Quantum Statistical Mechanics

General setting

$\mathcal{A}$ = algebra of observables $/\mathbb{C}$

C^*-algebra : involutive $*$ - anti-isomorphism
Banach algebra : complete in norm $\|\cdot\|$

$\|ab\| \leq \|a\| \cdot \|b\|$ $\qquad \|a^*a\| = \|a\|^2$

Gelfand-Naimark : $\mathcal{A} \hookrightarrow \mathcal{B}(\mathcal{H})$ bounded operators on Hilbert space (separable)

(if $\mathcal{A}$ is commutative $\iff$ $\mathcal{A} \cong C_0(X)$ X loc. comp. Hausdorff space)

in general $\mathcal{A}$ NC

Note: difference w/ typical QM setting: observables only self-adjoint $a^* = a$ elements
here all $\mathcal{A}$ is observables

State $\varphi : \mathcal{A} \to \mathbb{C}$ continuous linear functional

(not alg. homom.) with $\varphi(1) = 1$ normalization
$\varphi(a^*a) \geq 0$ positivity

(If $\mathcal{A}$ commutative $= C_0(X)$ a state is a probability measure on X)

Extremal state : φ s.t. not convex combination (non-trivial) of other states

Note: space $\mathcal{M}(\mathcal{A})$ of states on $\mathcal{A}$ is convex $\lambda\varphi_1 + (1-\lambda)\varphi_2 = \varphi$ $\quad \lambda \in [0,1]$ state

Time evolution on $\mathcal{A}$ $Aut(\mathcal{A})$ group of automorphisms of

$\sigma : \mathbb{R} \to Aut(\mathcal{A})$ continuous 1-param. semigroup of automorphisms

$\sigma_{t+s}(a) = \sigma_t(\sigma_s(a))$ $\sigma_t(ab) = \sigma_t(a)\,\sigma_t(b)$ ...

Representations: $\quad \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$

on a Hilbert space

Covariant representation: for $(\mathcal{A}, \sigma)$ dyn. system
$\exists H$ operator (densely defined unbounded self-adj.) on $\mathcal{H}$

$$\pi(\sigma_t(a)) = e^{itH}\, \pi(a)\, e^{-itH}$$

Hamiltonian

i.e. infinitesimal generator H implementing time evol. σ in repres. π $H = H_\pi$

Equilibrium state: $\quad \varphi : \mathcal{A} \to \mathbb{C}$ s.t.

$$\varphi(\sigma_t(a)) = \varphi(a) \quad \forall t \in \mathbb{R}$$

Gibbs equilibrium states: $\quad \beta = \frac{1}{kT}$ inverse temperature
k = Boltzmann constant
(units: set $k = 1$)

suppose H Hamiltonian for $(\mathcal{A}, \sigma, \pi)$ on $\mathcal{H}$ Hilbert space s.t.

$Tr(e^{-\beta H}) < \infty$ for suff. large $\beta > 0$ $\varphi_\beta(a) = \dfrac{Tr(\pi(a)\, e^{-\beta H})}{Tr(e^{-\beta H})}$

equil. state

$$\mathrm{Tr}(\pi(\sigma_t(a))e^{-\beta H}) = \mathrm{Tr}(e^{itH}\pi(a)e^{-itH}e^{-\beta H})$$
$$= \mathrm{Tr}(\pi(a)e^{-itH}e^{-\beta H}e^{itH}) = \mathrm{Tr}(\pi(a)e^{-\beta H})$$

Gibbs equilibrium states

Note: existence of Gibbs states depends on condition

$$Z(\beta) = \mathrm{Tr}(e^{-\beta H}) < \infty$$

partition function

measuring microstates of system (energy levels: spectrum of H) weighted by inverse T

$$\sum_{\lambda \in \mathrm{Spec}(H)} e^{-\beta\lambda} = \sum_{\lambda} m_\lambda e^{-\beta\lambda}$$

spectral multiplicities

At $T \to 0$ ($\beta \to \infty$) probability distribution centered at ground state $H = 0$ (frozen on ground state)

At T large (β small) larger contribution also from higher energy states (become accessible at high temperature)

What happens when $\mathrm{Tr}(e^{-\beta H}) = \infty$ not trace class ?

Generalization of Gibbs states: KMS_β-states

$\varphi \in KMS_\beta \qquad 0 < \beta < \infty \qquad$ if

$\forall a, b \in \mathcal{A} \quad \exists$ holom. function $F_{a,b}(z)$

- holom. on strip $I_\beta = \{ z \in \mathbb{C} \,,\, 0 < Im(z) < \beta \}$
- continuous on the boundary ∂I_β
- $F_{a,b}(t) = \varphi(a \sigma_t(b)) \qquad F_{a,b}(t + i\beta) = \varphi(\sigma_t(b) a)$

State that fails to be a trace by an amount controlled by interpolation by a holomorphic function

- An equivalent reformulation of KMS condition:

  $\exists\, \mathcal{A}_{an} \subset \mathcal{A}$ dense involutive subalgebra (of "analytic elements") s.t.
  invariant under σ

$$\varphi(a \sigma_{i\beta}(b)) = \varphi(ba)$$

- Relation to previous definition: for $a \in \mathcal{A}$ $b \in \mathcal{A}_{an}$

$$F_{a,b}(z) = \varphi(a \sigma_z(b))$$

- $\mathcal{A}_{an}$ consists of elements $a \in \mathcal{A}$ for which $\sigma_t(a)$ extends analytically to $\sigma_z(a)$ for $z \in I_\beta$

- KMS_β states are a generalization of Gibbs states: (5)

$$\varphi_\beta(a) = \frac{Tr(\pi(a)\, e^{-\beta H})}{Tr(e^{-\beta H})}$$

$$\begin{aligned}\varphi_\beta(ba) &= Z(\beta)^{-1}\, Tr(\pi(b)\pi(a)\, e^{-\beta H}) \\ &= Z(\beta)^{-1}\, Tr(\pi(a)\, e^{-\beta H}\pi(b) e^{\beta H}\, e^{-\beta H}) \\ &= Z(\beta)^{-1}\, Tr(\pi(a)\pi(\sigma_{i\beta}(b))) = \varphi_\beta(a\,\sigma_{i\beta}(b))\end{aligned}$$

