NONCOMMUTATIVE MIXMASTER COSMOLOGIES

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Abstract. In this paper we investigate a variant of the classical mixmaster universe model of anisotropic cosmology, where the spatial sections are noncommutative 3-tori. We consider ways in which the discrete dynamical system describing the mixmaster dynamics can be extended to act on the noncommutative torus moduli, and how the resulting dynamics differs from the classical one, for example, in the appearance of exotic smooth structures. We discuss properties of the spectral action, focussing on how the slow-roll inflation potential determined by the spectral action affects the mixmaster dynamics. We relate the model to other recent results on spectral action computation and we identify other physical contexts in which this model may be relevant.

Contents

1. Introduction

Noncommutative geometry has been used extensively as a method for the construction of models of particle physics, cosmology, and gravity coupled to matter. The point of view adopted in these models is conceptually similar to the extra dimensions of string theories, in the sense that one replaces a 4-dimensional spacetime manifold with a product (or fibration), where the fibers are the "extra dimensions". However, unlike in string theory, these extra dimensions are not manifolds but noncommutative spaces. Moreover, on this product geometry encompassing spacetime directions and noncommutative extra dimensions, one has an action functional, the *spectral action*, which is a natural action functional for pure gravity in noncommutative geometry. The key idea is that pure gravity on a noncommutative space that is the product of a spacetime manifold and a suitable noncommutative fiber looks like gravity coupled to matter from the spacetime point of view. More precisely, the asymptotic expansion for the spectral action functional delivers terms that contain gravity terms such as the Einstein–Hilbert action and a Lagrangian for the matter content of the model, which depends on the choice of the finite geometry. For a simple choice of a finite dimensional algebra as the fiber noncommutative space one recovers the Standard Model Lagrangian [14], extensions of the Minimal Standard Model that include right handed neutrinos [12] and supersymmetric extensions [8].

Unlike other high-energy physics models involving noncommutativity, such as those arising in some string theory compactifications, there is here no noncommutativity in the spacetime coordinates, only in the extra dimensions. Moreover, these particle physics models live naturally at unification energy. While there is, at this point, no clear picture of how the models should be extended to higher energies, it has been frequently proposed that, when moving towards the Planck scale, the model should incorporate more noncommutativity, which will eventually involve the spacetime coordinates as well. Moreover, one expects that more "seriously noncommutative" spaces (that is, not Morita equivalent to commutative ones) should appear as one approaches the Planck scale. In this paper we do not attempt to answer the question of how to extend the noncommutative geometry models of gravity coupled to matter towards the Planck scale, but we describe a toy model case for what a behavior of the type suggested would look like, in a geometry that is at the same time sufficiently simple to be explicitly computable, but sufficiently complex to exhibit a nontrivial behavior.

Our model is constructed by adapting a very well known example of classical cosmologies that exhibit a chaotic behavior, namely the mixmaster cosmological models of [5], [26], based on the Kasner metrics and on a discrete dynamical system related to the continuous fraction expansion, that governs the succession of mixmaster cycles and Kasner epochs. We input the noncommutativity in this model by replacing the spatial sections of this cosmology by noncommutative tori.

In §2 we recall the classical mixmaster universe model, which we formulate in the case where the spatial sections are 3-dimensional tori T^3. We recall the main properties of the discrete dynamical system that models the mixmaster dynamics, and its relation to the Kasner metrics and the Kasner epochs and cycles of this anisotropic and chaotic universe model.

In §3 we recall some basic properties of noncommutative 3-tori, as noncommutative algebras and as spectral triples (noncommutative Riemannian manifolds). We describe two different possible ways to extend the mixmaster dynamical system to act on the moduli of the noncommutative tori and not only on their metric structure. We show that one of these models leads naturally to the occurrence of exotic smooth structures in this noncommutative cosmological model.

In §4 we discuss inflation scenarios derived from the spectral action functional and we construct a toy model of deformations of parameters that gives rise to a transition from an early universe noncommutative mixmaster cosmology to a commutative inflationary cosmology.

Finally, in §5 we discuss other aspects of the model, related to properties of the spectral action, and we identify physical settings in which this type of model may be relevant.

2. The mixmaster universe

We review briefly in this section the basic properties of the mixmaster universe models in general relativity, and some aspects of the geometry that we need for the noncommutative generalization that follows.

