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Cusp forms and Modular Symbols

Preliminary: Continued fractions

notation

$$[k_1 \cdots k_n] := \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cdots}}}$$

can write this rational number as:

$$= \frac{P_n(k_1, \dots, k_n)}{Q_n(k_1, \dots, k_n)}$$

$k_i \geq 1$ integers

where P_n, Q_n are polynomials that can be constructed inductively

$$(*) \begin{cases} Q_{n+1}(k_1, \dots, k_n, k_{n+1}) = k_{n+1} Q_n(k_1, \dots, k_n) + Q_{n-1}(k_1, \dots, k_{n-1}) \\ P_n(k_1, \dots, k_n) = Q_{n-1}(k_2, \dots, k_n) \end{cases}$$

w/ initial condition for the recursion

$$Q_{-1} = 0 \qquad Q_1 = 1$$

in particular $(*)$ implies

$$[k_1, \dots, k_{n-1}, k_n + x_n] = \frac{P_{n-1}(k_1, \dots, k_{n-1}) x_n + P_n(k_1, \dots, k_n)}{Q_{n-1}(k_1, \dots, k_{n-1}) x_n + Q_n(k_1, \dots, k_n)}$$

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So

$$[k_1, \dots, k_{n-1}, k_n + x_n] = \begin{pmatrix} p_{n-1} & p_n \\ q_{n-1} & q_n \end{pmatrix} (x_n)$$

action by fractional linear transformations

<u>Continued fraction expansion</u> :

$\alpha \in (0,1) \cap \mathbb{R} \setminus \mathbb{Q}$ (irrational number)

- $\exists$ unique sequence of integers $k_n(\alpha) \geq 1$

  s.t. $\lim_{n \to \infty} [k_1(\alpha) \cdots k_n(\alpha)] = \alpha$

- $\exists$ unique sequence of real numbers $x_n(\alpha) \in (0,1)$

  s.t. $[k_1(\alpha), \dots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)] = \alpha$

if $\alpha \in (0,1] \cap \mathbb{Q}$ (rational)

$\exists$ finite sequence $k_i(\alpha) \geq 1$ s.t.

$\alpha = [k_1(\alpha) \cdots k_n(\alpha)]$

so $$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & k_1(\alpha) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n(\alpha) \end{pmatrix} x_n(\alpha)$$

Notation: $$p_n(\alpha) := P_n(k_1(\alpha), \dots, k_n(\alpha))$$
$$q_n(\alpha) := Q_n(k_1(\alpha), \dots, k_n(\alpha))$$

$\dfrac{p_n(\alpha)}{q_n(\alpha)}$ is called the sequence of convergents of the continued fraction expansion

$$\frac{p_n(\alpha)}{q_n(\alpha)} \xrightarrow[n\to\infty]{} \alpha$$ [base indep. rational approximation)

$$g_n(\alpha) := \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix}$$

<u>Reduced matrices</u>

reduced matrices of length n

$$\mathrm{Red}_n = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n \end{pmatrix} \;\middle|\; k_i \geq 1,\ k_i \in \mathbb{Z} \right\}$$

(*)

$$\mathrm{Red} := \bigcup_{n \geq 1} \mathrm{Red}_n \subset GL(2, \mathbb{Z})$$

reduced matrices

<u>semigroup</u>

Some facts about reduced matrices : (Lewis-Zagier)

(i) a matrix $A \in GL(2,\mathbb{Z})$ is reduced iff has non-negative entries w/ non-decreasing property:

$$Red = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2,\mathbb{Z}) \;\middle|\; 0 \le a \le b,\ 0 \le c \le d \right\}$$

(ii) unique representation of a reduced matrix in the form (*) above

(i.e. length n and k_i's uniquely def.)

$GL(2,\mathbb{Z})$ generated by

$$\sigma = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad \rho = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{with}$$

relations $$(\sigma^{-1}\rho)^2 = (\sigma^{-2}\rho^2)^6 = 1$$

the semigroup Red : subsemigroup of semigr. gen. by σ, ρ made of all words of the form $(n_i \ge 1)$

$$\sigma^{n_1 - 1}\rho\, \sigma^{n_2 - 1}\rho\, \sigma^{n_3 - 1}\rho \cdots \sigma^{n_k - 1}\rho$$

i.e. elements of this semigroup are products of elements of the form

$$\sigma^{n-1}\rho = \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix}$$ so same descr. of Red as above

