Action of $\mathcal{A}_F = \mathbb{C} \oplus \mathbb{H} \oplus M_3(\mathbb{C}) \subset \mathcal{A}_{LR}$ (1
on $\mathcal{H}_F$

$$a \begin{pmatrix} u_L \\ d_L \end{pmatrix} = q^t \begin{pmatrix} u_L \\ d_L \end{pmatrix} = \begin{pmatrix} \alpha u_L - \bar\beta d_L \\ \beta u_L + \bar\alpha d_L \end{pmatrix}$$

$$a \begin{pmatrix} u_R \\ d_R \end{pmatrix} = q(\lambda)^t \begin{pmatrix} u_R \\ d_R \end{pmatrix} = \begin{pmatrix} \lambda u_R \\ \bar\lambda d_R \end{pmatrix}$$

$$a \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = q^t \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} = \begin{pmatrix} \alpha \nu_L - \bar\beta e_L \\ \beta \nu_L + \bar\alpha e_L \end{pmatrix}$$

$$a \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} = q(\lambda)^t \begin{pmatrix} \nu_R \\ e_R \end{pmatrix} = \begin{pmatrix} \lambda \nu_R \\ \bar\lambda e_R \end{pmatrix}$$

Action on $\mathcal{H}_f$

$a\bar f = \lambda \bar f$ $\quad f$ lepton

$a\bar f = m \bar f$ $\quad f$ quark

Action on $\mathcal{H}_{\bar f}$

matrix acts on color indices

Involution

$$J_F \left(\Sigma \lambda_i f_i + \Sigma \mu_j \bar f_j \right) = \Sigma \bar\lambda_i \bar f_i + \Sigma \bar\mu_j f_j$$

$$J_F(\xi, \eta) = (\bar\eta, \bar\xi)$$

$$\gamma_F f_L = f_L \quad \gamma_F f_R = -f_R \quad \gamma_F \bar f_L = -\bar f_L \quad \gamma_F \bar f_R = \bar f_R$$

Hypercharges

Unitary group

$$U(\mathcal{A}) = \{ u \in \mathcal{A} \mid uu^* = u^*u = 1 \}$$

$$SU(\mathcal{A}_F) = \{ u \in U(\mathcal{A}_F) : \det(u) = 1 \}$$

in the given representation on $\mathcal{H}_F$

Prop: (1) Modulo a finite abelian group

$$SU(\mathcal{A}_F) \simeq U(1) \times SU(2) \times SU(3)$$

(2) Adjoint action of $U(1)$ multiplies basis vectors of $\mathcal{H}_F$ by powers of λ:

	$\lvert\uparrow\rangle \otimes \mathbf{1}^o$	$\lvert\downarrow\rangle \otimes \mathbf{1}^o$	$\lvert\uparrow\rangle \otimes \mathbf{3}^o$	$\lvert\downarrow\rangle \otimes \mathbf{3}^o$
$\mathbf{2}_L$	-1	-1	$\frac{1}{3}$	$\frac{1}{3}$
$\mathbf{2}_R$	0	-2	$\frac{4}{3}$	$-\frac{2}{3}$

<u>i.e.</u> correct hypercharge assignment to the particles:

$\nu_L \quad e_L \quad u_L \quad d_L$

$\nu_R \quad e_R \quad u_R \quad d_R$

(3) Adj. action of $SU(2)$ trivial on R-particles and $= \mathbf{2}$ rep. on L-particles

(4) Adj. action of $SU(3)$ trivial on leptons and $= \mathbf{3}$ rep on quarks

Proof: $u = (\lambda, q, m) \in U(\mathcal{A}_F)$

determinant

for action on $\mathcal{H}_f$ $\quad \det(\pi(u)) = 1$

because action of a unit quaternion $\in SU(2)$

for action on $\mathcal{H}_{\bar f}$ $= \underline{12}$ copies of $\mathbb{1} \oplus \mathbb{3}$

4×3

$(\mathbb{2}_L \oplus \mathbb{2}_R) \otimes -$; overall N = number of generations

$$\det(\pi'(u)) = \lambda^{12} \det(m)^{12} = \det(u)$$

$$\Rightarrow SU(\mathcal{A}_F) = SU(2) \times \underbrace{\overset{G}{\overbrace{U(1) \times_{\mu_{12}} U(3)}}}$$

$$G = \{ (\lambda, m) \in U(1) \times U(3) \mid \lambda^{12} \det(m)^{12} = 1 \}$$

exact sequence

$$1 \longrightarrow \mu_3 \longrightarrow U(1) \times SU(3) \longrightarrow G \longrightarrow \mu_{12} \longrightarrow 1$$

$(\lambda, m) \longmapsto \lambda \det(m)$

$(\lambda, m) \longmapsto (\lambda^3, \lambda^{-1} m)$

Ker of this map $= (\lambda, \lambda\, id)$ s.t. $\lambda^3 = 1$

$\cong \mu_3$

2) Thus, up to an ambiguity given by a choice of a 3rd root of unity

the $U(1) \subset SU(\mathcal{A}_F)$ identified

w/ elements of $\mathcal{A}_F$ of the form

$$u(\lambda) = (\lambda, 1, \lambda^{-1/3}\, id) \qquad \lambda \in \mathbb{C},\ |\lambda| = 1$$

Thus, action $Ad(u(\lambda))$:

$$Ad(u) = u(u^*)^\circ = u b^\circ \quad \text{for}$$

$$b = (\bar{\lambda}, 1, \lambda^{1/3}\, id)$$

On the $2_L \otimes$ - left action of u is trivial since $q = 1$ in $u(\lambda)$

right action of b is

$\bar{\lambda}$ on $2_L \otimes 1^\circ$ and $\lambda^{1/3}\, id_3$ on $2_L \otimes 3^\circ$

$\Rightarrow$ -1 and $\frac{1}{3}$ for $|\uparrow\rangle \otimes 1^\circ$, $|\downarrow\rangle \otimes 1^\circ$ and $|\uparrow\rangle \otimes 3^\circ$, $|\downarrow\rangle \otimes 3^\circ$

On the $2_R \otimes$ - left action of u is now by λ and $\bar{\lambda}$ on $|\uparrow\rangle$ and $|\downarrow\rangle$

same as before for right action of b

$\Rightarrow$ hypercharges

$1 + -1 = 0$ on $|\uparrow\rangle \otimes 1^\circ$ $\qquad$ $1 + \frac{1}{3} = \frac{4}{3}$ on $|\uparrow\rangle \otimes 3^\circ$

$= -2$ on $|\downarrow\rangle \otimes 1^\circ$ $\qquad$ $-1 + \frac{1}{3} = -\frac{2}{3}$ on $|\downarrow\rangle \otimes 3^\circ$

This assignment of hypercharges justifies identification of basis elements of $\mathcal{H}_F$ with elementary particles

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Classification of Dirac operators:

Input additional requirements:
Look for D_F that commute with

$$a = (\lambda, \lambda, 0) \qquad \lambda \in \mathbb{C}$$

This condition corresponds physically to photon being massless

(Preferably it should be deduced not assumed)

$T : |\uparrow\rangle_R \otimes 1^\circ \to 1 \otimes |\uparrow\rangle_R$

Notation:

$$D(Y) = \begin{pmatrix} S & T^* \\ T & \bar{S} \end{pmatrix} \quad \text{with} \quad T\nu_R = Y_R \bar{\nu}_R$$

Y_R $N \times N$-matrix

$$S = S_\ell \oplus (S_q \otimes id_3) \leftarrow \begin{cases} \ell = \text{lepton sector} \\ q = \text{quark sector} \end{cases}$$

$$S_\ell = \begin{pmatrix} 0 & 0 & Y_\nu^* & 0 \\ 0 & 0 & 0 & Y_e^* \\ Y_\nu & 0 & 0 & 0 \\ 0 & Y_e & 0 & 0 \end{pmatrix}$$

Y_α, Y_α^* are $N \times N$-matrices

$$S_q = \begin{pmatrix} 0 & 0 & Y_u^* & 0 \\ 0 & 0 & 0 & Y_d^* \\ Y_u & 0 & 0 & 0 \\ 0 & Y_d & 0 & 0 \end{pmatrix}$$

<u>Lemma</u> $P = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix}$ $P : \mathcal{H}_F = \mathcal{H}_f \oplus \mathcal{H}_{\bar{f}} \to \mathcal{H}_F$

<u>P belongs to commutant of $\mathcal{A}_F$ iff</u>

- $P_{11} : \mathcal{H}_f \to \mathcal{H}_f$ block diagonal

  blocks $\begin{cases} M_{12}(\mathbb{C}) & a = (\lambda, q, m) \text{ acts by } \lambda \\ M_{12}(\mathbb{C}) & \text{acts by } \bar{\lambda} \\ id_2 \otimes M_{12}(\mathbb{C}) & \text{acts by } q \end{cases}$

- $P_{12} : \mathcal{H}_{\bar{f}} \to \mathcal{H}_f$

  support in $\mathbb{1} \otimes 2^\circ_L \oplus \mathbb{1} \otimes 2^\circ_R$

  range in $|\uparrow\rangle_R \otimes \mathbb{1}^\circ \oplus |\uparrow\rangle_R \otimes 3^\circ$

- $P_{21} : \mathcal{H}_f \to \mathcal{H}_{\bar{f}}$ support in $|\uparrow\rangle_R \otimes \mathbb{1}^\circ \oplus |\uparrow\rangle_R \otimes 3^\circ$

  range in $\mathbb{1} \otimes 2^\circ_L \oplus \mathbb{1} \otimes 2^\circ_R$

- $P_{22} : \mathcal{H}_{\bar{f}} \to \mathcal{H}_{\bar{f}}$ of the form

$$P_{22} = P_\ell \oplus (id_3 \otimes P_q)$$

$$P_\ell : \mathbb{1} \otimes (2^\circ_L \oplus 2^\circ_R) \longrightarrow \mathbb{1} \otimes (2^\circ_L \oplus 2^\circ_R)$$

$$id_3 \otimes P_q : 3 \otimes (2^\circ_L \oplus 2^\circ_R) \longrightarrow 3 \otimes (2^\circ_L \oplus 2^\circ_R)$$

<u>Proof</u>: action of $\mathcal{A}_F$: $\begin{pmatrix} \pi(\lambda,q,m) & 0 \\ 0 & \pi'(\lambda,q,m) \end{pmatrix}$

on $\mathcal{H}_f$ w/ basis $|\uparrow\rangle, |\downarrow\rangle$ L,R of 8 part:

$$\pi(\lambda,q,m) = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & \bar\lambda & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & -\bar\beta & \bar\alpha \end{pmatrix} \otimes id_{12}$$

$(\mathbb{1}^\circ \oplus \mathbb{3}^\circ) \times \cancel{N}=3$ generations

if $P \in \mathcal{A}_F'$ then

P_{11} commutes with $\pi(\lambda,q,m)$

$\Rightarrow$ block diagonal with blocks for the subspaces where $\pi(\lambda,q,m)$ acts by λ, $\bar\lambda$, and q

$\Rightarrow M_{12}(\mathbb{C})$, $M_{12}(\mathbb{C})$, $id_2 \otimes M_{12}(\mathbb{C})$ blocks

P_{22} : commutes with action of $\pi'(\lambda,q,m)$ on $\mathcal{H}_{\bar f}$

given by multipl. by λ or by m (on leptons and on quarks)

$\Rightarrow P_{22} = P_\ell \oplus (id_3 \otimes P_q)$

Off-diagonal terms P_{12} and P_{21} intertwine actions $\pi(\lambda,q,m)$ and $\pi'(\lambda,q,m)$

but action of q and m are disjoint while action of λ occurs in both $\mathcal{H}_f$ and $\mathcal{H}_{\bar f}$

$\Rightarrow$ support and range have to intertwine these two pieces where action ...

<u>Theorem</u> D a Dirac operator for $(\mathcal{A}_F, \mathcal{H}_F, J_F, \gamma_F)$ satisfying

$$[D, \rho(\lambda, \lambda, 0)] = 0$$

- Then $\exists$ matrices Y_α $\alpha = \nu, e, u, d$ $N \times N$-matrices and symmetric $N \times N$-matrix Y_R such that $D = D(Y)$
- All $D(Y)$ are Dirac operators
- Two $D(Y)$ $D(Y')$ are conjugate by a unitary operator commuting with $\mathcal{A}_F, \gamma_F, J_F$ iff $\exists$ unitary matrices V_j, W_j s.t.

$$Y_e' = V_1 Y_e V_3^* \qquad Y_\nu' = V_2 Y_\nu V_3^*$$

$$Y_d' = W_1 Y_d W_3^* \qquad Y_u' = W_2 Y_u W_3^*$$

$$Y_R' = V_2 Y_R \overline{V_2}^*$$

This result in particular implies that
for $\mathcal{A} = \mathcal{A}_{LR}$ $\not\exists$ D with $D_{off} \neq 0$
and order-one condition

but maximal subalgebra satisfying this:

Prop: $\exists!$ (up to automorphisms of $\mathcal{A}_{LR}$)
involutive subalg $\mathcal{A} \subset \mathcal{A}_{LR}$ of max dim
s.t. $\exists D$ for $(\mathcal{A}, \mathcal{H}_F, D)$ with $D_{off} \neq 0$ and
$[[D, a], b^\circ] = 0 \quad \forall a, b \in \mathcal{A}$

$$\mathcal{A}_F = \{(\lambda, q_L, \lambda, m) \mid \lambda \in \mathbb{C}, q_L \in \mathbb{H}, m \in M_3(\mathbb{C})\}$$

i.e. (λ, q_L, q_R, m) where $q_R = q_R(\lambda) = \begin{pmatrix} \lambda & 0 \\ 0 & \bar{\lambda} \end{pmatrix}$

Proof:
Using previous result if $\exists D_{off} \neq 0$ then
$A \subset \mathcal{A}(T)$ support of T in range of e
range in range of e'

2 cases: π rep. of $\mathbb{H}$ on $\mathbb{C}^2$ & π' rep of $\begin{cases} \mathbb{C} & \text{on } \mathbb{C} \\ M_3(\mathbb{C}) & \text{on } \mathbb{C}^3 \end{cases}$

First case: $\lambda T \xi = T q \xi \quad \forall \xi \in E$ $\quad E =$ support T 1-dimensional

$\Rightarrow \lambda \xi - q \xi = 0 \quad \Rightarrow \mathcal{A}(T) =$ pullback of $\{(\lambda, q) \in \mathbb{C} \oplus \mathbb{H} \mid q\xi = \lambda \xi \ \forall \xi \in E\}$ under proj. $\mathcal{A}_{LR} \to \mathbb{C} \oplus \mathbb{H}$

i.e. corresp. to an embedding of $\mathbb{C}$ in $\mathbb{H}$

unique up to inner automorphisms of $\mathbb{H}$

(determined by mapping $i \in \mathbb{C}$ to an $x \in \mathbb{H}$ w/ $x^2 = -1$ and all these are conjugate)

$\Rightarrow$ determines a subalgebra $\mathcal{A}_F \subset \mathcal{A}_{LR}$ of codim $= 4$

Second case: π' rep of $M_3(\mathbb{C})$ on $\mathbb{C}^3$:

T has range $R(T)$ at most 2-dimensional (range of e')

$R(T)$ invariant under $\pi'(\mathcal{A})$ and $R(T)^\perp$ also

so $\pi'(M_3(\mathbb{C}) \cap \mathcal{A})$ contained in $\begin{pmatrix} \boxed{2\times 2} & \\ 0 \ 0 & \boxed{1} \end{pmatrix}$ matrices

real codimension 8 in $M_3(\mathbb{C})$

So this second case leads to a smaller dimensional algebra.

What Dirac operators for

$(\mathcal{A}_F, \mathcal{H}_F, J_F, \gamma_F)$

D_F classification

Moduli space of possible Dirac operators