2.1. **Kasner metrics.** The mixmaster universe models were developed (see [5], [26]) as interesting cosmological models exhibiting strong anisotropy and chaotic behavior. One considers anisotropic metrics

$$ds^2 = -dt^2 + a(t)^2\, dx^2 + b(t)^2\, dy^2 + c(t)^2 dz^2, \tag{2.1}$$

with the functions $a(t)$, $b(t)$, $c(t)$ of the Kasner form

$$ds^2 = -dt^2 + t^{2p_1} dx^2 + t^{2p_2} dy^2 + t^{2p_3} dz^2, \tag{2.2}$$

where the exponents p_1, p_2, p_3 satisfy the constraints $\sum_i p_i = 1$ and $\sum_i p_i^2 = 1$.

2.2. **Discrete dynamical system.** The mixmaster universe is a solution of the Einstein equation, built out of approximate solutions that look like Kasner metrics for certain intervals of time (Kasner eras), with a discrete dynamical system governing the transition from one era to the next in terms of the change of Kasner exponents in the metric, see [33] and [27], [28]. One sets

$$\begin{aligned} p_1 &= \frac{-u}{1+u+u^2} \\ p_2 &= \frac{1+u}{1+u+u^2} \\ p_3 &= \frac{u(1+u)}{1+u+u^2} \end{aligned} \tag{2.3}$$

The dynamics is discretized by starting, at the beginning of each Kasner era, with a value $u_n > 1$ of the parameter u. Each era is then subdivided into shorter cycles, determined by decreasing values u_n, $u_n - 1$, $u_n - 2, \ldots$. Within each of these cycles the metric is approximated with a Kasner metric (2.2) with exponents (2.3) with fixed $u = u_n - k$. This sequence of Kasner cycles stops when the next value $u_n - k$ becomes smaller than one (but still positive). Then one passes to the next Kasner era with the transformation $u \mapsto 1/u$ and restarts the sequence of Kasner cycles from this new value. Thus, the transformation of the parameter u that marks the passage from the beginning of one Kasner era to the the beginning of the next is the well known dynamical system

$$T : u_n \mapsto u_{n+1} = \frac{1}{u_n - [u_n]}, \tag{2.4}$$

which is the shift of the continued fraction expansion $Tx = 1/x - [1/x]$ with $x_{n+1} = Tx_n$ and $u_n = 1/x_n$.

Moreover, at the start of each new Kasner era, a permutation of the three spatial directions occurs, which reassigns the role of the direction responsible for the main dilation or contraction and of the two oscillating directions. As shown in [27], [28], in terms of the dynamical system (2.4) and the shift of the continued fraction expansion, this permutation can be described in the following way. Identify the three spatial directions with the three points of $\mathbb{P}^1(\mathbb{F}_2)$ via $1 \mapsto x$; $\infty \mapsto y$; $0 \mapsto z$. Then the shift $Tx = 1/x - [1/x]$ of the continued fraction expansion on $[0,1]$ extends to a discrete dynamical system on $[0,1] \times \mathbb{P}^1(\mathbb{F}_2)$ by

$$T : (x, s) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right). \tag{2.5}$$

This dynamical system (and its invertible two-sided extension) is in fact the dynamical system that gives the coding of geodesics on the modular curve $X_0(2) = \mathbb{H}/\Gamma_0(2)$, as shown in [27], [28], which leads to the description of solutions of the mixmaster universe dynamics in terms of geodesics on this modular curve, as recalled in Section 2.3 below.

Clearly, the description of mixmaster universe cosmologies in terms of this discrete dynamical system only leads to the construction of an approximate solution of the Einstein equation, and one can then argue with more subtle analytic methods in what sense this approximate solution is close to an actual solution, but we will not be dealing with the approximation problem in this paper.

2.3. **Mixmaster data and geodesics on the modular curve $X_0(2)$.** It was shown in [27], [28], that the solutions of the discretized mixmaster dynamics are in one-to-one correspondence with geodesics on the modular curve $X_0(2) = \mathbb{H}/\Gamma_0(2)$, with $\Gamma_0(2) \subset \mathrm{SL}_2(\mathbb{Z})$ the congruence subgroup of level two.

Recall from [27], [28] that every infinite geodesic on $X_0(2)$ not ending at the cusp can be coded by data $(\omega, s) = (\omega^-, \omega^+, s)$, with $\omega^\pm \in \mathbb{P}^1(\mathbb{R})$ and $s \in \mathbb{P}^1(\mathbb{F}_2)$, and where $\omega^\pm$ can be chosen with $\omega^+ \in [0,1] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$ and $\omega^- \in (-\infty, -1] \cap (\mathbb{R} \smallsetminus \mathbb{Q})$. These can be written in terms of continued fraction expansion as $\omega^+ = [k_0, k_1, k_2, \ldots, k_n, \ldots]$ and $\omega^- = [k_{-1}; k_{-2}, \ldots, k_{-n}, \ldots]$ and are acted upon by the shift as

$$T(\omega^+, s) = \left(\frac{1}{\omega^+} - \left[\frac{1}{\omega^+}\right], \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right)$$

$$T(\omega^-, s) = \left(\frac{1}{\omega^- + [1/\omega^+]}, \begin{pmatrix} -[1/\omega^+] & 1 \\ 1 & 0 \end{pmatrix} \cdot s \right).$$

Geodesics on $X_0(2)$ are parameterized by the orbits of the data (ω, s) under the action of the shift T.