<u>Note</u>: this semigroup if a <u>free</u> semigroup as only relations in $GL(2,\mathbb{Z})$ between σ & ρ involve σ^{-1}

so if had two different description of $\gamma \in$ Red

$$\gamma = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ 1 & m_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & m_\ell \end{pmatrix}$$

would have a relation

$$\sigma^{n_1-1}\rho \cdots \sigma^{n_k-1}\rho = \sigma^{m_1-1}\rho \cdots \sigma^{m_\ell-1}\rho$$

involving only non-neg powers of σ & ρ

(iii) reduced matrices are <u>hyperbolic</u>

$\Delta(\gamma) = Tr(\gamma)^2 - 4\det(\gamma)$

have 2 fixed pts on $\mathbb{R}$ $\Longleftarrow$ $\Delta(\gamma) > 0$, $Tr(\gamma) > 0$ (or $Tr(\gamma) > 2$)

and every conjugacy class of hyperbolic matrices in $GL(2,\mathbb{Z})$ contains representatives in Red

γ hyp matrix:

all with same length $l(\gamma)$

and # of rep's given by $\dfrac{l(\gamma)}{k(\gamma)}$

$k(\gamma)$ = max integer such that $\gamma = h^{k(\gamma)}$ for some h

(see Lewis-Zagier §2)

Modular symbols (Manin)

$\Gamma = PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\pm 1$

$G_0 \subset \Gamma$ fin. index subgroup

$G_0 \backslash \mathbb{H} = X_{G_0}$ mod curve

action by fractional lin. transf.

$\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) =: \overline{\mathbb{H}}$

$\overline{X}_{G_0} := G_0 \backslash \overline{\mathbb{H}}$

compactification

$\varphi : \overline{\mathbb{H}} \twoheadrightarrow \overline{X}_{G_0}$ quotient map

given a pair of points $\alpha, \beta \in \overline{\mathbb{H}}$

$\{\alpha, \beta\}_{G_0} \in H_1(\overline{X}_{G_0}, \mathbb{R})$ defined by

(*) $$\int_{\{\alpha,\beta\}} \omega := \int_\alpha^\beta \varphi^*(\omega)$$

for $\omega \in \Omega^1(\overline{X}_{G_0})$ hol. 1-form (differential of the first kind)

and integration along geodesic path in $\overline{\mathbb{H}}$ connecting pts α & β

<u>Note:</u>

if $G_0 \alpha = G_0 \beta$ geod curve from α to β in $\overline{\mathbb{H}}$ becomes closed curve in $\overline{X}_{G_0}$ (defines homology class in $H_1(\overline{X}_{G_0}, \mathbb{Z})$)

while

$$H_1(\overline{X}_{G_0}, \mathbb{R}) = \mathrm{Hom}(\underbrace{H^0(\overline{X}_{G_0}, \Omega^1)}_{\dim h^{1,0}}, \mathbb{C})$$

& (*) defines such as fixed unique path of integration = geod arc in $\overline{\mathbb{H}}$

call $\{\alpha, \beta\}_{G_0}$ the <u>modular symbol</u>

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Properties: 1) $\{\alpha,\beta\}_{G_0} + \{\beta,\gamma\}_{G_0} = \{\alpha,\gamma\}_G$

(as differ by int. on closed contractible curve in $\overline{H}$)

2) $\forall g \in G_0 \quad \{g\alpha, g\beta\}_{G_0} = \{\alpha,\beta\}_{G_0}$

(because integrating pullback of form on $\overline{X}_{G_0}$)

suppose $\alpha,\beta \in \mathbb{P}^1(\mathbb{Q})$ cusps (also called "parabolic points")

Known (Manin-Drinfeld) that for cusps α,β mod symbol $\{\alpha,\beta\}_{G_0}$ is a <u>rational</u> homology class in $H_1(\overline{X}_{G_0},\mathbb{Q})$

by (1) & (2) above can just look at case of $\{0,\alpha\}_{G_0}$ for $\alpha \in \mathbb{Q}$

then α has a finite contined fraction expansion

$\alpha = [k_1 \cdots k_n]$

$$g_k := \begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha) \\ q_{k-1}(\alpha) & q_k(\alpha) \end{pmatrix} \qquad \alpha = \frac{p_n(\alpha)}{q_n(\alpha)}$$

$k=1,\dots,n$

action by frac lin transf.

$$\left\{\frac{p_{k-1}(\alpha)}{q_{k-1}(\alpha)}, \frac{p_k(\alpha)}{q_k(\alpha)}\right\}_{G_0} = -\left\{g_k^{(\alpha)}(0),\ g_k^{(\alpha)}(i\infty)\right\}_{G_0}$$

