Interactions between Physics and Number Theory

Main topics:

- modular forms
- periods of modular forms
  modular symbols
- modular curves
- NC boundaries of alg geom spaces
- quantum statistical mechanics
- QSM systems associated to number theoretic objects
- number fields & QSM systems
- L-functions & zeta functions
- boundary of mod curves and quantum mod forms
- mock/quantum mod forms

Modular forms : preliminaries

Source: Don Zagier 1-2-3 of Modular Forms

$\mathbb{H}$ upper half plane $\mathbb{H} = \{ z \in \mathbb{C} \mid \mathrm{Im}(z) > 0 \}$

hyperbolic metric $ds^2 = \dfrac{dx^2 + dy^2}{y^2}$

isometry group: $SL(2,\mathbb{R})$ acting by fractional linear transformations

$\boxed{2}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : z \longmapsto \dfrac{az+b}{cz+d}$ fractional linear transformations

$PSL(2,\mathbb{R}) = SL(2,\mathbb{R}) / \pm 1$ orientation preserving isometries
(det = 1)
$Iso^+(\mathbb{H})$

$PGL(2,\mathbb{R}) = GL(2,\mathbb{R}) / \mathbb{R}^*$
also acts by fractional linear transformations on $\mathbb{H}$: all isometries including orientation reversing $Iso(\mathbb{H})$

$\Gamma \subset SL(2,\mathbb{R})$ discrete subgroups

functions that transform "well" under action of some Γ

in particular $\Gamma = SL(2,\mathbb{Z})$ full modular group

or $\Gamma \subset SL(2,\mathbb{Z})$ finite index subgroups

(in particular "congruence subgroups" etc)
these have relevance for theory of elliptic curves and mod. spaces of elliptic curves + "level structure"
$\rightsquigarrow$ modular curves

$\Gamma \backslash \mathbb{H}$ for $\Gamma = SL(2,\mathbb{Z})$

is the moduli space (= parametrizing space)
of isom. classes of elliptic curves

Elliptic curves (over $\mathbb{C}$) $\quad E = \mathbb{C}/\Lambda$

for some lattice $\Lambda \subset \mathbb{C}$

can always write Λ in the form

$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ for some $\tau \in \mathbb{H}$

up to a homothety (up to scaling $\Lambda \mapsto \lambda\Lambda$, $\lambda \in \mathbb{C}^*$)

Λ up to scaling: oriented basis $\omega_1, \omega_2 \in \mathbb{C}$

$\mathrm{Im}(\omega_1/\omega_2) > 0$

use $\lambda = \omega_2^{-1}$ for rescaling and $\tau = \frac{\omega_1}{\omega_2}$

then two lattices $\mathbb{Z} + \mathbb{Z}\tau$, $\mathbb{Z} + \mathbb{Z}\tau'$ isomorphic

iff τ, τ' in same orbit of action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$ by fractional linear transformations

(because same as $SL(2,\mathbb{Z})$ change of basis $\omega_1, \omega_2 \mapsto \omega_1', \omega_2'$)

Also $\left\{\begin{matrix}\text{lattices up to} \\ \text{isomorphisms}\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\text{complex elliptic} \\ \text{curves } E = \mathbb{C}/\Lambda \\ \text{up to} \\ \text{isomorphism}\end{matrix}\right\}$

$X_\Gamma = \Gamma \backslash \mathbb{H}$ modular curve $\Gamma = SL(2,\mathbb{Z})$

$=$ mod space of elliptic curves up to isomorphism

(4) <u>Modular functions</u>

$$f : X_\Gamma \longrightarrow \mathbb{C}$$
$$X_\Gamma = \Gamma \backslash \mathbb{H}$$
$$\Gamma = SL(2,\mathbb{Z})$$

- equivalently $f : \mathbb{H} \longrightarrow \mathbb{C}$ invariant under action of $SL(2,\mathbb{Z})$ by fractional lin. transf
$$f(\gamma z) = f(z)$$

- equivalently a function F on lattices $\Lambda \subset \mathbb{C}$ that is invariant under scaling
$$F(\lambda\Lambda) = F(\Lambda)$$ and
constant on isomorphic lattices

$$f(z) = F(\mathbb{Z} + \mathbb{Z}z) \quad \text{and} \quad F(\Lambda) = f(\omega_1/\omega_2)$$
$\{\omega_1, \omega_2\}$ oriented basis of Λ

* usually also assume that $f : \mathbb{H} \longrightarrow \mathbb{C}$ is a <u>meromorphic function</u>

* and assume a growth condition
$$f(x+iy) = O(e^{cy}) \quad \text{for } y \to \infty$$
exp. growth
$$f(x+iy) = O(e^{c/y}) \quad \text{for } y \to 0$$
for some $c > 0$