Symmetries $\quad G \subset Aut(\mathcal{A})$ subgroup of automorphisms

s.t. $\gamma\,\sigma_t(a) = \sigma_t(\gamma a) \quad \forall a \in \mathcal{A}$

$\gamma \circ \sigma_t = \sigma_t \circ \gamma$ compatible w/ time evolution

Inner automorphisms $\quad u \in \mathcal{U}(\mathcal{A})$ unitary $(u^* u = u u^* = 1)$

s.t. $\sigma_t(u) = u$

acting on $\mathcal{A}$ by adj. $\quad a \longmapsto u a u^*$

Action of Automorphisms-symmetries / Unitaries on

$\mathcal{E}_\beta \subset KMS_\beta$ extremal KMS-states

by pullback: $\quad \varphi \longmapsto \gamma^*\varphi \qquad \gamma^*\varphi(a) := \varphi(\gamma a)$

$\varphi(uau^*) = \varphi(au^*\sigma_{i\beta}(u)) =$

$\varphi(a\, uu^*) = \varphi(a)$

Note: u fixed by $\sigma_t \Rightarrow u \in \mathcal{A}_{an}$ trivial analytic extension of $\sigma_t(u) = u = \sigma_z(u)$

Inner autom-symm. act trivially on KMS_β states

Other symmetries : Endomorphisms $\rho \in Hom(\mathcal{A}, \mathcal{A})$

$\rho : \mathcal{A} \to \mathcal{A}$ $\quad \rho(1) = e$ idempotent (projector) in $\mathcal{A}$

$e^* = e = e^2$

(Note : more general endom. than unital ring homom.) $\quad \rho \circ \sigma_t = \sigma_t \circ \rho$

In order to have induced action on $\varphi \in \mathcal{E}_\beta \subset KMS_\beta$ need to have $\varphi(e) \neq 0$

then can define pullback : $\rho^*\varphi(a) = \dfrac{\varphi(\rho(a))}{\varphi(e)}$

(to preserve normalization condition)

Inner endomorphism symmetries :

u = isometry i.e. $u^*u = 1$ but $uu^* = e$ projector

$\sigma_t(u) = \lambda^{it} u$ eigenvector of the time evolution (some $\lambda > 0$)

$a \mapsto uau^*$

then $e = uu^*$ $\quad \varphi(e) \neq 0$ as before $\varphi(uu^*) = \lambda^{-\beta}$

$\varphi(uu^*) = \varphi(u^*\sigma_{i\beta}(u))$

$= \lambda^{-\beta} \varphi(u^*u) = \lambda^{-\beta}\varphi(1)$

$= \lambda^{-\beta}$

acts trivially on KMS : $\dfrac{\varphi(uau^*)}{\varphi(uu^*)} = \lambda^{\beta} \varphi(au^*\sigma_{i\beta}(u))$

$= \lambda^{\beta}\lambda^{-\beta}\varphi(a) = \varphi(a)$

KMS_∞ - states   weak limits of KMS_β - states

$$\varphi_\infty(a) = \lim_{\beta\to\infty} \varphi_\beta(a)$$ lowering temperature

Note: in the literature often different notion of KMS_∞ - states:

$\forall\, a,b \in \mathcal{A}$ $\exists\, F_{a,b}(z)$ holom. on $\mathbb{H}$

with $F_{a,b}(t) = \varphi_t(a\,\sigma_t(b))$

weaker notion (no other boundary condition)

Using weak limits of KMS_β states is better: gives stronger KMS condition

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- Action of symmetries on KMS_∞ - states:

  Typical situation in QSM systems we will consider:

  for all $\beta, \beta' > \beta_0$ (suff. large)

  $\mathcal{E}_\beta \cong \mathcal{E}_{\beta'}$ homeom. sets of ~~ex~~tremal KMS states

$\Rightarrow$ so when taking weak limits to define $\mathcal{E}_\infty$

get also $\mathcal{E}_\infty \cong \mathcal{E}_\beta$ suff. large β (stability)

then explicitly map $\mathcal{E}_\infty \xrightarrow{W_\beta} \mathcal{E}_\beta$ realizing homeom.

$$\varphi \in \mathcal{E}_\infty : \qquad W_\beta(\varphi)(a) = \frac{\operatorname{Tr}(\pi_\varphi(a)\, e^{-\beta H})}{\operatorname{Tr}(e^{-\beta H})}$$

here π_φ is GNS representation of $\mathcal{A}$ associated to state $\varphi \in \mathcal{E}_\beta$

- <u>GNS representations</u>: (Gelfand-Naimark-Segal construction)

$\mathcal{A}$ C^*-algebra $\varphi : \mathcal{A} \to \mathbb{C}$ state

<u>Thm</u>: given $(\mathcal{A}, \varphi)$ $\exists\, \mathcal{H}$ Hilb. space and rep $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ and cyclic vector $\xi \in \mathcal{H}$

$\{\pi(a)\xi : a \in \mathcal{A}\} \subset \mathcal{H}$ dense

s.t. $\varphi(a) = \langle \pi(a)\xi, \xi \rangle$

<u>Sketch of Pf</u>: define on $\mathcal{A}$ $\langle a, b \rangle := \varphi(b^* a)$

- $I = \{a : \varphi(a^* a) = 0\}$ subspace
  by C^*-algebra $\Rightarrow$ left ideal of $\mathcal{A}$
- $\mathcal{H} = \mathcal{A}/I$ with $\langle , \rangle$ Hilbert space
- $\pi(a)\; b + I = ab + I$ (because I ideal)
  (bounded: same C^*-alg. argument $\Rightarrow$ extends to closure $\mathcal{H}$ of $\mathcal{A}/I$)
- adjoint on $\mathcal{B}(\mathcal{H})$ agrees w/ $*$ on $\mathcal{A}$
- if $\mathcal{A}$ unital $\xi = 1 + I$ (if not unital need to use an "approximate unit" sequence (net))

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for QSM systems s.t. $W_\beta : \mathcal{E}_\infty \xrightarrow{\simeq} \mathcal{E}_\beta$ homeom. for suff. large $\beta > 0$

can define symmetries action on $\mathcal{E}_\infty$ by warming/cooling

$$\rho^* \varphi(a) = \lim_{\beta \to \infty} \frac{W_\beta(\varphi)(\rho a)}{W_\beta(\varphi)(e)}$$

for $W_\beta(\varphi)(e) \neq 0$

Main example : Bost-Connes $\mathbb{Q}$SM system (1995)

