Proof of classif. of Dirac operators:

- First check the $D = D(Y)$ are Dirac operators

<u>off-diag. part</u> $\begin{pmatrix} 0 & Y_R^* \\ Y_R & 0 \end{pmatrix}$ anti-commuting with $\gamma = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 0 & Y_R^* \\ Y_R & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & Y_R^* \\ -Y_R & 0 \end{pmatrix}$$
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & Y_R^* \\ Y_R & 0 \end{pmatrix} = \begin{pmatrix} 0 & -Y_R^* \\ Y_R & 0 \end{pmatrix}$$

also commutes w/ J_F since

$\overline{Y_R \xi} = Y_R^* \bar{\xi}$ since Y_R is a symmetric matrix

and $J_F = \begin{pmatrix} 0 & c \\ c & 0 \end{pmatrix}$ c = complex conjugation

order-one condition for this part ok: $[[D_{off}, a], b] = 0$

Y_R acts only on $|\uparrow\rangle_R \otimes \mathbb{1}^\circ \longrightarrow \mathbb{1} \otimes |\uparrow\rangle_R$

and $\pi(a)\, |\uparrow\rangle_R \otimes \mathbb{1}^\circ = q(\lambda)^t\, |\uparrow\rangle_R \otimes \mathbb{1}^\circ = \lambda\, |\uparrow\rangle_R \otimes \mathbb{1}^\circ$

$\pi'(a)\, \mathbb{1} \otimes |\uparrow\rangle_R = \lambda\, \mathbb{1} \otimes |\uparrow\rangle_R$

So $[D_{off}, a] = 0$

<u>diagonal part</u> $\begin{pmatrix} S & 0 \\ 0 & \bar{S} \end{pmatrix}$ commuting w/ J_F, anti-commuting w/ γ_F — by construction given how S defined

- Commutation with $(\lambda, \lambda, 0)$
- Order one condition

Notice that S does not have components intertwining $|\uparrow\rangle$ and $|\downarrow\rangle$ since

$$\begin{pmatrix} 0 & 0 & Y_\nu^* & 0 \\ 0 & 0 & 0 & Y_e^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$$

no terms interchanging $\uparrow \downarrow$

$S_\ell : \ell_R \otimes \mathbb{1}^\circ \oplus \ell_L \otimes \mathbb{1}^\circ \longrightarrow \ell_R \otimes \mathbb{1}^\circ \oplus \ell_L \otimes \mathbb{1}^\circ$

(same for S_q)

$\Rightarrow$ commutation w/ $\pi(\lambda, \lambda, 0)$

order one: $[S, \pi(a)] = P$

is an operator of the form

$$P = P_\ell \oplus (id_3 \otimes P_q)$$

because $S = S_\ell \oplus (id_3 \otimes S_q) = \begin{pmatrix} 0 & 0 & Y_\nu^* & 0 \\ 0 & 0 & 0 & Y_e^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix} \oplus id_3 \otimes \begin{pmatrix} 0 & 0 & Y_u^* & 0 \\ 0 & 0 & 0 & Y_d^* \\ Y_u & 0 & 0 & 0 \\ 0 & Y_d & 0 & 0 \end{pmatrix}$

and $\pi(a) = \pi(\lambda, q, m) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar{\beta} & \bar{\alpha} \end{pmatrix} \otimes id_{12}$

on the basis

$|\uparrow\rangle_R, |\downarrow\rangle_R, |\uparrow\rangle_L, |\downarrow\rangle_L$

then see that operators of the form

$P = P_\ell \oplus (id \otimes P_q)$ commute with

$b^\circ = J_F b^* J_F$ for all $b \in \mathcal{A}_F$

$\pi(b^\circ) = \pi'(b^*) = \pi'(\bar{\lambda}, \bar{q}, m^*)$

$= \bar{\lambda} \oplus \underline{m^* \otimes id}$ commuting w/ $P_\ell \oplus (\underline{id_3 \otimes P_q})$

(Physically: color is unbroken)

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then show that all Dirac operators are of the form $D(Y)$:

$D = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix}$ because self-adjoint and $DJ_F = J_F D$

$T = T^t$ symmetric

On $\mathcal{H}_f \subset \mathcal{H}_F$ grading γ_F given by $\gamma_F \xi = v \xi$ (not on $\mathcal{H}_{\bar{f}}$)