The data (ω, s) in turn determine a solution of the mixmaster dynamics, by assigning the $[u_n]$ to be the digits k_n of the continued fraction expansion of $\omega^\pm$ and the alternation of the spatial directions being determined by the action of T on the element $s \in \mathbb{P}^1(\mathbb{F}_2)$, according to the identification mentioned above between points of $\mathbb{P}^1(\mathbb{F}_2)$ and spatial axes.

2.4. **Mixmaster tori.** Observe that the metric (2.2) can be considered equally on a spacetime whose spatial sections are topologically a flat space $\mathbb{R}^3$ or whose sections that are topologically tori T^3. We will focus on the latter possibility for our noncommutative model. Thus, we assume that the spacetime manifold is topologically a cylinder $T^3 \times \mathbb{R}$, endowed with the Lorentzian metric of the Kasner form (2.2).

In the case of a torus T^3, the evolution in time (T^3_t, g_t) with the Kasner metric $g_t = t^{2p_1}dx^2 + t^{2p_2}dy^2 + t^{2p_3}dz^2$ has volume $\mathrm{Vol}(T^3_t) = t\mathrm{Vol}(T^3)$, since $\mathrm{Vol}(T^3_t) = t^{p_1+p_2+p_3}\mathrm{Vol}(T^3)$, where we are assuming that $p_1 + p_2 + p_3 = 1$.

Consider the Dirac operator $\partial\!\!\!/_t$ on (T^3_t, g_t), associated to a choice of a spin structure $\mathfrak{s}$ on T^3. On the 3-torus T^3 there are eight different spin structures $\mathfrak{s}_j$. We recall the following result (see [2], [23]) on the Dirac spectrum.

Let $T^3 = \mathbb{R}^3/\Lambda$ be a 3-dimensional torus, with Λ a lattice in $\mathbb{R}^3$. Let $\{\tau_1, \tau_2, \tau_3\}$ be a basis for Λ and let $\Lambda^\vee$ be the dual lattice with dual basis $\{w_1, w_2, w_3\}$. The eight spin structures are classifies by eight vectors $\{\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3) \,|\, \mathfrak{s}_i \in \{0, 1\}\}$, where the value of each $\mathfrak{s}_i$ distinguishes whether the spin structure on each of the three directions v_i is twisted or untwisted. In fact, on the circle S^1, the spinors for the two possible spin structures can be identified with

$$\Gamma(S^1, \mathbb{S}) = \{\psi : \mathbb{R} \to \mathbb{C} \,|\, \psi(t + 2\pi) = \pm\psi(t)\},$$

and the Dirac operator $-i\frac{d}{dt}$ has eigenfunctions $\psi_k(t) = \exp(2k\pi i t)$ for the trivial spin structure and $\psi_k(t) = \exp((2k + 1)\pi i t)$ for the other one.

On $T^3 = \mathbb{R}^3/\mathbb{Z}^3$, the Dirac operator $\partial\!\!\!/$ is of the form

$$\partial\!\!\!/ = -i(\sigma_1 \frac{\partial}{\partial x} + \sigma_2 \frac{\partial}{\partial y} + \sigma_3 \frac{\partial}{\partial z}) = \begin{pmatrix} -i\frac{\partial}{\partial z} & -\frac{\partial}{\partial y} - i\frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} - i\frac{\partial}{\partial x} & i\frac{\partial}{\partial z} \end{pmatrix}. \tag{2.6}$$

More generally, on $T^3 = \mathbb{R}^3/\Lambda$, one can write the Dirac operator in the form (up to a possible overall additive shift)

$$\begin{aligned} \partial\!\!\!/ &= -i\sum_{j=1}^{3}(\tau_j \cdot \underline{\partial})\sigma_j = -i((\tau_1 \cdot \underline{\partial})\sigma_1 + (\tau_2 \cdot \underline{\partial})\sigma_2 + (\tau_3 \cdot \underline{\partial})\sigma_3) \\ &= \begin{pmatrix} -i\partial_{\tau_3} & -\partial_{\tau_2} - i\partial_{\tau_1} \\ \partial_{\tau_2} - i\partial_{\tau_1} & i\partial_{\tau_3} \end{pmatrix}, \end{aligned} \tag{2.7}$$

where $\underline{\partial} = (\partial_1, \partial_2, \partial_3) = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$, $\partial_{\tau_j} = \tau_j \cdot \underline{\partial}$, and σ_j are the Pauli matrices.