[$\Rightarrow$ f extends to a <u>meromorphic function</u> on compactif. $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$]

Note : need to use meromorphic to extend to compactif.
as no holomorphic functions
on a compact Riemann surface

More general class of functions

modular forms

think again as functions on lattices
$F(\Lambda)$ now instead of invariance under scaling
require an equivariance condition

$$F(\lambda\Lambda) = \lambda^{-k} F(\Lambda)$$

weight k
(k integer)

in terms of the function
$f: \mathbb{H} \to \mathbb{C}$ $\quad f(\tau) = F(\mathbb{Z}+\mathbb{Z}\tau)$

this scaling condition gives

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$$

$\forall z \in \mathbb{H}$ and all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

(more generally can require this for some fixed subgroup $\Gamma \subset SL(2,\mathbb{Z})$)

then one can require
holomorphic functions

- usually <u>modular forms</u> assumes holomorphic (unless explicitly said that meromorphic)

<u>Note</u>: modular functions can be realized as quotients of two modular forms ("mod forms are like "integers" & mod functions like "rationals")

For fixed $\Gamma \subset SL(2,\mathbb{Z})$

$$\mathcal{M}_*(\Gamma) = \bigoplus_k \mathcal{M}_k(\Gamma)$$

$\mathcal{M}_k(\Gamma) =$ $\mathbb{C}$-vector space of holomorphic modular forms for Γ of weight k

$\begin{cases} \text{as vector space } \mathcal{M}_k(\Gamma) \text{ fin dim (trivial if } k<0) \\ \text{as algebra } \mathcal{M}_*(\Gamma) \text{ finitely generated over } \mathbb{C} \end{cases}$

(see later)

---

for modular forms assume a <u>growth condition</u>

$f(z) = O(1)$ as $y \to \infty$

$f(z) = O(y^{-k})$ as $y \to 0$

7 translation & Fourier series

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{Z})$$

if f is a modular form for $SL(2,\mathbb{Z})$ of any weight : invariance

periodic period = 1

$$f(Tz) = f(z+1) \underline{= f(z)}$$

so in the variable $q = e^{2\pi i \tau}$

can expand in Fourier series (function on the circle)

$$f(z) = \sum_{n=0}^{\infty} a_n e^{2\pi i n z} = \sum_{n=0}^{\infty} a_n q^n$$

Note: no $n<0$ term because assuming <u>growth condition</u>

<u>Note</u>: for $\gamma = -\mathbb{1} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in SL(2,\mathbb{Z})$

$$\frac{az+b}{cz+d} = z \quad \text{and} \quad (cz+d)^k = (-1)^k$$

$$\text{so} \quad f(\gamma z) = (cz+d)^k f(z) \Rightarrow \underbrace{f(z) = (-1)^k f(z)}$$

so modular forms of odd weight k vanish
so assume even k

8

for k even automorphy factor $(cz+d)^k$
same for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and for $-\mathbb{1}\cdot\gamma$

So consider mod forms for $PSL(2,\mathbb{Z})$

$PSL(2,\mathbb{Z})$ is generated by

$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

relations $S^2 = (ST)^3 = 1$

($PSL(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3\mathbb{Z}$ w/ S, ST gen.)

$\underbrace{S : z \mapsto \frac{-1}{z}}$ $\qquad$ $\underbrace{T : z \mapsto z+1}_{\text{this just the periodicity}}$

$\underbrace{f\left(\frac{-1}{z}\right) = z^k f(z)}_{\text{modularity reduces to this property}}$

---

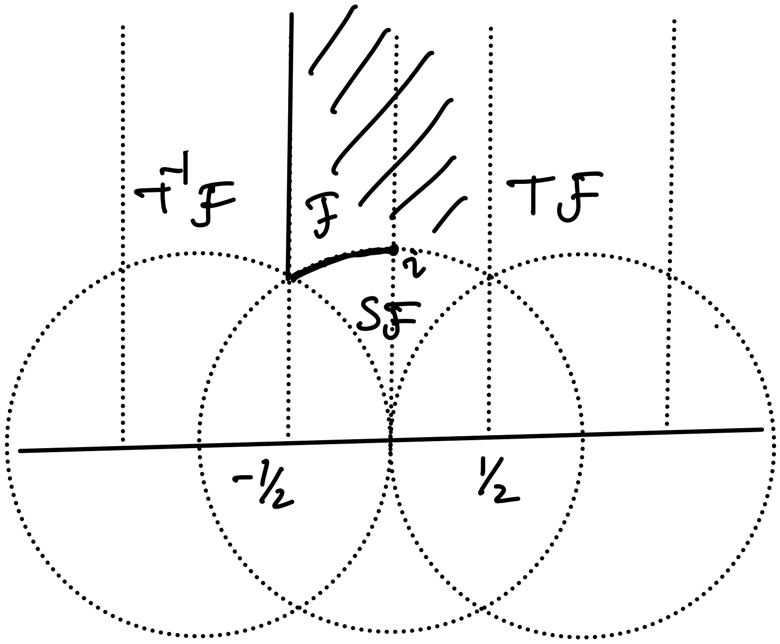
group action G on a set S

fundam. domain $F \subset S$ for the action
subset that meets each orbit of G
in S in exactly one point