9 so get

$$\int_0^\alpha \varphi^*(\omega) = \sum_{k=1}^n \int_{p_{k-1}(\alpha)/q_{k-1}(\alpha)}^{p_k(\alpha)/q_k(\alpha)} \varphi^*(\omega) = -\sum_{k=1}^n \int_{\{g_k(0),\, g_k(i\infty)\}_{G_0}} \omega \qquad \boxed{*}$$

the k-th integral in sum depends only on the class

$[g_k] \in G_0\backslash PSL(2,\mathbb{Z})$ because of invariance (2) above

$\mathbb{P}_0 := G_0\backslash PSL(2,\mathbb{Z})$

<u>Note</u>: on $SL(2,\mathbb{Z})$ vs $GL(2,\mathbb{Z})$: cont. fr. expansion (semigr. Red etc.) in $GL(2,\mathbb{Z})$ while mod grp $SL(2,\mathbb{Z})$ and $G_0 \subset PSL(2,\mathbb{Z})$

GL_2- cont. fr.

replace G_0 with its lift to $SL(2,\mathbb{Z})$ and write G for subgr. of $GL(2,\mathbb{Z})$ gen. by G_0 and $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Then $\mathbb{P}_0 \simeq G\backslash GL(2,\mathbb{Z}) = \mathbb{P}$ $\quad$ ($g \mapsto g^{-1}$ if write $\mathbb{P} = GL(2,\mathbb{Z})/G$)

$\Gamma = GL(2,\mathbb{Z})$ acts on the right on $\mathbb{P} = G\backslash\Gamma$

$$\xi : \mathbb{P} \longrightarrow H_1(\overline{X}_G, \mathbb{Q})$$

$$\xi(s) = \{g(0), g(i\infty)\}_G$$

$s = [g]$ in $\mathbb{P} = G\backslash GL(2,\mathbb{Z})$

and $\boxed{*}$ $\Longrightarrow$ $\{0,\alpha\}_G = -\sum_{k=1}^n \xi(\underbrace{Gg_k}_{s_k \in \mathbb{P}})$

$\alpha \in \mathbb{Q}$

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(Manin) perfect pairing between space $\mathcal{S}_{G_0,2}$ of cusp modular forms of weight 2 and modular symbols

$$\langle \cdot , \cdot \rangle : \mathcal{S}_{G_0,2} \times H_1(\bar{X}_{G_0}, \mathbb{R}) \longrightarrow \mathbb{C}$$

$$\langle \psi, \{\alpha, \beta\}_{G_0} \rangle = \int_{\{\alpha,\beta\}_{G_0}} \psi(z)\, dz$$

ω = 1-form holom on $\bar{X}_{G_0}$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G_0$

$\omega = \psi(z)\, dz$

$$\omega\left(\frac{az+b}{cz+d}\right) = \psi\left(\frac{az+b}{cz+d}\right) d\left(\frac{az+b}{cz+d}\right)$$

invariance for ω (defined on quotient) means

$$= \psi\left(\frac{az+b}{cz+d}\right) \cdot \left(\frac{a}{cz+d} - \frac{(az+b)c}{(cz+d)^2}\right) dz$$

$$= \psi\left(\frac{az+b}{cz+d}\right) \frac{acz + ad - acz - bc}{(cz+d)^2}\, dz$$

$$= (cz+d)^{-2}\, \psi\left(\frac{az+b}{cz+d}\right) \cdot dz$$

$$\psi\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 \psi(z)$$ so weight 2 mod form

(and convergence $\rightsquigarrow$ zero at cusps)

$\psi \in S_{G_0,2}$

$\Rightarrow$ periods of modular forms

What happens at points $\alpha \in \mathbb{R} \setminus \mathbb{Q}$?