Then on the 3-torus $T^3 = \mathbb{R}^3/\Lambda$, the Dirac operator $\partial\!\!\!/$ on the spin structure $\mathfrak{s}$ has spectrum

$$\mathrm{Spec}(\partial\!\!\!/) = \{\pm 2\pi\|w + \frac{1}{2}\sum_{j=1}^{3}\mathfrak{s}_j w_j\| \,|\, w \in \Lambda^\vee\}. \tag{2.8}$$

In particular, in the case of the torus $T^3_t = \mathbb{R}^3/\Lambda_t$, where the dual lattice $\Lambda_t^\vee$ is spanned by the basis of vectors $\{t^{p_1}e_1, t^{p_2}e_2, t^{p_3}e_3\}$, with e_i the standard orthonormal basis, the Dirac operator is given by

$$\partial\!\!\!/ = -i(\sigma_1 t^{-p_1}\frac{\partial}{\partial x} + \sigma_2 t^{-p_2}\frac{\partial}{\partial y} + \sigma_3 t^{-p_3}\frac{\partial}{\partial z}) \tag{2.9}$$

$$= \begin{pmatrix} -it^{-p_3}\frac{\partial}{\partial z} & -t^{-p_2}\frac{\partial}{\partial y} - it^{-p_1}\frac{\partial}{\partial x} \\ t^{-p_2}\frac{\partial}{\partial y} - it^{-p_1}\frac{\partial}{\partial x} & it^{-p_3}\frac{\partial}{\partial z} \end{pmatrix}.$$

The spectrum of $\partial\!\!\!/_t$, for the spin structure $\mathfrak{s}$, will then be of the form

$$\mathrm{Spec}(\partial\!\!\!/_t) = \{\pm 2\pi \|(t^{-p_1}k, t^{-p_2}m, t^{-p_3}n) + \frac{1}{2}\sum_{j=1}^{3}\mathfrak{s}_j t^{-p_j}e_j\| \,|\, (k,m,n)\in\mathbb{Z}^3\}. \tag{2.10}$$

2.5. **Mixmaster dynamics on classical tori.** The dependence of the Kasner exponents on the u-parameter as in (2.3) and the discrete dynamical system (2.4), with the permutations (2.5), correspondingly determine a sequence of 3-tori $T^3_{u_n}$ with Dirac operators as in (2.10), where the exponents p_i are functions of $u = u_n$ through (2.3) and the permutation (2.5) of the coordinate axes.

As in Section 2.3, passing of each Kasner era is marked by the transition $u_n \mapsto u_{n+1}$ and $s_n \mapsto s_{n+1}$, where $u_n = 1/x_n$ and $x_{n+1} = Tx_n$, and with $s_n \in \mathbb{P}^1(\mathbb{F}_2)$, with $s_{n+1} = Ts_n$ with the map T as in (2.5). In particular, the action of T that gives the transition from the nth to the $(n+1)$st Kasner era is given by the action of the matrix

$$\gamma_n = \begin{pmatrix} -[u_n] & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}), \tag{2.11}$$

which simultaneously acts on changing the metric of the torus and on permuting its three generators.

3. Noncommutative 3-tori and mixmaster cosmologies

We now reconsider the model of mixmaster universe described above, in a setting where the three spatial coordinates that give the 3-torus T^3_t are replaced by a *noncommutative* 3-torus $\mathbb{T}^3_\Theta$. We describe these noncommutative spaces as Riemannian geometries, in the noncommutative sense, that is, as *spectral triples*. We then propose different possible ways in which the discrete dynamical system that describes the evolution of a classical mixmaster cosmology can be extended to involve also an action on the parameters (the moduli) of the noncommutative torus itself, so that not only the metric structure, but also the underlying noncommutative space, evolves along with the succession of Kasner epochs.

3.1. **3-tori as spectral geometries.** In noncommutative geometry the analog of a Riemannian spin manifold is described by the data of a *spectral triple*. These consist of a triple $(\mathcal{A}, \mathcal{H}, D)$, where $\mathcal{A}$ is an involutive, dense subalgebra of a C^*-algebra closed under holomorphic functional calculus, together with a (faithful) representation $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ as bounded operators on a separable Hilbert space $\mathcal{H}$, and a "Dirac operator" D. The latter is a self-adjoint (unbounded) operator, densely defined on $\mathcal{H}$, with compact resolvent and satisfying the condition of having bounded commutators with the elements of the algebra, $[\pi(a), D] \in \mathcal{B}(\mathcal{H})$, for all $a \in \mathcal{A}$.