Algebra $\mathcal{A}_{\mathbb{Q}}$ generators $e(r)$ $r \in \mathbb{Q}/\mathbb{Z}$, μ_n $n \in \mathbb{N}$

relations $e(0) = 1$ $e(r+s) = e(r)e(s)$

$\mu_n \mu_m = \mu_{n \cdot m}$ $\mu_n^* \mu_m = \mu_m \mu_n^*$ if $(n,m) = 1$

$\mu_n^* \mu_n = 1$ $\mu_n \mu_n^* = \cancel{\pi_n}$ projector

$\frac{1}{n} \sum_{ns=r} e(s) = \rho_n(e(r))$ $\sigma_n(e(r)) = e(nr)$

endom. of $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$

$\mu_n^* e(r) = \sigma_n(e(r)) \mu_n^*$ $e(r)\mu_n = \mu_n \sigma_n(e(r))$

$\mu_n e(r) = \rho_n(e(r)) \mu_n$

$\Rightarrow$ $\mu_n e(r) \mu_n^* = \rho_n(e(r))$ $\mu_n^* e(r) \mu_n = \sigma_n(e(r))$

$\pi_n = \frac{1}{n} \sum_{ns=0} e(s)$ $\qquad \sigma_n \circ \rho_n = id \qquad \rho_n \circ \sigma_n = \pi_n$

$\mathcal{A}_{\mathbb{Q}} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes_\rho \mathbb{N}$ semigroup crossed product

Semigroup action of $\mathbb{N}$ on $\mathbb{Q}[\mathbb{Q}/\mathbb{Z}]$ by

$\rho_n(e(r)) = \frac{1}{n} \sum_{ns=r} e(s)$ implemented by isometries μ_n

i.e. $\rho_n(e(r)) = \mu_n e(r) \mu_n^*$

<u>Note</u> on crossed product algebras

Group crossed product

$\mathcal{A}$ $\qquad G \subset Aut(\mathcal{A})$ $\qquad$ or more generally action $\alpha: G \to Aut(\mathcal{A})$

$\mathcal{A} \rtimes_\alpha G$ (suppose G discrete)

$\sum_g a_g U_g$ w/ product

$\underbrace{a_g U_g \; b_{g'}}_{(U_g b_{g'} U_g^*) U_g} U_{g'} = a_g \, \alpha_g(b_{g'}) \, U_{gg'}$

action $\alpha_g(a) = U_g a U_g^*$ implemented by unitaries U_g

ie. covariant representations:

$\pi: \mathcal{A} \to B(\mathcal{H})$ s.t. $\pi(\alpha_g(a)) = U_g \, \pi(a) \, U_g^*$

$U: G \to \mathcal{U}(\mathcal{H})$ unitaries for semigroup crossed product $\quad S$ with (abelian) semigroup in BC case

$$\rho : S \to \mathrm{End}(\mathcal{A})$$

covariant rep: $\quad \pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$

$\mu : S \to \mathcal{I}(\mathcal{H})$ isometries ($\mu^*\mu = 1 \quad \mu\mu^* = e$ projector)

$$\pi(\rho_s(a)) = \mu_s \, \pi(a) \, \mu_s^*$$

$\forall s \in S \quad \forall a \in \mathcal{A}$

Bost - Connes algebra $\quad \mathcal{A}_{\mathbb{Q}} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ rational algebra

complexify $\quad \mathcal{A}_{\mathbb{C}} = \mathcal{A}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{C} = \mathbb{C}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$

then C^*-algebra completion :

group ring : Γ discrete group

$\mathbb{C}[\Gamma]$ finite combinations $\sum_{\gamma \in \Gamma} a_\gamma \, \gamma \qquad a_\gamma \in \mathbb{C}$

or equivalently fin. supported functions

$$f : \Gamma \to \mathbb{C} \qquad (f(\gamma) = a_\gamma)$$

w/ product $\quad a_\gamma \gamma \; b_{\gamma'} \gamma' = a_\gamma b_{\gamma'} \, \gamma\gamma'$

i.e. convolution product

$$(f_1 * f_2)(\gamma) = \sum_{\gamma = \gamma_1 \cdot \gamma_2} f_1(\gamma_1) \, f_2(\gamma_2)$$

associative not commutative (Γ not abelian)

and involution $\quad f^*(\gamma) = \overline{f(\gamma^{-1})}$

~~[illegible]~~ Reduced group C^*-algebra:

representation (left-regular) of

$\mathbb{C}[\Gamma]$ on $\ell^2(\Gamma)$ $\qquad \xi \in \ell^2(\Gamma)$

$$L : \mathbb{C}[\Gamma] \to B(\ell^2(\Gamma))$$

$$(L_\gamma \xi)(\gamma') = \xi(\gamma\gamma')$$

completion of $\mathbb{C}[\Gamma]$ in norm of $B(\ell^2(\Gamma))$

$\Rightarrow$ C^*-algebra $C^*_r(\Gamma) \subset B(\ell^2(\Gamma))$

reduced group C^*-algebra

if Γ is abelian: then Pontrjagin duality $\hat{\Gamma} = \mathrm{Hom}(\Gamma, U(1))$

(Fourier transform)

$$\Rightarrow \quad \begin{array}{c} C^*_r(\Gamma) \cong C(\hat{\Gamma}) \\ \| \\ C^*(\Gamma) \end{array}$$

abelian C^*-algebra of continuous functions on Pontrjagin dual group

e.g. $\quad C^*(\mathbb{Z}) = C(S^1)$

Case of $\mathbb{Q}/\mathbb{Z}$: $\qquad \mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ $\quad C^*$-algebra completion:

$$C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{\mathbb{Z}})$$

not Pontrjagin: just notation for

$$\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} \quad \text{ordered by divisibility}$$

$\mathbb{Q}/\mathbb{Z}$ is abelian group of (abstract) roots of unity

i.e. $\{\, \zeta = \exp(2\pi i \tfrac{a}{b}) \;;\; \tfrac{a}{b} \in \mathbb{Q}/\mathbb{Z} \,\}$

$$\zeta^m \xleftarrow{\ \sigma_m\ } \zeta$$

$$\mathbb{Z}/N\mathbb{Z} \twoheadleftarrow \mathbb{Z}/N\cdot m\,\mathbb{Z} \qquad \zeta = e^{2\pi i \frac{l}{N\cdot m}} \quad l = 1, \dots, N\cdot m$$

$e^{2\pi i \frac{k}{N}}$

$k = 1, \dots, N$

Pontryagin

$$\widehat{\mathbb{Q}/\mathbb{Z}} = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, U(1)) = \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$$

an element of $\mathbb{Q}/\mathbb{Z}$ goes to a torsion element of $U(1)$ i.e. to $\mathbb{Q}/\mathbb{Z}$

an element

$\gamma \in \mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z})$ is a compatible family of homom. $\gamma_N : \mathbb{Z}/N\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$