9 Fundamental domain: interior given by

$$\mathcal{F} = \{ z \in \mathbb{H} \mid |z| > 1 \; ; \; |\mathrm{Re}(z)| < \tfrac{1}{2} \}$$

is a fundamental domain for the action of $SL(2,\mathbb{Z})$ on $\mathbb{H}$



Pf: first show given any $z \in \mathbb{H}$ can find

1 a $z^* \in \mathcal{F}$ $z^* \sim z$ under $SL(2,\mathbb{Z})$ action

given $z \in \mathbb{H}$ $\{ m + nz \mid m, n \in \mathbb{Z} \} = \Lambda_z$ lattice

nonzero vector of min length in Λ_z

$v = cz + d$

Note: c, d rel. prime otherwise can divide by common divisor and get shorter v

So if c, d rel. prime $\exists\, a, b$ s.t.

$$\gamma_1 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$$

Note that under fractional lin. transf.

$$\mathrm{Im}(\gamma_1 z) = \frac{\mathrm{Im}(z)}{|cz+d|^2}$$

so if $v = cz+d$ shortest vector then
$\mathrm{Im}(\gamma_1 z)$ as large as possible in the orbit

$z^* = T^n \gamma_1 z = \gamma_1 z + n$ where n
chosen so that z^* is in $|\mathrm{Re}(z^*)| < \frac{1}{2}$

to see $z^* \in \mathcal{F}$ need to check don't
have $|z^*| < 1$ but this true
since $\mathrm{Im}(-1/z^*) = \dfrac{\mathrm{Im}(z^*)}{|z^*|^2}$
would then be $> \mathrm{Im}(z^*)$
contradicting $\mathrm{Im}(z^*) \left(= \mathrm{Im}(\gamma_1 z)\right)$ maximal
in orbit

2 then show that if two different points
$z_1 \sim z_2$ under $SL(2,\mathbb{Z})$ orbit
cannot both be in $\mathcal{F}$

suppose that $z_1 \neq z_2$ $z_1, z_2 \in \mathcal{F}$ with
$z_2 = \gamma z_1$ $\gamma \in SL(2,\mathbb{Z})$ $\gamma \neq \pm 1$

~~fist~~ first γ cannot be $\gamma = T^n$ because then
cannot both have $|\mathrm{Re}(z_i)| < \frac{1}{2}$

so if $\gamma \neq T^n$ have $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ w/ $c \neq 0$

<u>Note</u>: all $z \in \mathcal{F}$ have $\mathrm{Im}(z) > \frac{\sqrt{3}}{2}$

so $$\frac{\sqrt{3}}{2} < \mathrm{Im}(z_2) = \frac{\mathrm{Im}(z_1)}{|cz_1+d|^2} \leq \frac{\mathrm{Im}(z_1)}{c^2\,\mathrm{Im}(z_1)^2} < \frac{2}{c^2\sqrt{3}}$$

$$\frac{\sqrt{3}}{2} \leq \frac{2}{c^2\sqrt{3}} \qquad 0 < c^2 \leq \frac{4}{3} \Rightarrow c = \pm 1$$
$c \in \mathbb{Z}$

have then $\mathrm{Im}(z_2) = \dfrac{\mathrm{Im}(z_1)}{|\pm z_1 + d|^2} \leq \dfrac{\mathrm{Im}(z_1)}{|z_1|^2}$

where $|z_1| > 1$ so cannot be
$\mathrm{Im}(z_1) \leq \mathrm{Im}(z_2)$

but same reasoning would apply to
$z_1 = \gamma^{-1} z_2$ so also cannot be
$\mathrm{Im}(z_2) \leq \mathrm{Im}(z_1)$

so z_1, z_2 cannot both be in $\mathcal{F}$

Finite dimensionality of $M_k(\Gamma)$

for $\Gamma = SL(2,\mathbb{Z})$ use properties of fund. domain

$f : \mathbb{H} \to \mathbb{C}$ mod form of weight k

not a <u>function</u> on $\Gamma \backslash \mathbb{H}$ because of automorphy factor
(not mod function)

but order of vanishing is well defined

$\mathrm{ord}_p(f)$ = order of vanishing at $P \in \Gamma\backslash\mathbb{H} = X_\Gamma$