$v = (-1, 1, 1) \in \mathcal{A}_F$

$D\gamma_F = -\gamma_F D \Rightarrow$

$$D = -\tfrac{1}{2}\gamma_F [D, \gamma_F]$$

$\gamma_F = \begin{pmatrix} g & 0 \\ 0 & -\bar{g} \end{pmatrix}$ so $S = -\frac{1}{2} g [S, g] = -\frac{1}{2} v [S, v]$

where $v[S,v]$ first block of matrix $v[D,v]$

$\Rightarrow$ order-one condition:

S commutes w/ all b^0 $\Rightarrow$ $S = S_\ell \oplus \underline{\underline{id_3}} \otimes S_q$

color unbroken

then the form

$$S_\ell = \begin{pmatrix} 0 & 0 & Y_\nu^* & 0 \\ 0 & 0 & 0 & Y_e^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$$ and similarly for S_q

follow from anticommuting w/ γ_F and commuting with $\pi(\lambda, \lambda, 0) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar{\lambda} & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \bar{\lambda} \end{pmatrix}$

To check what T should be like:

if an operator $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ commutes with elements of $\mathcal{A}_F^o = J_F \mathcal{A}_F J_F$

then need:

P_{12} support $\mathbb{1} \otimes |\uparrow\rangle_R^o \oplus 3 \otimes |\uparrow\rangle_R^o$ ← here $a^o \in \mathcal{A}_F^o$ acts by $q(\lambda)^t |\uparrow\rangle_R = \lambda |\uparrow\rangle_R$

range $2_L \otimes \mathbb{1}^o \oplus 2_R \otimes \mathbb{1}^o$ ← here $a^o \in \mathcal{A}_F^o$ acts by λ

P_{21} support $2_L \otimes \mathbb{1}^o \oplus 2_R \otimes \mathbb{1}^o$ ← action of $\mathcal{A}_F^o$ by λ

$\mathbb{1} \otimes |\uparrow\rangle_R^o \oplus 3 \otimes |\uparrow\rangle_R^o$ ← action of $\mathcal{A}_F^o$ by $q(\lambda)^t |\uparrow\rangle_R = \lambda |\uparrow\rangle_R$

Then: $\pi(e)$ for $e = (0,1,0) \in \mathcal{A}_F$

$\quad \parallel$

projection onto $\gamma_F = 1$ eigenspace

while $\pi'(e) = 0$

$[D, e]$ commutes w/ $\mathcal{A}_F^\circ$ by order-one

$\Rightarrow \quad \pi'(e)T - T\pi(e) = -T\pi(e)$

supp $\subset \mathbb{H}_L \otimes \mathbb{1}^\circ \oplus \mathbb{H}_R \otimes \mathbb{1}^\circ$

range $\subset \underbrace{\mathbb{1} \otimes |\uparrow\rangle_R^\circ \oplus \mathbb{H} \otimes |\uparrow\rangle_R^\circ}$ (where $\gamma_F = 1$)

$\Rightarrow$ anticommuting w/ γ_F:

supp.$(T\pi(e)) = \{\gamma_F = -1\}$

but since $\pi(e)$ proj. on $\gamma_F = 1$

$\Rightarrow \quad T\pi(e) = 0$

Then $\pi(e_3) = \pi((0,0,1)) \in \mathcal{A}_F$

find that also $Te_3^\circ = 0$

$\pi(e_3^\circ) =$ proj. onto $\cdot \otimes \mathbf{3}^\circ$ subspace of $\mathcal{H}_f$

$\pi'(e_3^\circ) = 0 \quad \Rightarrow [T, e_3^\circ] = T e_3^\circ$

but $[T, e_3^\circ] = 0$ since T commutes w $\pi(\lambda,\lambda,0)$ and with $\pi(\lambda,\lambda,0)^\circ$

$Te_3^\circ = 0 \quad \Rightarrow \quad \text{supp}(T) \subset \mathbb{H}_R \otimes \mathbb{1}^\circ$

and since T symmetric: range$(T) \subset \mathbb{1} \otimes \mathbb{H}_R^\circ$

(λ, q, m) acts on these as $(\lambda, \lambda, 0)$

so T commutes w/ $\mathcal{A}_F$ and $\mathcal{A}_F^\circ$

by previous Lemma and T

$\Rightarrow$ supp & range T: $|\uparrow\rangle_R \otimes \mathbb{1}^\circ \longrightarrow \mathbb{1} \otimes |\uparrow\rangle_R^\circ \quad = Y_R$

