_\nu = A_\infty \text{ diagonal}\right)$

deformation parameters
the positions of sing. pts $a_1, \dots, a_k$

$$d\psi = -\sum_{\nu=1}^{k} \frac{A_\nu \psi}{\lambda - a_\nu}\, da_\nu$$

$$\frac{\partial \psi}{\partial a_\nu} = \frac{-A_\nu}{\lambda - a_\nu}\psi$$

Schlesinger equations:

$\Rightarrow$ isomonodromy deformation equation

$$\begin{cases} \dfrac{\partial A_\nu}{\partial a_\nu} = -\displaystyle\sum_{\mu \neq \nu} \frac{[A_\mu, A_\nu]}{a_\mu - a_\nu} & \nu = 1, \dots, k \\[2ex] \mu \neq \nu \quad \dfrac{\partial A_\nu}{\partial a_\mu} = \dfrac{[A_\mu, A_\nu]}{a_\mu - a_\nu} & \mu \neq \nu = 1, \dots, k \end{cases}$$

this nonlinear system describes the isomonodromic deformations of the linear system
$$\frac{d\Psi}{\partial\lambda} = A(\lambda)\Psi$$

---

case of 4 Fuchsian points at $\{0, 1, \infty, x\}$

$$\begin{cases} \dfrac{\partial\Psi}{\partial\lambda} = \left(\dfrac{A_0}{\lambda} + \dfrac{A_1}{\lambda-1} + \dfrac{A_2}{\lambda-x}\right)\Psi \\[2ex] \dfrac{\partial\Psi}{\partial x} = -\dfrac{A_2}{\lambda-x}\Psi \end{cases}$$

the Schlesinger equation that gives the compatibility condition for these:

$$\frac{\partial}{\partial x}\left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + \frac{A_2}{\lambda-x}\right) - \frac{\partial}{\partial\lambda}\left(-\frac{A_2}{\lambda-x}\right)$$
$$= \left[-\frac{A_2}{\lambda-x}, \left(\frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + \frac{A_2}{\lambda-x}\right)\right]$$

$$\Rightarrow \begin{cases} \dfrac{dA_0}{dx} = \dfrac{1}{x}[A_2, A_0] \\[2ex] \dfrac{dA_1}{dx} = \dfrac{1}{x-1}[A_2, A_1] \\[2ex] \dfrac{dA_2}{dx} = -\dfrac{dA_0}{dx} - \dfrac{dA_1}{dx} \end{cases}$$

nontrivial part of this system of equations:

(24) single nonlinear second order scalar equation for last (12th) coeff of

$$A(\lambda,x) = \frac{A_0}{\lambda} + \frac{A_1}{\lambda-1} + \frac{A_2}{\lambda-x} \qquad \frac{W(x)\,(\lambda-u(x))}{\lambda(\lambda-1)(\lambda-x)}$$

This coeff has a zero at a pt $u(x)$ depending on x through equation

$$u'' = \frac{1}{2}\left(\frac{1}{u}+\frac{1}{u-1}+\frac{1}{u-x}\right)(u')^2$$
$$-\left(\frac{1}{x}+\frac{1}{x-1}+\frac{1}{u-x}\right)u'$$
$$+\frac{u(u-1)(u-x)}{x^2(x-1)^2}\left(\alpha_1+\beta_1\frac{x}{u^2}+\gamma_1\frac{x-1}{(u-1)^2}+\delta_1\frac{x(x-1)}{(u-x)^2}\right)$$

relation between parameters:

$$\alpha_1 = \frac{1}{2}(2\delta-1)^2 \qquad \beta_1 = -2\alpha^2$$

$$\gamma_1 = 2\beta^2 \qquad \delta_1 = \frac{1}{2}(1-4\gamma^2)$$

for $\pm\alpha$, $\pm\beta$, $\pm\gamma$ eigenvalues of A_0, A_1, A_2

$\pm\delta$ in A_∞ diag.