(aut factor does not contribute zeros)

depends only on the orbit of P so well def on

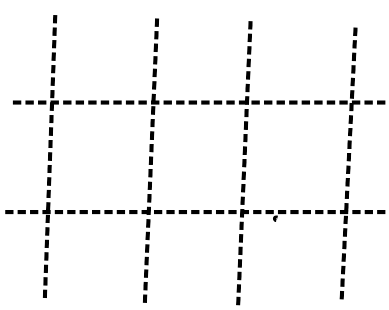
but: elliptic points that have nontrivial stabilizer under Γ-action where this counting has some multiplicity

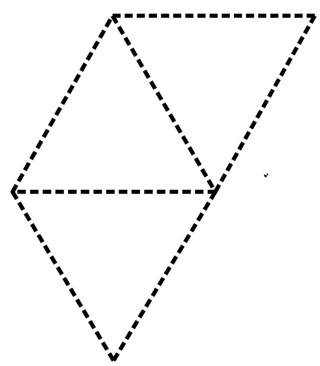
and also compactification of $X_\Gamma = \Gamma\backslash\mathbb{H}$ by cusp points cusps

stabilizers:
$$\begin{cases} \omega = \frac{1}{2}(-1 + i\sqrt{3}) = e^{2\pi i/3} \\ i \end{cases}$$

these points in the boundary of $\mathcal{F}$ have nontrivial stabilizer under the $SL(2,\mathbb{Z})$ action

square and triangular lattice that have extra symmetries





stabilized by cyclic subgroups of order two & three

$\mathbb{Z}/2\mathbb{Z}$ $\mathbb{Z}/3\mathbb{Z}$

So in the quotient $\Gamma \backslash \mathbb{H}$

these points are <u>singular</u> (orbifold) points

(they have metric neighborhoods that are not disks but quotients of disks by cyclic group : cone pts)

n_p = order of the stabilizer group of P

$$n_p = \begin{cases} 3 & P = \omega \\ 2 & P = i \\ 1 & \text{all other pts } P \in X_\Gamma \end{cases}$$

Compactification

Expansion $f(z) = \sum_{n \geq 0} a_n q^n \qquad q = e^{2\pi i z}$

of modular forms

implies that well defined lim for $q \to 0$

and f extends holomorphically to $q = 0$

so define $\overline{X_\Gamma} = X_\Gamma \cup \{\infty\}$

$q = 0$

can also think of $X_\Gamma = \Gamma \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

where action of $\Gamma = SL(2,\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{Q})$ by fract. lin. tr.
single orbit

$\mathbb{P}^1(\mathbb{Q}) = \Gamma \cdot \{\infty\}$ orbit

$\mathrm{ord}_\infty(f)$ = smallest $n \geq 0$ s.t. $a_n \neq 0$ in $f(z) = \sum_n a_n q^n$

Relation between order and zeros:

$f \in M_k(\Gamma)$ $\Gamma = SL(2,\mathbb{Z})$ $f \not\equiv 0$

$$\Rightarrow \sum_{p \in X_\Gamma} \frac{1}{n_p} \mathrm{ord}_p(f) + \mathrm{ord}_\infty(f) = \frac{k}{12}$$

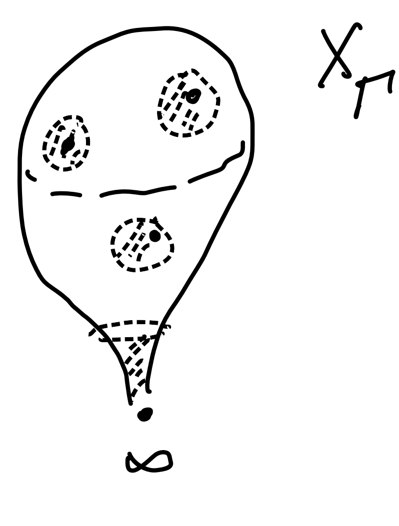
Pf: all zeroes of f in X_Γ

remove ε-neighborhoods (disjoint)

and remove neighb. of ∞

$D = X_\Gamma \setminus \mathcal{U}_\varepsilon(x_i) \cup \mathcal{U}(\infty)$

$\qquad\qquad\qquad\qquad \{\mathrm{Im}(z) > \varepsilon^{-1}\}$



Then Cauchy residue thm

$$\int_{\partial D} d\log f(z) = \int_{\partial D} \frac{f'(z)}{f(z)}\,dz = 0$$

View D as closed set in fund-domain $\mathcal{F}$

so ∂D contains

1. horiz. line from $-\frac{1}{2}+i\varepsilon^{-1}$ to $\frac{1}{2}+i\varepsilon^{-1}$
2. vertical line from w to $-\frac{1}{2}+i\varepsilon^{-1}$ and from $w+1$ to $\frac{1}{2}+i\varepsilon^{-1}$
3. arc of $|z|=1$ from w to $w+1$