$\gamma_N \in \mathbb{Z}/N\mathbb{Z}$ w/ compatibility

$$\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} \subset \prod_N \mathbb{Z}/N\mathbb{Z} \text{ given by sequences}$$

$$\{\gamma_N\}_{N\in\mathbb{N}} \text{ s.t. } \sigma_m(\gamma_{N\cdot m}) = \gamma_N \qquad (\zeta_{N\cdot m}^m = \zeta_N)$$

<u>Note</u>: $\hat{\mathbb{Z}} = \varprojlim_N \mathbb{Z}/N\mathbb{Z} = \prod_p \varprojlim_m \mathbb{Z}/p^m\mathbb{Z} = \prod_p \mathbb{Z}_p$

profinite completion of $\mathbb{Z}$ — prod. of p-adic integers

$\hat{G} = \varprojlim_H G/H$ — $H \trianglelefteq G$ normal subgroups of finite index

$G \subset \hat{G}$ dense

$$\left(\mathbb{Z}/n\mathbb{Z} \twoheadrightarrow \mathbb{Z}/m\mathbb{Z} \right) \quad m \mid n$$

with topology of proj. limit (and fin. top. on $\mathbb{Z}/n\mathbb{Z}$)

$\hat{\mathbb{Z}}$ Cantor set

compact / Hausdorff / perfect / totally disconnect

So Bost-Connes C^*-algebra

$$C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N} \cong C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$$

Semigroup action

$$\rho_n(e(r)) = \frac{1}{n} \sum_{ns=r} e(s) = \mu_n e(r) \mu_n^*$$

seen on functions $f \in C(\hat{\mathbb{Z}})$

$f(x) \quad x \in \hat{\mathbb{Z}}$

$$\rho_n(f)(x) = \begin{cases} f(\frac{1}{n}x) & \text{if } x \in n\hat{\mathbb{Z}} \quad \text{range of multipl. by } n \\ 0 & x \notin n\hat{\mathbb{Z}} \end{cases}$$

Representations of $C^*(\mathbb{Q}/\mathbb{Z}) \rtimes \mathbb{N}$

on $\ell^2(\mathbb{N}) = \mathcal{H}$ covariant rep of $\mathbb{C}[\mathbb{Q}/\mathbb{Z}]$ $C^*(\mathbb{Q}/\mathbb{Z})$

$\mathbb{N} \longrightarrow B(\ell^2(\mathbb{N}))$

$n \longmapsto \mu_n$ $\{\varepsilon_m\}_{m \in \mathbb{N}}$ standard o.n. basis of $\ell^2(\mathbb{N})$

$\mu_n \varepsilon_m = \varepsilon_{nm}$

$\mu_n \mu_{n'} = \mu_{nn'}$ $\langle \mu_n^* \varepsilon_m, \varepsilon_{m'} \rangle \Rightarrow \mu_n^* \varepsilon_m = \begin{cases} 0 & n \nmid m \\ \varepsilon_{m'} & m = nm' \end{cases}$ $\varepsilon_{m'} = \varepsilon_{m/n}$

$\mu_n^* \varepsilon_m = \varepsilon_{m/n}$ or zero : $\langle \varepsilon_m, \mu_n \varepsilon_{m'} \rangle = \langle \varepsilon_m, \varepsilon_{nm'} \rangle = \begin{cases} 0 & m \neq nm' \\ 1 & m = nm' \end{cases}$

$\mu_n^* \mu_n = 1$ first multipl. then divide by n

$\mu_n \mu_n^* = \pi_n$ projector $\quad \pi_n \varepsilon_m = \begin{cases} 0 & n \nmid m \\ \varepsilon_m & n \mid m \end{cases}$

Then rep. of $C^*(\mathbb{Q}/\mathbb{Z})$ on $\ell^2(\mathbb{N})$

$\alpha : \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}$ i.e. $\alpha \in \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^*$

i.e. concrete realization of abstract roots of unity $\mathbb{Q}/\mathbb{Z}$ as concrete roots of 1 in $\mathbb{C}$

invertible elements of $\mathrm{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}$

w/ ring structure given by composition

$R_\alpha : C^*(\mathbb{Q}/\mathbb{Z}) \to \mathcal{B}(\ell^2(\mathbb{N}))$

$$R_\alpha(e(r))\, \varepsilon_m = \zeta_r^m\, \varepsilon_m$$

$\zeta_r = \alpha(e(r)) \in \mathbb{C}$ root of unity (if $r = \frac{a}{b}$: $\zeta_r = \zeta_b^a$) prim. order b

$$R_\alpha(e(r+s)) = R_\alpha(e(r))\, R_\alpha(e(s))$$
$$R_\alpha(e(0)) = 1$$

Then covariance relation; should have

$$\mu_n R_\alpha(e(r))\, \mu_n^* = \frac{1}{n} \sum_{ns=r} R_\alpha(e(s))$$

acting on ε_m basis: l.h.s.

$\zeta_r^{m/n}\, \varepsilon_m$ if $n|m$ or 0

r.h.s.

$\frac{1}{n} \sum_{ns=r} \zeta_r^m\, \varepsilon_m$

if $n|m$ is 1
if $n \nmid m$ is 0

sum of roots of 1

Time evolution of Bost–Connes system

$$\begin{cases} \sigma_t(e(r)) = e(r) & \text{nothing on } C^*(\mathbb{Q}/\mathbb{Z}) \text{ part} \\ \sigma_t(\mu_n) = n^{it}\mu_n & \text{phase factor on isometries} \end{cases}$$

$\alpha \in \hat{\mathbb{Z}}^*$ compatible symmetries automorphisms

$\alpha\, e(r) = e(\alpha(r))$ $\qquad \alpha \in \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) = \hat{\mathbb{Z}}^*$

$\alpha\, \mu_n = \mu_n$ nothing on $\mathbb{N}$ part

So $\alpha \circ \sigma_t = \sigma_t \circ \alpha$

Covariant rep. of time evolution and Hamiltonian

$$\sigma_t(\mu_n) = R_\alpha(\sigma_t(\mu_n))\, \varepsilon_m = n^{it}\, \varepsilon_{nm}$$

take $H\varepsilon_m = \log(m)\cdot \varepsilon_m$ Hamiltonian

$e^{itH}\varepsilon_m = m^{it}\varepsilon_m$ (indep of R_α same for all these representations)

$$e^{itH} R_\alpha(\mu_n)\, e^{-itH}\varepsilon_m = e^{itH}\mu_n\, e^{-itH}\varepsilon_m$$

$$= (n.m)^{it}\, m^{-it}\, \varepsilon_{n.m} = n^{it}\mu_n\,\varepsilon_m = R_\alpha(\sigma_t(\mu_n))\,\varepsilon_m$$

Partition function: $Z(\beta) = \mathrm{Tr}(e^{-\beta H}) = \sum_m \langle \varepsilon_m, e^{-\beta H}\varepsilon_m\rangle = \sum_{m\geq 1} m^{-\beta} = \zeta(\beta)$
Riemann zeta function

Partition function $Z(\beta) = \sum_{n \geq 1} n^{-\beta} = \zeta(\beta)$

- convergent series for $\beta > 1$
- analytic contin. to merom. funct. for $\beta \in \mathbb{C}$
- poles : pole at $\beta = 1$ simple w/ residue 1

So when inverse temperature $\beta > 1$ have KMS_β states of Gibbs form

$\pi_\alpha : \mathcal{A}_{BC} \to \mathcal{B}(\ell^2(\mathbb{N}))$ $\alpha \in \hat{\mathbb{Z}}^*$

covariant rep's as before

$\pi_\alpha(e(r))\, \varepsilon_m = \zeta_r^m\, \varepsilon_m$

$\zeta_r = \alpha(r)$

$\pi_\alpha(\mu_n)\, \varepsilon_m = \mu_n \varepsilon_m = \varepsilon_{nm}$

$$\varphi_{\beta,\alpha}(a) = \frac{Tr(\pi_\alpha(a)\, e^{-\beta H})}{Tr(e^{-\beta H})}$$

$H \varepsilon_m = \log(m)\, \varepsilon_m$
Hamiltonian implementing σ
(same for all α)

$$= \zeta(\beta)^{-1}\, Tr(\pi_\alpha(a)\, e^{-\beta H})$$

More explicitly : to see values of $\varphi_{\beta,\alpha}$

linear basis for algebra $\mathcal{A}_{BC}$

from relations of algebra presentation :

can always write elements of $\mathcal{A}_{BC}$ as limits (closure) of lin. combinations of elements of the form $e(r)\, \mu_n\, \mu_m^*$

– evaluating $\varphi_{\beta,\alpha}$ states on basis elements

$e(r)\,\mu_m\,\mu_m^*$

if $n \neq m$

$$\varphi_{\beta,\alpha}(e(r)\mu_n\mu_m^*) = \varphi_{\beta,\alpha}(\mu_n\mu_m^*\,\sigma_{i\beta}(e(r)))$$

$$= \varphi_{\beta,\alpha}(\mu_n\mu_m^*\,e(r)) = \varphi_{\beta,\alpha}(\cancel{\mu_n}\,e(r)\,\sigma_{i\beta}(\mu_n\mu_m^*))$$

$$\Longleftarrow$$

$$= \left(\frac{n}{m}\right)^{-\beta}\varphi_{\beta,\alpha}(e(r)\mu_n\mu_m^*)$$

$\varphi_{\beta,\alpha}(e(r)\mu_n\mu_m^*) = 0$

so state is supported on abelian subalg. $C^*(\mathbb{Q}/\mathbb{Z})$

(Note: $\mu_n\mu_n^* = \cancel{\pi_n} = \frac{1}{n}\sum_{s:\,ns=0} e(s)$ is also in $C^*(\mathbb{Q}/\mathbb{Z})$)

– Values on lin. basis $e(r)$ of $C^*(\mathbb{Q}/\mathbb{Z})$

$$\varphi_{\beta,\alpha}(e(r)) = \zeta(\beta)^{-1}\sum_{m\geq 1}\langle \varepsilon_m, \pi_\alpha(e(r))\,e^{-\beta H}\varepsilon_m\rangle$$

$$= \zeta(\beta)^{-1}\sum_{m\geq 1}\zeta_r^m\, m^{-\beta}$$

Polylogarithm function

$$Li_s(z) = \sum_{n\geq 1}\frac{z^n}{n^s}$$

special case $s=1$:

$Li_1(z) := -\log(1-z)$

dilog is an integral of log etc.

recursive property: $Li_{s+1}(z) = \int_0^z \frac{Li_s(t)}{t}\,dt$

Gibbs states of Bost-Connes system
are (combinations of) polylogs at roots of unity
normalized by Riemann zeta function

$$\varphi_{\beta,\alpha}(e(r)) = \frac{Li_\beta(\zeta_r)}{\zeta(\beta)}$$ with $\zeta_r = \alpha(r)$ root of 1

- What about KMS_β states when $0 < \beta \leq 1$?

Partition function $Z(\beta) = \sum_{n \geq 1} n^{-\beta}$ not convergent $e^{-\beta H}$ not trace class

$\Rightarrow$ no Gibbs states

Thm (Bost-Connes):

for $0 < \beta \leq 1$ $\quad \mathcal{E}_\beta$ single pt. with
(unique KMS_β state)

$$\varphi_\beta(e(r)) = f_{-\beta+1}(b) \big/ f_1(b)$$

$r = \frac{a}{b}$

where $f_k(b) = \sum_{d|b} \mu(d) \left(\frac{b}{d}\right)^k$

μ = Möbius function

f_1 = Euler totient function

$$f_k(n) = n^k \prod_{\substack{p \mid n \\ p \text{ prime}}} (1 - p^{-k})$$

- Möbius function and Möbius inversion

f, g arithmetic functions $: \mathbb{N} \to \mathbb{R}$ (or $\mathbb{C}$)
st.

$$g(n) = \sum_{d \mid n} f(d) \quad \text{then} \quad f(n) = \sum_{d \mid n} \mu(d)\, g\!\left(\tfrac{n}{d}\right)$$

for all $n \in \mathbb{N}$

- Möbius function $\mu(n) = \displaystyle\sum_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} e^{2\pi i \frac{k}{n}} = \sum_{\substack{1 \le k \le n \\ \gcd(k,n)=1}} \xi_{\frac{k}{n}}$

Sum of roots of 1 primitive order n

Values: $\mu : \mathbb{N} \to \{\pm 1, 0\}$ depending on prime factorization of n

$$\begin{cases} \mu(n) = 1 & \text{if } n \text{ square free \& even \# prime factors} \\ \mu(n) = -1 & n \text{ square free \& odd \# prime factors} \\ \mu(n) = 0 & \text{if } n \text{ not square free (has some } p^2 \text{ prime factor)} \end{cases}$$

- multiplicative $\mu(ab) = \mu(a)\mu(b)$

- $\displaystyle\sum_{d \mid n} \mu(d) = \begin{cases} 1 & n = 1 \\ 0 & n > 1 \end{cases}$ also $\displaystyle\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}$ from Euler product

Can also rewrite the unique high temperature KMS_β state ($0<\beta\leq 1$) as

$$\varphi_\beta(e(\tfrac{a}{b})) = b^{-\beta} \prod_{\substack{p \text{ prime} \\ p\mid b}} \frac{(1-p^{\beta-1})}{(1-p^{-1})}$$

How does one get the classification of all extremal KMS states?

– Note: previous observation: for all $\varphi\in KMS_\beta$

$$\varphi(e(r)\mu_n\mu_m^*) = 0 \qquad n\neq m$$

so all supported on $C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{\mathbb{Z}})$

So $\boxed{\text{probability measures on } \hat{\mathbb{Z}} \text{ with some specific } \beta\text{-dependent scaling property}}$

w/resp. to semigroup action

$$\mu(s\cdot Z) = n(s)^{-\beta}\,\mu(Z)$$

General setting

$n: S \to \mathbb{R}_+^*$ semigroup homom. defining time evol.

$$\sigma_t(\mu_s) = n(s)^{it}\mu_s$$

$$\mu_s\mu_s^* = e_s \;:\; \sigma_t(e_s) = n(s)^{-\beta} e_s$$

$$s\cdot \chi_Z = \mu_s\,\chi_Z\,\mu_s^*$$

Will prove a general theorem on KMS-states classification that covers also Bost-Connes case

Other equivalent descriptions of the Bost-Connes algebra

Hecke algebra: (warning on different meanings of term "Hecke algebra")

$ax+b$ -group (affine transf.)

R ring $\qquad P_R$ group of matrices

$P_R = \{ \begin{pmatrix} 1 & b \\ 0 & a \end{pmatrix} : a, b \in R \quad a \in R^* = GL_1(R)$ invertible elements $\}$

[Notation: R ring $R^\times = R \setminus \{0\}$ non-zero elt's $\qquad R^* = GL_1(R)$ invertible elt's]

if $R \subset \mathbb{R}$ (e.g. $R = \mathbb{Z}$ or $R = \mathbb{Q}$)

also define $P_R^+ \subset P_R$ with $a > 0$

$P_{\mathbb{Z}}^+ \subset P_{\mathbb{Q}}^+$ inclusion of countable groups

$\Gamma_0 \subset \Gamma$ general case

Commensurability condition:

all orbits of left action of Γ_0 on Γ/Γ_0 are finite

(same for right action of Γ_0 on $\Gamma_0 \backslash \Gamma$)

$\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$ Hecke algebra

$f : \Gamma_0 \backslash \Gamma \to \mathbb{Q}$ finitely supported and Γ_0-invariant $\forall \gamma \in \Gamma \; \gamma_0 \in \Gamma_0$ $f(\gamma\gamma_0) = f(\gamma)$

convolution product (think of f as defined on double quotient $f: {}_{\Gamma_0}\backslash\Gamma/\Gamma_0 \to \mathbb{Q}$)

$$(f_1 * f_2)(\gamma) = \sum_{\tilde{\gamma} \in \Gamma_0\backslash\Gamma} f_1(\gamma\tilde{\gamma}^{-1}) f_2(\tilde{\gamma})$$

<u>time evolution</u>; trivial if $\Gamma_0 \subset \Gamma$ normal

Regular representation of $\mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0)$

$$\mathcal{H} = \ell^2(\Gamma_0\backslash\Gamma)$$

$$\underline{(\pi(f)\xi)}(\gamma) = \sum_{\tilde{\gamma} \in \Gamma_0\backslash\Gamma} f(\gamma\tilde{\gamma}^{-1})\, \xi(\tilde{\gamma})$$

(rep. w/ resp. to convol. prod $*$)

involution: $f^*(\gamma) = \overline{f(\gamma^{-1})}$

involutive alg. rep.

C^*-alg. closure in π-representation in $B(\ell^2(\Gamma_0\backslash\Gamma))$

for $\gamma \in \Gamma$

$L(\gamma) = \#$ orbit of ~~right~~ Γ_0 action on $\gamma \in \Gamma_0\backslash\Gamma$

$R(\gamma) = \#$ orbit of ~~right~~ Γ_0 action on $\gamma \in \Gamma/\Gamma_0$

$= L(\gamma^{-1})$

time evolution:

$$\sigma_t(f)(\gamma) = \left(\frac{L(\gamma)}{R(\gamma)}\right)^{it} f(\gamma)$$

$$f \in \mathcal{H}_{\mathbb{C}}(\Gamma, \Gamma_0) = \mathcal{H}_{\mathbb{Q}}(\Gamma, \Gamma_0) \otimes_{\mathbb{Q}} \mathbb{C}$$

Proved in Bost-Connes '95 paper:

C^*-closure of $\mathcal{H}_{\mathbb{C}}(P^+_{\mathbb{Q}}, P^+_{\mathbb{Z}})$ in $\mathcal{B}(\ell^2(P^+_{\mathbb{Z}} \backslash P^+_{\mathbb{Q}}))$
w/ this time evol.

$\cong (\mathcal{A}_{BC}, \sigma_t)$ Bost-Connes system

$\mathcal{A}_{BC} = C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}$ $\quad \sigma_t(f) = f$
$\quad \sigma_t(\mu_n) = n^{it} \mu_n$

<u>Note</u>: Symmetries of BC system

$G = \hat{\mathbb{Z}}^* \subset \mathrm{Aut}(\mathcal{A}_{BC}, \sigma)$ $\qquad (\hat{\mathbb{Z}}^* = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z}))$

$\gamma(e(r)) = e(\gamma(r))$
$\gamma(\mu_n) = \mu_n \Rightarrow \gamma \circ \sigma_t = \sigma_t \circ \gamma$

action on KMS_β (for $0 < \beta \leq 1$ unique state: fixed)
for $\beta > 1$ Gibbs states $\varphi_{\beta,\alpha}$ with $\gamma^* \varphi_{\beta,\alpha} = \varphi_{\beta, \gamma\alpha}$ in $\hat{\mathbb{Z}}^*$

4. Zero temperature KMS states of the Bost-Connes system and Galois theory

$$\lim_{\beta\to\infty} \varphi_{\beta,\alpha}(e(r)) = \lim_{\beta\to\infty} \frac{Li_\beta(\zeta_r)}{\zeta(\beta)}$$

$$= \lim_{\beta\to\infty} \frac{Tr(\pi_\alpha(e(r))e^{-\beta H})}{Tr(e^{-\beta H})} = \lim_{\beta\to\infty} \frac{\langle \varepsilon_1, \pi_\alpha(e(r))\varepsilon_1\rangle + \sum_{n>1} \langle \varepsilon_n, \pi_\alpha(e(r)) n^{-\beta}\varepsilon_n\rangle}{\zeta(\beta) = 1 + \sum_{n>1} n^{-\beta}}$$

$$Ker(H) = Span\{\varepsilon_1\} \subset \ell^2(\mathbb{N}) \quad (H\varepsilon_1 = \log(1)\varepsilon_1 = 0.)$$

$$= \langle \varepsilon_1, \pi_\alpha(e(r))\varepsilon_1\rangle = \zeta_r = \varphi_{\infty,\alpha}(e(r))$$

$$\zeta_r = \alpha(r) \qquad \alpha\in\hat{\mathbb{Z}}^* \quad r\in\mathbb{Q}/\mathbb{Z}$$

$\Rightarrow$ Arithmetic subalgebra $\mathcal{A}_\mathbb{Q} = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}$ of BC alg

$\{\varphi_{\infty,\alpha}\} = \Sigma_\infty$ extremal KMS-states

values of $\varphi_{\infty,\alpha}$ on $a\in\mathcal{A}_\mathbb{Q}$ are
algebraic numbers $\bar{\mathbb{Q}}$ (not $\mathbb{C}$)
and in fact in $\mathbb{Q}^{ab} \cong \mathbb{Q}^{cycl}$ gen. by roots of 1

Action of symmetries $Aut(\mathcal{A}_{BC},\sigma) \cong \hat{\mathbb{Z}}^*$ on algebra
$\Rightarrow$ action of $Gal(\mathbb{Q}^{ab}/\mathbb{Q}) = Gal(\bar{\mathbb{Q}}/\mathbb{Q})^{ab} \cong \hat{\mathbb{Z}}^*$

Kronecker-Weber theorem — explicit class field theory for $\mathbb{Q}$

The explicit class field theory problem

- Understanding abelian extensions of number fields & function fields K
- $\exists$ correspondence

  abelian extensions $\leftrightarrow$ open subgroups of idele class group of K

- Would like: explicit generators of K^{ab} (max abelian extension of K) and explicit description of action of Galois group $\mathrm{Gal}(K^{ab}/K) \simeq \mathrm{Gal}(\bar{K}/K)^{ab}$ on generators

  Hilbert 12 th problem (explicit class field theory problem)

- <u>Kronecker-Weber</u>: $K = \mathbb{Q}$, $\mathbb{Q}^{ab} \simeq \mathbb{Q}^{cycl}$

  $\mathrm{Gal}(\mathbb{Q}^{cycl}/\mathbb{Q}) \simeq \hat{\mathbb{Z}}^*$

  expl. generators roots of 1 $\leftarrow$ torsion pts of $\mathbb{G}_m$ special values of exp

  and explicit action on roots of 1 by $\hat{\mathbb{Z}}^* = \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$

- Complex multiplication elliptic curves;

  $K = \mathbb{Q}(\tau)$ τ CM point of $\mathbb{H}$

  imaginary quadratic fields

  K^{ab} gen. by special values $f(\tau)$ of modular functions w/ explicit $\mathrm{Gal}(K^{ab}/K)$ action

- More general CM fields (Complex multiplication abelian varieties)

  F totally real field (all embeddings into $\mathbb{C}$ real)

  K totally imaginary extension of F (cannot be embedded in $\mathbb{R} \hookrightarrow \mathbb{C}$)

- First important open case: real quadratic fields $\mathbb{Q}(\sqrt{d})$

  (Stark conjectures)

  Manin: NC tori w/ real multiplication

  | Hecke alg. cond $\Rightarrow$ [illegible]

Explicit class field theory and QSM

- K number field or function field

- $(\mathcal{A}_K, \sigma)$ QSM system
  - partition function $\zeta_K(s)$ Dedekind (number field) or suitable L-funct. for f.f.
  - Symmetries $\simeq Gal(K^{ab}/K)$
  - $\mathcal{E}_\infty = \{\varphi_{\infty,\alpha}\}$ extremal zero temperature KMS

$\exists\ \mathcal{A}_{K,\mathbb{Q}}$ arithm. subalgebra w/

$$\varphi_{\infty,\alpha}(\mathcal{A}_{K,\mathbb{Q}}) \subset K^{ab}$$

action of symmetries on $\mathcal{A}_{K,\mathbb{Q}}$ $\simeq$ $Gal(K^{ab}/K)$ action on K^{ab}

Why useful approach?

- generators for $\mathcal{A}_{K,\mathbb{Q}}$ algebra (easier problem than generators for K^{ab} field)
- action of symmetries $Symm(\mathcal{A}_{K,\mathbb{Q}}, \sigma)$ of algebra (easier problem than explicit ~~gal~~ action for gen. of field)

- $K = \mathbb{Q}$ (Bost-Connes)
- $K = \mathbb{Q}(\tau)$ imaginary quadratic (Connes-Marcolli-Ramachandran)
- partial results for K other number fields
  K function fields

Ha-Paugam, Laca-Larsen-Neshveyev, Cornelissen-Li-Marcolli
Jacob, Consani-Marcolli, Neshveyev-Rust

Bosonic and Fermionic Fock spaces

Bosonic: $V = \ell^2(\mathcal{P})$ $\quad \mathcal{P}$ = set of primes

$$\mathcal{H}_B = \bigoplus_{n\geq 0} S^n V$$

$S^n V$ = n-th fold symmetric product of V

o.n. basis $\{\varepsilon_p\}_{p\in\mathcal{P}}$ of V

elements of $S^\ell V$ spanned by $\varepsilon_{p_1}\cdots\varepsilon_{p_\ell}$ (any order) possibly w/ multiplicities

identify $\varepsilon_{p_1}\cdots\varepsilon_{p_\ell}$ with ε_n

$n = p_1\cdots p_\ell$ primary decomposition

$\langle \varepsilon_{p_i}, \varepsilon_{p_j}\rangle = \delta_{ij}$ $\qquad \langle \varepsilon_n, \varepsilon_m\rangle = \prod_{i,j} \langle \varepsilon_{p_i}, \varepsilon_{p_j'}\rangle = \prod_{i,j}\delta_{ij} = \delta_{n,m}$

$$\mathcal{H}_B = \bigoplus_{n\geq 0} S^n \ell^2(\mathcal{P}) \simeq \ell^2(\mathbb{N})$$

Bosonic Fock space of $\ell^2(\mathcal{P})$

Fermionic:

$$\mathcal{H}_F = \bigoplus_{n\geq 0} \Lambda^n V$$

exterior powers

$\Lambda^n V$ spanned by $\varepsilon_{p_1}\wedge\cdots\wedge\varepsilon_{p_n}$ $\quad p_i\in\mathcal{P}$

$\langle \varepsilon_{p_1}\wedge\cdots\wedge\varepsilon_{p_\ell}, \varepsilon_{p_1'}\wedge\cdots\wedge\varepsilon_{p_\ell'}\rangle = \delta_{n,m}$ $\qquad n = p_1\cdots p_\ell$, $m = p_1'\cdots p_\ell'$

but now only n, m with square free primary decomposition

because $\varepsilon_{p_i}\wedge\varepsilon_{p_j} = -\varepsilon_{p_j}\wedge\varepsilon_{p_i}$ & $\varepsilon_p\wedge\varepsilon_p = 0$

Möbius function $\mu : \mathbb{N} \to \mathbb{R}$
$\{\pm 1, 0\}$

$\mathcal{H}_F \cong \mathcal{H}_B / \mathrm{Ker}(\mu)$ $\qquad$ $\mu : \varepsilon_n \longmapsto \mu(n)$

induced $\bar{\mu}$ on $\mathcal{H}_F$ sign function ± 1 values
(only square-free n)

$\Pi_F : \mathcal{H}_B \twoheadrightarrow \mathcal{H}_F$ projection

Fermionic BC algebra:

$\mathcal{A}_{\mathbb{Q}}$ BC algebra $\qquad$ $\Pi_F \mathcal{A}_{\mathbb{Q}} \Pi_F$ compression
$\qquad$ $= \mathcal{A}_{\mathbb{Q},F}$ acting on $\mathcal{H}_F$

generators $e(r)$ $r \in \mathbb{Q}/\mathbb{Z}$
$\tilde{\mu}_n$ n square-free $\qquad$ $\tilde{\mu}_n^* \tilde{\mu}_n = P_n$, $\tilde{\mu}_n \tilde{\mu}_n^* = Q_n$ projections

Same rel's for $e(r)$'s

$\tilde{\mu}_n \tilde{\mu}_m = \tilde{\mu}_m \tilde{\mu}_n = 0$ if $(n,m) \neq 1$
$= \tilde{\mu}_{nm}$ if $(n,m) = 1$

$\tilde{\mu}_n^* \tilde{\mu}_m = \tilde{\mu}_m \tilde{\mu}_n^*$ if $(n,m) = 1$

$\tilde{\mu}_n^* e(r) = \sigma_n(e(r)) \tilde{\mu}_n^*$ $\qquad$ ρ_n, σ_n endom's as in BC case

$\tilde{\mu}_n e(r) = \rho_n(e(r)) \tilde{\mu}_n$ $\qquad$ $\tilde{\mu}_n = \Pi_F \mu_n \Pi_F$

$e(r) \tilde{\mu}_n = \tilde{\mu}_n \sigma_n(e(r))$

$\tilde{D} = F|\tilde{D}|$ with $F = \mu$ $\quad |\tilde{D}| = \exp H$

$D = F\,|D|$ w/ $F = \mu$ $\quad |D| = H$

$\Rightarrow$ operators on $\mathcal{H}_F$ with trivial ker, comp. resolv.

- $\tilde{D}$ fin. summable

$$\zeta_{\tilde{D}}(\beta) = \mathrm{Tr}_{\mathcal{H}_F}\left(|\tilde{D}|^{-\beta}\right) = \frac{\zeta(\beta)}{\zeta(2\beta)}$$

$$\eta_{\tilde{D}}(\beta) = \mathrm{Tr}_{\mathcal{H}_F}\left(F|\tilde{D}|^{-\beta}\right) = \zeta(\beta)^{-1}$$

- D θ-summable: $\mathrm{Tr}_{\mathcal{H}_F}\left(F e^{-\beta|D|}\right) = \mathrm{Tr}_{\mathcal{H}_F}\left(F|\tilde{D}|^{-\beta}\right)$

$$\rightarrow \sum_{n\geq 1} |\mu(n)|\, n^{-\beta} = \frac{\zeta(\beta)}{\zeta(2\beta)}$$

$$\rightarrow \sum_{n\geq 1} \mu(n)\, n^{-\beta} = \frac{1}{\zeta(\beta)}$$

Möbius inversion formula

$\rightarrow$ Witten index of SUSY Riemann gas

$$\Delta = \mathrm{Tr}\left((-1)^F e^{-\beta H}\right)$$

sign on $\mathcal{H}_F$ implementing the supersymmetry

Data $(\mathcal{A}_{\mathbb{Q},F}, \mathcal{H}_F, D)$ θ-summable σ-spectral triple

w/ $\sigma \in \mathrm{Aut}(\mathcal{A}_{\mathbb{Q},F})$ $\quad \sigma(a) = F a F$

eta function of $\tilde{D}$ and polylogarithms:

$$\varphi_{\beta,\alpha}(a) = \frac{Tr(F\pi_\alpha(a)e^{-\beta H})}{Tr(e^{-\beta H})} = \frac{\eta_{a,\tilde{D}}(\beta)}{\zeta(\beta)}$$

$$Tr(F\pi_\alpha(a)e^{-\beta H}) = \sum_{n\geq 1}\mu(n)n^{-\beta}\langle\varepsilon_n|\pi_\alpha(a)|\varepsilon_n\rangle$$

$$= \sum_{n\geq 1}\mu(n)\frac{\zeta_r^n}{n^\beta}$$

Möbius transform of Polylog function

Möbius inversion of polylog

$$Li_\beta(\zeta) = \sum_{n\geq 1}\frac{\zeta^n}{n^\beta}$$

Note: for $\varphi_{\beta,\alpha}$ RMS_β of BC system

$$\varphi_{\beta,\alpha}(e(r)) = \zeta(\beta)^{-1}\sum_{n\geq 1}\frac{\zeta_r^n}{n^\beta}$$ can rewrite equivalently as

$$\varphi_{\beta,\alpha}(e(r)) = \sum_{n\geq 1}\frac{b_n}{n^\beta} \qquad b_n = \sum_{d|n}\mu\left(\frac{n}{d}\right)\zeta_r^d$$

Using relation of Dirichlet series:

$$\sum_{n\geq 1}\frac{a_n}{n^\beta} = \zeta(\beta)\sum_{n\geq 1}\frac{b_n}{n^\beta} \qquad b_n = \sum_{d|n}\mu\left(\frac{n}{d}\right)a_d$$