$\Rightarrow$ A characterization of $\mathcal{A}_F$:

$$\mathcal{A}_F = \{ a \in \mathcal{A}_{LR} : [D_F^{off}, a] = 0 \}$$

Moduli space of Dirac operators

$$\mathcal{C}_q = \{ (Y_d, Y_u) \in GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) : \text{mod equiv relation}$$
$$Y_d' = W_1 Y_d W_3^* \quad Y_u' = W_2 Y_u W_3^*$$
$$\text{for } W_1, W_2, W_3 \in U(3) \}$$

$$= U(3) \times U(3) \backslash GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) / U(3)$$

$$\mathcal{C}_\ell = \{ (Y_e, Y_\nu, Y_R) : Y_e, Y_\nu \in GL_3(\mathbb{C}) ; Y_R \text{ symmetric}$$

symm complex matrices

$$U(3) \times U(3) \backslash GL_3(\mathbb{C}) \times GL_3(\mathbb{C}) \times \mathcal{S} / U(3)$$

modulo equiv. relation

$$Y_e' = V_1 Y_e V_3^* \quad Y_\nu' = V_2 Y_\nu V_3^*$$
$$Y_R' = V_2 Y_R \overline{V_2}^* \quad V_1, V_2, V_3 \in U(3) \}$$

Each equivalence class in $\mathcal{C}_q$ contains a representative of the form (Y_d, Y_u) with Y_u diagonal w/ positive entries and Y_d positive $= C \delta_\downarrow C^*$ $\quad \delta_\downarrow = \text{diag}$ and $C \in SU(3)$

In fact: choosing $W_2, W_3 \Rightarrow Y_u$ diag & positive
choosing $W_1 \Rightarrow Y_d$ positive

$$Y_u = \delta_\uparrow \qquad Y_d = C\,\delta_\downarrow\,C^*$$

Parameters: $C \in SU(3)$: 8 real parameters

$(\delta_\uparrow, C\delta_\downarrow C^*) \sim (\delta_\uparrow, C'\delta_\downarrow C'^*)$ iff
$\exists$ A, B diagonal $\subset SU(3)$ s.t.

$AC = C'B$ $\qquad \dim_{\mathbb{R}} = 8 - 4$ double coset space of $C \in SU(3)$ mod diag A, B

$C = R_{23}(\theta_2)\, d(\delta)\, R_{12}(\theta_1)\, R_{23}(-\theta_3)$

$d(\delta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -e^{i\delta} \end{pmatrix}$

3 angles + one phase

$\Rightarrow$ Real dim $C_q = 3 + 3 + 4 = 10$

Real dim C_ℓ :

$\pi : C_\ell \longrightarrow C_q$ fibration (forgetting Y_R)

$(Y_e, Y_\nu, Y_R) \longmapsto (Y_e, Y_\nu)$

fibre = symmetric complex 3×3 matrices
mod action of $\lambda \in \mathbb{C}$

$$Y_R \longmapsto \lambda^2 Y_R \qquad \dim_{\mathbb{R}} \text{fibre} = 12 - 1 = 11$$

(scalar λ; see above passing from $U(3)$ to $SU(3)$)

$\Rightarrow \dim_{\mathbb{R}} \mathcal{C}_\ell = 10 + 11 = 21$

Physically: parameters of the standard model

- Minimal Standard model

  - 3 charged lepton masses (no neutrino masses)
  - 6 quark masses
  - 3 gauge coupling constants
  - 3 quark mixing angles
  - 1 complex phase
  - 1 Higgs mass
  - 1 coupling constant of quartic interaction of the Higgs
  - 1 QCD vacuum angle

  19 parameters

- With right handed neutrinos additional:

  - 3 neutrino masses
  - 3 lepton mixing angles
  - 1 lepton phase of mixing matrix
  - 11 Majorana terms (matrix Y_R) for the right handed neutrinos

  $19 + 18 = 37$

Matrix
$$C = \begin{pmatrix} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e_\delta & c_1 c_2 s_3 + s_2 c_3 e_\delta \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e_\delta & c_1 s_2 s_3 - c_2 c_3 e_\delta \end{pmatrix}$$