w/ some ε-neighborhoods removed

4. boundaries of ε-neighborhoods of the zeros (total angle 2π if not elliptic pts and π or $\frac{2\pi}{3}$ if i or w)

contributions to integral

2): zero contrib: f same value on both and opposite orientation so cancellation

1): $2\pi i \,\mathrm{ord}_\infty(f)$

4): $2\pi i \,\mathrm{ord}_p(f)$ if $n_p = 1$

$\frac{2\pi i \,\mathrm{ord}_p(f)}{n_p}$ as integrating over $\frac{1}{n_p}$ part of full circle of disk neighborhood

3): $\frac{\pi i k}{6}$ right/left arcs and $d\log f(Sz) = d\log f(z) + k\frac{dz}{z}$ $\square$

Consequence of this: $\Gamma = SL(2,\mathbb{Z})$

$$\dim M_k(\Gamma) \leq \begin{cases} [k/12]+1 & \text{if } k \not\equiv 2 \mod 12 \\ [k/12] & \text{if } k \equiv 2 \mod 12 \end{cases}$$

Pf: $m = [k/12]+1$ take m distinct non-elliptic pts $P_i \in X_\Gamma$

for $f_1, \dots, f_{m+1} \in M_k(\Gamma)$ $\exists$ lin. comb. f vanishing at all P_i (lin alg.)

but then by order counting above $f \equiv 0$ since $m > \frac{k}{12}$ $\Rightarrow$ So the f_i are lin. dep.

$\Rightarrow \dim M_k(\Gamma) \leq m$

- if $k \equiv 2 \mod 12$ improve by one as to satisfy ord zero relation need at least simple zero at i and double at ω giving $\frac{1}{2} + \frac{2}{3} = \frac{7}{6}$

together with $\frac{k}{12} - \frac{7}{6} = m-1$ other zeros

$\Rightarrow$ get same but with $\dim M_k(\Gamma) \leq m-1$

[17] for $k=12$ $\quad \dim M_{12}(\Gamma) \leq 2$ $\quad \Gamma = SL(2,\mathbb{R}$

and given two f, g lin indep in $M_{12}(\Gamma)$

$$\frac{f}{g} : \Gamma\backslash\mathbb{H} \xrightarrow{\cong} \mathbb{P}^1(\mathbb{C})$$

Pf:

if f, g lin indep then

for $(\lambda, \mu) \neq (0,0)$ $\quad \lambda f + \mu g$ has

one zero by zero-counting formula

so $\frac{f}{g}$ takes every value $\frac{\mu}{\lambda} \in \mathbb{P}^1(\mathbb{C})$ exactly once

Note: factor $\frac{1}{12}$ in zero-counting formula

vs $\frac{1}{4\pi} \cdot \mathrm{Vol}(\Gamma\backslash\mathbb{H})$ in hyperbolic metric

$$d\mu = \frac{dx\,dy}{y^2} \qquad \mathrm{Vol}(\Gamma\backslash\mathbb{H}) = \int_{\mathcal{F}} d\mu \quad =$$

$$= \int_{-\frac{1}{2}}^{1/2} \left(\int_{\sqrt{1-x^2}}^{\infty} \frac{dy}{y^2} \right) dx$$

$$= \int_{-1/2}^{1/2} \frac{dx}{\sqrt{1-x^2}} = \arcsin(x) \Big|_{-\frac{1}{2}}^{\frac{1}{2}} = \frac{\pi}{3}$$

Similar argument: if $\Gamma \subset SL(2,\mathbb{Z})$ has fundamental domain $\mathcal{F}_\Gamma$ of finite volume

$$V(\Gamma) := \mathrm{Vol}(\mathcal{F}_\Gamma) = \int_{\mathcal{F}_\Gamma} d\mu < \infty$$

then $\dim M_k(\Gamma) \leq \frac{k\,V(\Gamma)}{4\pi} + 1$

also gives finite dimensionality of space of modular forms

In particular always $M_k(\Gamma) = 0 \quad k < 0$
$M_0(\Gamma) = \mathbb{C}$ (holom!)