"invisible" to the compactification by cusps

$$\overline{X}_{G_0} = G_0 \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$$

geodesics in $\mathbb{H}$ w/ endpoints $\alpha, \beta \in \mathbb{P}^1(\mathbb{R})$ (but not nec. in $\mathbb{P}^1(\mathbb{Q})$)

take one endpoint $\alpha = 0$ and just look at $\gamma_{0,\beta}$ geodesic in $\mathbb{H}$

mod symbol $\{0,\beta\}$ not defined but can define an "averaged" asymptotic behavior

LIMITING MODULAR SYMBOLS:

$$\{\{ *, \beta \}\}_{G_0} := \lim_{y \to \beta} \frac{1}{T(x_0, y)} \{x_0, y\}_{G_0}$$

mod symbol in $H_1(\overline{X}_{G_0}, \mathbb{R})$ determ.. by this geod. arc

looking at asympt. behavior when approaching β so where from does not matter

fixed base pt on geodesic $\gamma_{\alpha,\beta}$

T = length of geodesic arc from x_0 to y along geod $\gamma_{\alpha,\beta}$

- Not clear w/ this definition if / when limit exists
- if it exists how to compute it (in terms of basis of $H_1(\overline{X}_{G_0}, \mathbb{R})$)

Some properties of limiting mod symbols

① • if β is a quadratic irrationality

(i.e. a pt. in a real quadratic field $K \subset \mathbb{R}$)

i.e. equiv. condition: if $\exists$ hyperb. matrix $\gamma \in G_0$ s.t. β is one of the two fixed pts of γ

(the other fixed pt is the Galois conj of β w/ $Gal(K/\mathbb{Q})$ of real quadr field)

$\beta^{\pm}$ the two fixed pts (w β = one or other)

$\lambda_\gamma^{\pm}$ eigenvalues of γ $\quad 0 < \lambda_\gamma^{+} < 1$

action of γ on $\mathbb{H}$ shifts along geod between $\beta^{\pm}$

amount of shift produced by action of γ along geod $\to$ $\ell_\gamma := \log \lambda_\gamma^{-}$

(= length of closed geodesic in quotient $\overline{X}_{G_0}$)

$\{x, \gamma x\}_{G_0}$ closed loop in $\overline{X}_{G_0}$ for any x on geod between $\beta^{\pm}$

will get : $\{\{*, \beta\}\}_{G_0} = \dfrac{\{0, \gamma(0)\}_{G_0}}{\ell_\gamma}$

> quadratic irrationalities have eventually periodic cont. fr. expansion

also equivalent expression :

$$= \frac{\sum_{k=1}^{\ell} \{ g_k^{-1}(\beta)\, g(0),\; g_k^{-1}(\beta)\, g(i\infty) \}_{G_0}}{\lambda(\beta)\, \ell}$$

ℓ = length of periodic cont. fr. expansion of β quadr. irrat.

② for "general pts" in $\mathbb{R}$ (a set of full measure)

lim modular symbol "weakly converges" to zero

Multifractal decomposition of the limiting mod symbol

③ $f_1, \dots, f_g$ basis of vector space $S_{G,2}$ (identified w/ dual of $H_1(\bar{X}_G, \mathbb{R})$)

$\mathrm{Re}(f_i)$, $\mathrm{Im}(f_i)$ basis of $2g$-dim real vector space $\simeq H^1(\bar{X}_G, \mathbb{R}) \simeq \mathbb{R}^{2g}$

(assume here $g \geq 1$)

where g = genus of comp. curve $\bar{X}_G$

level sets of lim mod symbol

$\alpha \in \mathbb{R}^{2g}$

$$E_\alpha = \{ \beta : \langle \underline{f}, \{\{*, \beta\}\}_{G_0} \rangle = \alpha \in \mathbb{R}^{2g} \}$$

(i.e. $\lim_{x \to \beta} \frac{1}{T} \int_{\{x_0, x\}_G} f_i(z) dz = \alpha_i$)

i.e. for all pts in E_α lim mod symb $\{\{*, \beta\}\}_{G_0}$ is a class $h_\alpha \in H_1(\bar{X}_{G_0}, \mathbb{R})$ w/ $\langle \underline{f}, h_\alpha \rangle = \alpha$

(Kesselböhmer – Stratmann):

$\exists$ convex differentiable function

$\beta_{G_0} : \mathbb{R}^{2g} \to \mathbb{R}$ s.t.

$\forall \alpha \in \nabla \beta_{G_0}(\mathbb{R}^{2g}) \subset \mathbb{R}^{2g}$

$$\dim_H (E_\alpha) = \hat{\beta}_{G_0}(\alpha)$$

w/ $\hat{\beta}_{G_0}$ Legendre transform

$$\hat{\beta}_{G_0}(\alpha) = \inf_{v \in \mathbb{R}^{2g}} (\beta_{G_0}(v) - \langle \alpha, v \rangle)$$

4. exceptional set in $\mathbb{R}$
on which limit does not exist

More precisely how to obtain (some of)
these results on lim mod symbols: