Some basic notions of metric noncommutative geometry

Def: A spectral triple $(\mathcal{A}, \mathcal{H}, D)$

- $\mathcal{H}$ = Hilbert space
- $\mathcal{A}$ = unital involutive algebra

  $a \mapsto a^*$ conjugate-linear $(\lambda a)^* = \bar{\lambda} a^*$

  $(a^*)^* = a$ and $(ab)^* = b^* a^*$

- Representation $\rho: \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ (as $*$-algebras morphisms)

  i.e. $\rho(a^*) = \rho(a)^*$ ← adjoint operator in $\mathcal{B}(\mathcal{H})$

$\mathcal{B}(\mathcal{H})$ = bounded operators on $\mathcal{H}$

Banach algebra w/ norm

$$\|T\| = \sup_{\|x\|=1} \|Tx\| = \sup_{x \neq 0} \frac{\|Tx\|}{\|x\|}$$

Completion in this norm of $\rho(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$
$\Rightarrow$ C^*-algebra

- D a self adjoint $(D^* = D)$ operator (unbounded) on $\mathcal{H}$ densely defined $\mathrm{Dom}(D) \subset \mathcal{H}$

  such that D has compact resolvent

  $(D-\lambda)^{-1}$ compact for $\lambda \notin \mathrm{Sp}(D)$

  equivalently $(1+D^2)^{-1/2}$ compact operator

  (limit of finite rank operators)

  and with compatibility condition with the algebra

  Commutators $[D, \rho(a)]$ are bounded operators $\in \mathcal{B}(\mathcal{H})$ for all $a \in \mathcal{A}$ (not in the C^*-alg.)

Def: $(A, \mathcal{H}, D)$ is even if

$\mathcal{H}$ is $\mathbb{Z}/2\mathbb{Z}$-graded $\quad \mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$

with $\quad \gamma$ = chirality projection = grading operator

$$\gamma(x^+) = x^+ \quad \gamma(x^-) = -x^-$$

such that

$$[\gamma, \rho(a)] = 0 \quad \forall a \in \mathcal{A} \qquad \text{commutation}$$

$$\gamma D + D\gamma = 0 \qquad \text{anticommutation}$$

(i.e. $\quad D = \begin{pmatrix} 0 & D^+ \\ D^- & 0 \end{pmatrix}$)

$(\mathcal{A}, \mathcal{H}, D)$ is odd if it does not admit such γ
(ie if it is not even)

Def: $(\mathcal{A}, \mathcal{H}, D)$ is finitely summable

if the characteristic values of $(D-\lambda)^{-1} = R_\lambda(D)$
resolvent

satisfy an estimate

$$\lambda_n = O(n^{-\alpha})$$

for some $\alpha > 0$

characteristic values $\lambda_n(T)$
= eigenvalues of $(T^*T)^{1/2}$

Know discrete spectrum accumulating at zero because of compact resolvent hypothesis

Not always the case that spectral triples are finitely summable

Finite summability equiv. to (suppose $Ker(D)=0$)

$$\mathrm{Tr}(|D|^{-r}) < \infty \quad \text{for some } r>0$$

If $\mathrm{Tr}(|D|^{-r})<\infty$ for all $r \geq r_0 \Rightarrow r_0$-summable

" " " $r > r_0 \Rightarrow r_0^+$-summable

Not always satisfied: e.g.

T = tree all vertices valence $q \in \mathbb{N}$ (infinite tree)

$V(T)$ = set of vertices, $v_0 \in V(T)$ = base pt

$\mathcal{H} = \ell^2(V(T))$ o.n. basis $\{\varepsilon_v\}_{v\in V(T)}$

$D\, \varepsilon_v := d(v,v_0)\, \varepsilon_v$

$Sp(D) = \{0,1,2,\dots\} = \mathbb{N}$

multiplicity of eigenspaces

E_λ = eigenspace of eigenvalue λ

$$\dim E_n = \#\{v \in V(T) : d(v_0,v) = n\} = q(q-1)^{n-1} \sim q^n$$

So $\mathrm{Tr}(|D|^{-\alpha}) \sim \sum_n n^{-\alpha} q^n$ not convergent

But it has property that $\mathrm{Tr}(e^{-tD^2}) < \infty$ for $t>0$

$$\mathrm{Tr}(e^{-tD^2}) \sim \sum_n e^{-t\, n^{2\alpha} q^n}$$

these triples with $\mathrm{Tr}(e^{-tD^2})<\infty$ are called θ-summable

In case of a finitely summable $(\mathcal{A}, \mathcal{H}, D)$

if $\mathrm{Tr}(|D|^{-\alpha}) < \infty$ for all $\alpha > n$

Say that the <u>metric dimension</u> of $(\mathcal{A}, \mathcal{H}, D)$ is n

Various different notions of dimension for spectral triples

- metric dimension (a positive integer or a real number)
- KO-dimension (a positive integer modulo 8)
- dimension spectrum (a subset of $\mathbb{C}$)

<u>Def:</u> A real structure of KO-dimension $n \in \mathbb{Z}/8\mathbb{Z}$ on $(\mathcal{A}, \mathcal{H}, D)$ is a conjugate-linear isometry

$J : \mathcal{H} \to \mathcal{H}$ such that

$$\begin{cases} J^2 = \varepsilon \\ JD = \varepsilon' DJ \\ J\gamma = \varepsilon'' \gamma J \end{cases}$$

$\varepsilon, \varepsilon', \varepsilon''$ either ± 1 (commuting or anticommuting)

	0	1	2	3	4	5	6	7
ε	1	1	-1	-1	-1	-1	1	1
ε'	1	-1	1	1	1	-1	1	1
ε''	1		-1		1		-1	

Notice that this set of three numbers $\varepsilon, \varepsilon', \varepsilon'' \in \{\pm 1\}$ fixes uniquely n mod 8

Properties of a real structure J (Def)

(1) $[b, a^\circ] = 0 \quad \forall b \in \mathcal{A} \quad \forall a \in \mathcal{A}$

where $a^\circ = J a^* J^{-1}$

<u>Notice</u>: if think of $\mathcal{H}$ as a (right) $\mathcal{A}$ module
then $a \mapsto a^\circ$ defines another (left)
$\mathcal{A}$-action on $\mathcal{H}$ making it into a bimodule

the commutation $[b, a^\circ]$ ensures that
these can be regarded as a right and left action

(2) order one condition

$$[[D, a], b^\circ] = 0 \quad \text{for all } a \in \mathcal{A}, \ b \in \mathcal{A}$$

(similar $[[D, a^\circ], b]$ follows from (1) & (2))

Note: $\not{\partial}_x$ order one
$[[\not{\partial}_x, a], b] = -i[\gamma(da), b] = 0$
$J b^\circ J^{-1} = b$
but not for order 2
Laplacian $[[\Delta, a], b] = 2g^{-1}(da, db) \neq 0$

$(\mathcal{A}, H, D)$ = real spectral triple if $\exists J$ with these properties

The real part of a real spectral triple

$(\mathcal{A}, \mathcal{H}, D, J)$ spectral triple with a real structure

then

(1) $$\mathcal{A}_J := \{ a \in \mathcal{A} : aJ = Ja \}$$

defines involutive commutative real subalgebra of $Z(\mathcal{A})$ = center of $\mathcal{A}$

(2) $(\mathcal{A}_J, \mathcal{H}, D)$ real spectral triple w/ J

(3) All $a \in \mathcal{A}_J$ commute with algebra generated by $\sum_i a_i [D, b_i]$ $a_i, b_i \in \mathcal{A}$

Pf: J isometry so $(JaJ^{-1})^* = Ja^* J^{-1}$

So if $JaJ^{-1} = a$ (i.e. $a \in \mathcal{A}_J$)

then $Ja^* J^{-1} = a^*$ $\Rightarrow a^* \in \mathcal{A}_J$

$\Rightarrow \mathcal{A}_J$ $*$-subalgebra of $\mathcal{A}$

real subalgebra

$$J \lambda a J^{-1} = \bar{\lambda} \, JaJ^{-1} = \bar{\lambda} a$$

$\Rightarrow \lambda a \in \mathcal{A}_J$ for $\lambda \in \mathbb{R}$

Contained in the center:

$\forall a \in \mathcal{A}_J \, \forall b \in \mathcal{A}$ since $[b, a^\circ] = 0$ $\quad a^\circ = Ja^* J^{-1} = a^*$

$[b, a^*] = 0 \Rightarrow \mathcal{A}_J \subset Z(\mathcal{A})$

Dimension spectrum of a spectral triple
$(\mathcal{A}, \mathcal{H}, D)$ finitely summable
(of metric dim m)

$\delta(T) = [\,|D|, T\,]$ derivation

$\mathcal{B}$ = algebra generated by $\delta^k(a) = \underbrace{[\,|D|, [\,|D|, \ldots [\,|D|}_{k\text{-times}}, a]] \ldots]$
and $\delta^k([D,a])$ for $a \in \mathcal{A}$

<u>Zeta functions</u>: for $b \in \mathcal{B}$

$$\zeta_b(s) = \mathrm{Tr}(b|D|^{-s}) \qquad \mathrm{Re}(s) > m$$

$$\mathrm{Dim\,Sp}(\mathcal{A}, \mathcal{H}, D) := \{ z \in \mathbb{C} : \zeta_b(z) \text{ pole} \}$$
$\mathrm{Re}(z) \geq 0$

Simple dimension spectrum when all simple poles

<u>Connes-Moscovici local index formula</u> (we'll see later)

Prototype example of spectral triple

$$(\mathcal{A}, H, D) = (C^\infty(X),\ L^2(X,S),\ \partial\!\!\!/_X)$$

X = compact smooth spin manifold

$H = L^2(X,S)$ square-integrable sections of the spinor bundle

$\partial\!\!\!/_X$ = Dirac operator

Inner fluctuations of the Dirac operator

$\mathcal{A}$ unitary involutive algebra

$\mathcal{B}$ " " " is Morita equivalent to $\mathcal{A}$

iff $\exists\ \mathcal{E}$ (right) $\mathcal{A}$-module

finitely generated projective

such that

$$\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$$

Not clear from this that it is an equivalence relation fin. gen. projective $\mathcal{E} = p\mathcal{A}^m$

and $\mathcal{B} = p\mathcal{M}_m(\mathcal{A})p$ with $p = p^*$ $p^2 = p$

- $\mathcal{A}, \mathcal{B}$ Morita equivalent if $\exists$ $\mathcal{B}$-$\mathcal{A}$ bimodule $\mathcal{E}$ and an $\mathcal{A}$-$\mathcal{B}$ bimodule $\mathcal{F}$ such that $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{F} \simeq \mathcal{B}$ and $\mathcal{F} \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A}$

(viewed respectively, as a $\mathcal{B}$-$\mathcal{B}$ and an $\mathcal{A}$-$\mathcal{A}$ bimodule)

from this clear that it is an equivalence relation

previous formulation:

row vectors

$$\mathrm{End}_{\mathcal{A}}(\mathcal{E}) \cong \mathcal{E} \otimes_{\mathcal{A}} \mathcal{E}^t \qquad \mathcal{E}^t = \mathcal{A}^m p$$

column vectors

So partic. case: if $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\mathcal{E})$ $\Rightarrow$ equiv. through $\mathcal{E}$ and $\mathcal{E}^t$ (and $\mathcal{E}^t \otimes_{\mathcal{B}} \mathcal{E} \simeq \mathcal{A}$)

Conversely: if $\mathcal{B} = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{F}$ then can also show $\mathcal{B} = \mathrm{End}_{\mathcal{A}}(\tilde{\mathcal{E}})$ for some $\tilde{\mathcal{E}}$

<u>Representations</u> $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$

then $\rho' : \mathcal{B} \to \mathcal{B}(\mathcal{H}')$ $\quad \mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$

conversely given $\rho' : \mathcal{B} \to \mathcal{B}(\mathcal{H}')$

get $\rho : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ for $\mathcal{H} = \mathcal{F} \otimes_{\mathcal{B}} \mathcal{H}'$

Can transfer back and forth representations

When one considers norm completions so $\mathcal{A}, \mathcal{B}$ are C^*-algebras then also on the bimodules $\mathcal{E}, \mathcal{F}$ extra structure:

Hilbert C^*-modules

Inner fluctuations:

$(\mathcal{A}, \mathcal{H}, D)$ and $\mathcal{B}$ morita equivalent to $\mathcal{A}$ through $\mathcal{E}, \mathcal{F}$

$(\mathcal{B}, \mathcal{H}' = \mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}, ?)$

<u>Notice</u> $D'(\xi \otimes \eta) = \xi \otimes D\eta$ does not work:

taking $\mathcal{E} \otimes_{\mathcal{A}} \mathcal{H}$ tensor product over $\mathcal{A}$

So for $a \in \mathcal{A}$ $\quad \xi a \otimes \eta = \xi \otimes a\eta$

but $D'(\xi a \otimes \eta) = \xi \otimes D(a\eta) \neq \xi \otimes a D\eta$

because $[D, a] \neq 0$

Need to correct this by something

A connection on $\mathcal{E}$

$$\nabla : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1_D$$

$$\Omega^1_D = \left\{ \sum_j a_j [D, b_j] \; : \; a_j, b_j \in \mathcal{A} \right\}$$

a bimodule over $\mathcal{A}$ $\qquad \omega = \sum_j a_j [D, b_j]$

$$a\omega = \sum_j a a_j [D, b_j]$$

$$\omega a = \sum_j a_j [D, b_j] a = \sum_j a_j (D b_j a - b_j D a)$$

$$\sum_j a_j (D b_j a - b_j a D + b_j a D - b_j D a)$$

$$\sum_j a_j [D, b_j a] + \sum_j a_j b_j [D, a] \in \Omega^1_D$$

∇ should satisfy a Leibnitz rule

$$\nabla(\xi a) = (\nabla \xi) a + \xi \otimes da \qquad da = [D, a]$$

Then set $D'(\xi \otimes \eta) := \xi \otimes D\eta + (\nabla \xi)\eta$

then have the correct behavior:

$$D'(\xi \otimes a\eta) = \xi \otimes D(a\eta) + (\nabla \xi) a\eta \; =$$

$$\xi \otimes a D\eta + (\nabla \xi) a\eta + \xi \otimes D(a\eta) - \xi \otimes a D\eta$$

$$\parallel$$

$$D'(\xi a \otimes \eta) = \xi a \otimes D\eta + \nabla(\xi a)\eta = \xi a \otimes D\eta + (\nabla \xi) a\eta + \xi \otimes [D, a]\eta$$

In particular have self-Morita equivalences of $\mathcal{A}$

(Note: if $\mathcal{A}, \mathcal{B}$ commutative $\Rightarrow$ Morita equiv $\equiv$ isomorphism ~~of~~

but if non-commutative $\exists$ interesting Morita equivalences)

Self Morita equivalences of $\mathcal{A}$ with $\mathcal{E} = \mathcal{A}$

so same $\mathcal{H} = \mathcal{A} \otimes_{\mathcal{A}} \mathcal{H}$

and only the Dirac operator changes

$$D \mapsto D' = D + A$$

$A = A^*$ self-adjoint element of Ω^1_D

$$A = \sum_j a_j [D, b_j]$$

"inner fluctuations of D"

When $(\mathcal{A}, \mathcal{H}, D)$ has also a real structure J

define left module structure on $\mathcal{H}$

(or right if see action of $\mathcal{A}$ as left mod str.)

by $b\xi = \rho(b^0)\xi$ (or $\xi b = \rho(b^0)\xi$)

So that $\mathcal{H}$ becomes a bimodule

Then have adjoint action

$$\mathcal{H} \ni \xi \mapsto Ad(u)(\xi) = u \xi u^*$$

$u \in \mathcal{A}$ unitary $uu^* = u^* u = 1$

Then inner fluctuations of D by

$$D \mapsto D + A + \varepsilon' JAJ^{-1}$$

So that it still has properties of D w/respect to real structure: $JD = \varepsilon' DJ$

"Gauge transformations" of Dirac operator

$u \in \mathcal{A} \quad u^* u = u u^* = 1$

$A = A^* \in \Omega^1_D$

then
$$Ad(u) \circ (D + A + \varepsilon' JAJ^{-1}) \circ Ad(u^*)$$
$$= D + \gamma_u(A) + \varepsilon' J \gamma_u(A) J^{-1}$$

for $\gamma_u(A) = u[D, u^*] + u A u^*$

Direct computation writing explicitly
$$Ad(u) = u J u J^{-1}$$

Also inner fluctuations of inner fluctuations are still inner fluctuations

$D \mapsto D + A = D'$ or $D \mapsto D + A + \varepsilon' JAJ^{-1} = D'$

and then $D' \mapsto D' + B = D''$ or $D' \mapsto D' + B + \varepsilon' JBJ^{-1} = B$

then $D' + B = D + A + B$ is also inner fluctuation

i.e. $\Omega^1_{D'} \subset \Omega^1_D$

follows from Ω^1_D bimodule over $\mathcal{A}$ and $[a, Jb^* J^{-1}] = 0$ and $[[D, a], Jb^* J^{-1}] = 0$

Simplest kind of spectral triples

- $\mathcal{A}$ = finite dimensional algebra
- $\mathcal{H}$ = finite dim vector space
- D = self-adjoint matrix
- J = antilinear map of $\mathcal{H}$

$$J^2 = \varepsilon \qquad JD = \varepsilon' DJ \qquad J\gamma = \varepsilon'' \gamma J$$

- $[[D, a], b^\circ] = 0 \qquad \forall a \in \mathcal{A} \quad \forall b \in \mathcal{A}$

$$[a, b^\circ] = 0 \qquad b^\circ = J b^* J^{-1}$$

The other conditions are trivially satisfied in finite dimensional case

(Chamseddine-Connes-M. : ATMP 2007):

Start with "left-right symmetric algebra"

$$\mathcal{A}_{LR} := \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

$\mathbb{H}$ = quaternions (real algebra)

$$q = \alpha + \beta j \qquad \alpha, \beta \in \mathbb{C}$$

$$= x + yi + zj + wk \qquad ij = k \quad i^2 = j^2 = k^2 = -1$$

basis $\{1, i\sigma^\alpha\}_{\alpha=1,2,3}$ corresp. to the three imaginary units

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

involution on

$$\mathcal{A}_{LR} = \mathbb{C} \oplus \mathbb{H}_L \oplus \mathbb{H}_R \oplus M_3(\mathbb{C})$$

$$(\lambda, q_L, q_R, m) \longmapsto (\bar{\lambda}, \bar{q}_L, \bar{q}_R, m^*)$$

where $q = x + yi + zj + wk \mapsto \bar{q} = x - yi - zj - wk$
$x, y, z, w \in \mathbb{R}$

<u>Notation</u>: $\mathbb{C} \oplus M_3(\mathbb{C})$ "integer spin part" of $\mathcal{A}_{LR}$
$\mathbb{H}_L \oplus \mathbb{H}_R$ "half ~~integer~~ spin part" of $\mathcal{A}_{LR}$

General fact as before: if $\mathcal{M}$ is a bimodule over $\mathcal{A}$ then for $u \in \mathcal{A}$ $u^*u = uu^* = 1$ have

$$Ad(u)(\xi) = u \xi u^* \quad \text{adjoint action}$$

In the particular case of $\mathcal{A}_{LR}$
consider the element $s = (1, -1, -1, 1)$

<u>Def</u>: A bimodule $\mathcal{M}$ over $\mathcal{A}_{LR}$ is <u>odd</u> if $Ad(s) = -1$

If consider odd bimodules then can pass from the algebra $\mathcal{A}_{LR}$ to a complex algebra

$\mathcal{A}_{LR}^{o}$ = opposite algebra of $\mathcal{A}_{LR}$

$$\mathcal{B} = (\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{o})_p = p(\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{o})p$$
$$= \{ pap : a \in \mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{o} \}$$

compressed by projector p

$$p = p^* \quad p^2 = p$$

$$p = \frac{1}{2}(1 - s \otimes s^{o})$$

<u>Note</u>: $$\mathcal{B} = (\mathbb{H}_L \oplus \mathbb{H}_R) \otimes_{\mathbb{R}} (\mathbb{C} \oplus M_3(\mathbb{C}))^{o}$$
$$\oplus \ (\mathbb{C} \oplus M_3(\mathbb{C})) \otimes_{\mathbb{R}} (\mathbb{H}_L \oplus \mathbb{H}_R)^{o}$$

(the other two pieces of $\mathcal{A}_{LR} \otimes_{\mathbb{R}} \mathcal{A}_{LR}^{o}$ killed by p)

but $\mathbb{H} \otimes_{\mathbb{R}} \mathbb{C} = M_2(\mathbb{C})$ and $\mathbb{H} \otimes_{\mathbb{R}} M_3(\mathbb{C}) = M_6(\mathbb{C})$

so just have

$$\mathcal{B} = \bigoplus_{1}^{4} (M_2(\mathbb{C}) \oplus M_6(\mathbb{C}))$$ complex algebra

Odd bimodule $\mathcal{M}$ for $\mathcal{A}_{LR}$ $\Longleftrightarrow$ representation $\mathcal{M}$ of $\mathcal{B}$

<u>Notation</u> $\mathbf{3}$ = 3-dim standard rep. of $M_3(\mathbb{C})$ on $\mathbb{C}^3$

$\mathbb{1}$ = 1-dim rep. of $\mathbb{C}$ on $\mathbb{C}$

$\mathbf{2}$ = 2-dim rep. of $\mathbb{H}$ on $\mathbb{C}^2$ $\quad q = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$