Examples of modular forms:

- $E_k(z)$ Eisenstein series of weight $k>2$
- $\Delta(z)$ discriminant function of weight 12

Notation: $f|_k \gamma(z) := (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$

and more generally $\gamma \in SL(2,\mathbb{R})$

$f|_k \gamma_1\gamma_2 = (f|_k \gamma_1)|_k \gamma_2$ on holom. functions on $\mathbb{H}$ w/ growth condition

then $M_k(\Gamma)$

$\quad\quad \parallel$

fixed by $f \mapsto f|_k \gamma$ action

Constructing invariants:

G finite group $\quad$ V vector space

$v_0 \in V$ given vector $\quad v := \sum_{g \in G} \underbrace{v_0|g}_{\text{denoting action of } G \text{ on } V}$

by construction is G-invariant

if G infinite same idea but convergence issue if v_0 <u>fixed</u> by an infinite subgroup $G_0 \subset G$

$\Rightarrow$ $v_0|g$ depends on g only through coset $G_0 g \in G_0\backslash G$

$\Rightarrow$ <u>smaller</u> sum

$$\sum_{g \in G_0\backslash G} v_0|g$$

better chances at convergence

$\Gamma \subset SL(2,\mathbb{R})$ discrete subgroup (Fuchsian group)

v_0 rational function $\Rightarrow$ Poincaré series

in particular for $v_0 = 1$ constant function

$\Gamma_0 := \Gamma_\infty$ = stabilizer of cusp at ∞

$$\Rightarrow \sum_{\gamma \in \Gamma_\infty\backslash\Gamma} 1|_k \gamma$$

Eisenstein series

$\Gamma = SL(2,\mathbb{Z})$ (or for more gen. Γ discr. $\subset SL(2,\mathbb{R})$)

for $\underbrace{\qquad\qquad}$:

any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$ sends ∞ to $\frac{a}{c}$

$\Rightarrow$ $\gamma \in \mathrm{Stab}(\infty)$ iff $c = 0$

$\Rightarrow$ $\gamma = \pm\begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} = \pm T^n$

k even (no mod forms odd deg) $\Rightarrow$ take $\overline{\Gamma} = PSL(2,\mathbb{Z})$

$\Rightarrow \bar{\Gamma}_\infty =$ infinite cyclic group gen. by T

$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad T^n \cdot \gamma = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$= \begin{pmatrix} a+nc & b+nd \\ c & d \end{pmatrix} = \gamma'$

γ' same bottom row as γ

also check that this is iff

same bottom row $\Rightarrow \gamma' = T^n \cdot \gamma$

every pair of coprime integers $(c,d)=1$
occurs as bottom row of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$

$\Rightarrow$ can rewrite the Eisenstein series as

$$E_k(z) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} 1|_k \gamma = \sum_{\gamma \in \bar{\Gamma}_\infty \backslash \bar{\Gamma}} 1|_k \gamma =$$

$\left(\begin{array}{l} \Gamma = SL(2,\mathbb{Z}) \\ \bar{\Gamma} = PSL(2,\mathbb{Z}) \end{array} \right)$

$$= \frac{1}{2} \sum_{\substack{c,d \in \mathbb{Z} \\ (c,d)=1}} \frac{1}{(cz+d)^k}$$

because
(c,d) and $(-c,-d)$ are same element of $\Gamma \backslash \bar{\Gamma}$ so don't overcount

- absolutely convergent for $k>2$
- $E_k(z)$ is then a mod form weight k, all even $k \geq 4$

22

Also describe the Eisenstein series as functions on lattices $\Lambda \subset \mathbb{C}$ w/ homogeneity cond of weight k

$$F(\lambda \Lambda) = \lambda^{-k} F(\Lambda)$$

construct such a function as a series

$$G_k(\Lambda) = \frac{1}{2} \sum_{\lambda \in \Lambda \setminus \{0\}} \lambda^{-k}$$

(avoids counting λ & $-\lambda$ as separate)

for $\Lambda = \mathbb{Z} + \mathbb{Z} z$

$$G_k(z) = \frac{1}{2} \sum_{\substack{m,n \in \mathbb{Z} \\ (m,n) \neq (0,0)}} \frac{1}{(mz+n)^k} \quad \left(\begin{matrix} \text{abs conv.} \\ \text{loc unif. conv} \\ k > 2 \end{matrix} \right)$$

$$\Rightarrow G_k \in M_k(\Gamma) \qquad \Gamma = SL(2,\mathbb{Z})$$

Relation to $E_k(z)$:

any $(m,n) \neq (0,0)$ is $(m,n) = r \cdot (c,d)$
with some $r \in \mathbb{N}$ $\quad r = \gcd(m,n)$
and $(c,d) = 1$ rel. prime

$$\Rightarrow \quad G_k(z) = \zeta(k) E_k(z) \qquad \zeta(k) = \sum_{r>0} \frac{1}{r^k}$$

sometimes different normalization

$$\mathbb{G}_k(z) = \frac{(k-1)!}{(2\pi i)^k} G_k(z)$$

} so that Fourier coeff's are in $\mathbb{Q}$

(normalized eigenfunct. of Hecke operators)

The ring $M_*(\Gamma)$ for $\Gamma = SL(2,\mathbb{Z})$ is freely generated by E_4 and E_6

Pf: to see that E_4 & E_6 are alg. independent:

- $E_4(z)^3$ & $E_6(z)^2$ of weight 12 cannot be proportional

if had $E_6(z)^2 = \lambda E_4(z)^3$ then

$f(z) = \dfrac{E_6(z)}{E_4(z)}$ merom. modular of weight 2 would have

$f(z)^2 = \lambda E_4$ and $f^3 = \lambda^{-1} E_6$

so would be holomorphic

but if $f(z)^2$ holom. $\Rightarrow$ $f(z)$ has no poles

$\Rightarrow$ cannot be $\dim M_2(\Gamma) \leq 0$ as dim counting gives

... but (any two mod. forms f_1, f_2 same weight not proportional must be alg. independent because given any polyn.

$P(X,Y) \in \mathbb{C}[X,Y]$ with $P(f_1(z), f_2(z)) \equiv 0$
(alg. dependent)

then by weight have $P_d(f_1, f_2) \equiv 0$

homogeneous terms of degree d

but also have

$$\frac{P_d(f_1,f_2)}{f_2^d} = p(f_1/f_2)$$ for some one-variable polynomial p

fin. many roots

$\left(P_d(f_1,f_2) \equiv 0 \iff f_1/f_2 = \text{constant} \right)$

$\Rightarrow$ E_4^3 and E_6^4 alg. independent

E_4 and E_6 alg. independent

- then dim of weight k part of $M_*(\Gamma)$ $\Gamma = SL(2,\mathbb{Z})$ that E_4 & E_6 generate equals expected one (skip details)

What happens w/ Eisenstein series in weight $k=2$?

- absolute conv. of series for $k>2$ only
- but Fourier series of $G_k(z)$ conv. (to holom. fnct) also for $k=2$

25 So can define Eisenstein series weight 2

$\mathbb{G}_2$, G_2, E_2 by their Fourier series

(skip explicit derivation of Fourier fn $\mathbb{G}_k(\tau)$)

$$\mathbb{G}_2(z) = -\frac{1}{24} + \sum_{n=1}^{\infty} \sigma_1(n)\, q^n$$

$$G_2(z) = -4\pi^2 \mathbb{G}_2(z) \qquad E_2(z) = \frac{6}{\pi^2} G_2(z)$$

where $\sigma_{k-1}(n)$ = sum of the $(k-1)$-th powers of the positive divisors of n

then these satisfy

$$G_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 G_2(z) - \pi i c(cz+d)$$

quasimodular form

while taking

$$\begin{cases} G_2^*(z) = G_2(z) - \dfrac{\pi}{2y} \\ E_2^*(z) = E_2(z) - \dfrac{3}{\pi y} \\ \mathbb{G}_2^*(z) = \mathbb{G}_2(z) + \dfrac{1}{8\pi y} \end{cases}$$

almost modular forms

(almost holomorphic mod forms)

ring of almost mod fms

polyn ring / $\mathbb{C}$ in generators

$E_2(\tau)$

$E_4(\tau)$

$E_6(\tau)$

quasimod form is holom. part of an almost modular form

(determined by its holom. part

Discriminant function

$$\Delta(z) = e^{2\pi i z} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z})^{24}$$

$z \in \mathbb{H}$ so $|e^{2\pi i z}| < 1$ so ∞-prod converges
$\Rightarrow$ holomorphic nowhere vanishing function on $\mathbb{H}$

[$\Delta(z)$ is a modular form of weight 12 for $\Gamma = SL(2,\mathbb{Z})$]

Pf: take log derivative (since nonzero)

$$\frac{1}{2\pi i} \frac{d}{dz} \log \Delta(z) = 1 - 24 \sum_{n=1}^{\infty} \frac{n\, e^{2\pi i n z}}{1 - e^{2\pi i n z}}$$

$$= 1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)\, e^{2\pi i m z}$$

by expanding $\dfrac{e^{2\pi i n z}}{1 - e^{2\pi i n z}}$ as geometric series $\displaystyle\sum_{r=1}^{\infty} e^{2\pi i r n z}$

& exchange order of summation and keep track of multiplicities & counting of variable $m = rn$

then $1 - 24 \sum_{m=1}^{\infty} \sigma_1(m)\, e^{2\pi i m z} = E_2(z)$

then using the quasi modularity / almost modularity of weight 2 Eisenstein ser.

get transformation for Δ :

$$\frac{1}{2\pi i} \frac{d}{dz} \log\left(\frac{\Delta\left(\frac{az+b}{cz+d}\right)}{(cz+d)^{12}\Delta(z)} \right) = \frac{1}{(cz+d)^2} E_2\left(\frac{az+b}{cz+d}\right)$$

$$- \frac{12}{2\pi i} \frac{c}{cz+d} - E_2(z) = 0$$

So get $(\Delta|_{12}\gamma)(z) = C(\gamma)\,\Delta(z)$

for some $C(\gamma) \in \mathbb{C}$
$\neq 0$

to get modularity need $C(\gamma) = 1$

Note that $C : \Gamma \to \mathbb{C}$ is homomorphism
(because $\Delta \mapsto \Delta|_{12}\gamma$ group action of Γ)

so check $C(\gamma)$ for $\gamma = T$ and $\gamma = S$ suffices

- $C(T) = 1$ because $\Delta(z)$ power series in $e^{2\pi i z}$
  $\Rightarrow$ periodic in $z \mapsto z+1$
- $\Delta(-1/z) = C(S)\, z^{12}\, \Delta(z)$ but $\Delta(i) \neq 0$
  $\Rightarrow C(S) = 1$

$\dim \mathcal{M}_{12}(\Gamma) \leq 2$ for $\Gamma = SL(2,\mathbb{Z})$

so $\Delta(z)$ has to be a linear combination of $E_4(z)^3$ and $E_6(z)^2$

check first terms of Fourier expansion

$E_4^3 = 1 + 720q + \cdots$

$E_6^2 = 1 - 1008\, q + \cdots$

$\Delta(z) = q + \cdots$

so $\Delta(z) = \frac{1}{1728}\left(E_4(z)^3 - E_6(z)^2 \right)$

<u>Consequence</u>: every mod form for $\Gamma = SL(2,\mathbb{Z})$ is polyn. in E_4 & E_6 more explicitly:

induction on weight:

$f(z)$ mod of weight $k \geq 4$ $\quad f(z) = \sum_n a_n q^n$

take integers $a, b \geq 0$ s.t. $4a + 6b = k$

then take $h(z) = \dfrac{\left(f(z) - a_0 E_4(z)^a E_6(z)^b \right)}{\Delta(z)}$

is holomorphic in $\mathbb{H}$ since Δ nowhere vanishing and <u>also at ∞</u>

because $f(z) - a_0 E_4(z)^a E_6(z)^b$ has constant term $= 0$ in $\sum_n b_n q^n$ expansion and exp of Δ starts w/ q

$\Rightarrow h$ is a mod form of weight $k-12$

by induction on weight

then h is a polyn. in E_4, E_6

and so also $f = a_0 E_4^a E_6^b + \Delta \cdot h$ also is

Note : the isom. $\Gamma \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) = \overline{X}_\Gamma \xrightarrow{\simeq} \mathbb{P}^1(\mathbb{C})$

$\Gamma = SL(2,\mathbb{Z})$

is realized by the function

$$j(z) = \frac{E_4(z)^3}{\Delta(z)} = q^{-1} + 744 + 196884\, q + \cdots$$

"modular invariant"

since Δ nowhere vanishing on $\mathbb{H}$ $\Rightarrow$ j finite on $\mathbb{H}$

so isom $\Gamma \backslash \mathbb{H} \simeq \mathbb{C}$ and matches cusp to pt at ∞

so $\overline{X}_\Gamma \xrightarrow{\simeq} \mathbb{P}^1(\mathbb{C})$

---

cusp forms : modular forms

$$f(z) = \sum_n a_n q^n \qquad q = e^{2\pi i z}$$

where $a_0 = 0$

since constant term of Fourier exp. of Eisenstein series $G_k(z)$ is $\neq 0$

every mod form for $SL(2,\mathbb{Z})$ is a sum of an Eisenstein series and a cusp form

---

estimates on growth of coefficients:
for $G_k(z)$ explicit form and growth like

$$n^{k-1} \leq \sigma_{k-1}(n) \leq \zeta(k-1)\, n^{k-1}$$

for the cusp forms there is also an estimate:

$f(z)$ cusp form weight k for $SL(2,\mathbb{Z})$
$= \sum_{n=1}^{\infty} a_n q^n$ then $|a_n| \leq C\, n^{k/2}$
where C only depends on f

---

cusp forms for other $\Gamma \subset SL(2,\mathbb{Z})$:

now not all $\mathbb{P}^1(\mathbb{Q})$ identified to single pt in $\overline{X}_\Gamma = \Gamma \backslash \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

but can have more than one cusp

so would have to check the Fourier expansion at each cusp and vanishing of constant term in each of these expansions

easier equivalent condition:

cusp forms (for Γ) of weight k: mod forms for which

$$y^{k/2} f(x+iy) \text{ is bounded}$$