$s_i = \sin(\theta_i)$
$c_i = \cos(\theta_i)$
$e_\delta = e^{i\delta}$

Cabibbo-Kobayashi Maskawa matrix

for Leptons, Pontecorvo-Maki-Nakagawa-Sakata matrix

The product geometry

M compact smooth 4-dimensional $\underbrace{\ }_{\text{Spin}}$ manifold

$(C^\infty(M), L^2(M,S), \not{D}_M)$ spectral triple

(M, g) g: Riemannian metric
(Euclidean signature)

Levi-Civita connection

$\nabla_\mu e_a = \omega_{\mu a}^{\ \ b} e_b$ $\{e_a\}$ = basis of frame bundle

$g^{\mu\nu} = e^\mu_a e^\nu_a \eta^{ab}$ vierbein

$\partial_\mu e^a_\nu - \partial_\nu e^a_\mu - \omega_{\mu b}^{\ \ a} e^b_\nu + \omega_{\nu b}^{\ \ a} e^b_\mu = 0$

$\omega_{\mu a}^{\ \ b}$ solutions of this eq.

M spin mfld : lifting of $SO(n)$- frame bundle to 2:1-covering $Spin(n)$-bundle

Clifford algebra bundle $Cl(M)$

$Cl(M)_x = Cliff_{\mathbb{C}}(T^*_x M)$

$Cl(V)$ = TV mod rel: $uv + vu = 2\langle u,v\rangle$
(quantization of exterior algebra ΛV)

has $\mathbb{C}$-rep. dim 2^n
$\rightsquigarrow$ assoc. Spinor bundle

$Spin(n) = \{\det = 1$ elts of $Cl(n)\}$

$\gamma : C(M, Cl(M)) \to B(H)$

$\gamma(dx^\nu) = \gamma^\nu = \gamma^a e^\mu_a$ $H = L^2(M,S)$

$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2g(dx^\mu, dx^\nu) = -2g^{\mu\nu}$ Clifford alg. relations

∇^S spin connection

$$\nabla^S_\mu = \partial_\mu + \omega^S_\mu = \partial_\mu + \frac{1}{2}\omega_{\mu\, ab}\gamma^a\gamma^b$$

Dirac operator $\gamma \circ \nabla = \not{D}_M$

$$\not{D}_M = \gamma(dx^\mu)\nabla^S_\mu = \gamma^\mu(x)(\partial_\mu + \omega^S_\mu) = \gamma^a e^\mu_a(\partial_\mu + \omega^S_\mu)$$

when $\dim M = \text{even}$

<u>grading</u> $\gamma = \gamma^{n+1} := i^{n/2}\gamma^1 \cdots \gamma^n$

(γ^5 in $\dim = 4$)

$\gamma^2 = id$ $\gamma^* = \gamma$

<u>real structure</u> J given by "charge conjugation operator"

$\psi \in C^\infty(M, S)$ $J\psi = C\psi = i\gamma^2\gamma^0\bar{\psi}$

Observation: the spectral triple $(C^\infty(M), L^2(M,S)\ \not{D}_M)$ recovers the Riemannian metric

<u>Lemma</u>: $\text{dist}(x,y) = \sup\{|f(x) - f(y)| : f \in C^\infty(M) \text{ with } \|[D,f]\| \leq 1\}$

Product geometry

$(A_1, \mathcal{H}_1, D_1) \cup (A_2, \mathcal{H}_2, D_2)$

dim 4 $\left\{ \begin{array}{l} \text{dim } 0 \\ \text{KO-dim } 6 \end{array} \right.$

$$(\mathcal{A}, \mathcal{H}, D, J, \gamma) = (A_1 \otimes A_2,\ \mathcal{H}_1 \otimes \mathcal{H}_2,\ D_1 \otimes 1 + \gamma_1 \otimes D_2$$
$$J_1 \otimes J_2,\ \gamma_1 \otimes \gamma_2)$$

Note: choosing $D_1 \otimes 1 + \gamma_1 \otimes D_2$ or $D_1 \otimes \gamma_2 + 1 \otimes D_2$
same as they are unitarily equivalent
(but different in case of mfld w/ boundary:
Chamseddine / Connes)

resulting $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ KO-dim 10 mod 8
$= 2$ mod 8

Take here $(A_1, \mathcal{H}_1, D_1, J_1, \gamma_1) = (C^\infty(M), L^2(M, S), \not{\partial}_M, J_M, \gamma_M)$

and $(A_2, \mathcal{H}_2, D_2, J_2, \gamma_2) = (A_F, \mathcal{H}_F, D_F, J_F, \gamma_F)$

with D_F a choice of Dirac in the moduli space described before