A smooth compact Riemannian spin manifold X is a special case of a spectral triple, where the data $(\mathcal{A}, \mathcal{H}, D)$ are given by $\mathcal{A} = \mathcal{C}^\infty(X)$, $\mathcal{H} = L^2(X, \mathbb{S})$, with $\mathbb{S}$ the spinor bundle, and $D = \partial\!\!\!/_X$ the Dirac operator.

Thus, the mixmaster tori $T^3_t = \mathbb{R}^3/\Lambda_t$ described above are spectral triples with $\mathcal{A} = \mathcal{C}^\infty(T^3)$, and with $\mathcal{H} = L^2(T^3_t, \mathbb{S}_\mathfrak{s})$, where $\mathbb{S}_\mathfrak{s}$ is the spinor bundle for the spin structure $\mathfrak{s} = (\mathfrak{s}_1, \mathfrak{s}_2, \mathfrak{s}_3)$ as above, with

$$L^2(T^3_t, \mathbb{S}_\mathfrak{s}) = \{\psi : \mathbb{R}^3 \to \mathbb{C}^2 \,|\, \psi \in L^2,\ \ \psi(\underline{x} + t^{p_j} v_j) = (-1)^{\mathfrak{s}_j}\psi(\underline{x})\}, \tag{3.1}$$

for $\underline{x} = (x, y, z) \in \mathbb{R}^3$ and $\{\tau_j\}$ the basis of Λ, and with Dirac operator

$$\partial\!\!\!/ = -i\sum_{j=1}^{3} t^{-p_j}(\tau_j \cdot \underline{\partial})\sigma_j = -i(t^{-p_1}(\tau_1 \cdot \underline{\partial})\sigma_1 + t^{-p_2}(\tau_2 \cdot \underline{\partial})\sigma_2 + t^{-p_3}(\tau_3 \cdot \underline{\partial})\sigma_3). \tag{3.2}$$

Here we scaled the spatial coordinates $x_i \mapsto t^{p_i} x_i$, so that $\partial_{x_i} \mapsto t^{-p_i}\partial_{x_i}$, and fixed the τ_i, instead of scaling $\Lambda \mapsto \Lambda_t$ as before: the resulting (3.2) is the same.

3.2. **Noncommutative 3-tori and their spectral geometry.** A noncommutative 3-torus is the universal C^*-algebra $\mathcal{A}_\Theta$ generated by three unitaries U_1, U_2, U_3 with the relations

$$U_j U_k = \exp(2\pi i \Theta_{jk}) U_k U_j, \tag{3.3}$$

where $\Theta = (\Theta_{jk})$ is a skew-symmetric matrix

$$\Theta = \begin{pmatrix} 0 & \theta_3 & -\theta_2 \\ -\theta_3 & 0 & \theta_1 \\ \theta_2 & -\theta_1 & 0 \end{pmatrix}. \tag{3.4}$$

It can also be described as the twisted group C^*-algebra $C^*_r(\mathbb{Z}^3, \sigma_\Theta)$, where the $U(1)$-valued 2-cocycle σ_Θ is given by (see [18])

$$\sigma_\Theta(\underline{x}, \underline{y}) = \exp(\pi i \langle \underline{x}, \Theta \underline{y}\rangle). \tag{3.5}$$

A detailed discussion of the main properties of noncommutative 3-tori can be found in [4].

Recall that, for a noncommutative 3-torus, the algebra of smooth functions is defined as

$$\mathcal{A}^\infty_\Theta = \{X \in \mathcal{A}_\Theta,\ X = \sum_{m,n,k\in\mathbb{Z}} a_{m,n,k}\, U_1^m U_2^n U_3^k \,|\, a = (a_{m,n,k}) \in \mathcal{S}(\mathbb{Z}^3, \mathbb{C})\}, \tag{3.6}$$

which is the noncommutative analog of functions on the 3-torus with rapidly decaying Fourier coefficients.

It was recently proved by Venselaar in [41] that all equivariant real spectral triples on rank n noncommutative tori are isospectral deformations, in the sense of [15], where spin structures are Dirac operators on commutative flat n-dimensional tori T^n. The case of rank-two tori was previously shown in [37].

Thus, these spectral triples will all be of the form $(\mathcal{A}^\infty_\Theta, L^2(T^3, \mathbb{S}), \partial\!\!\!/)$, where $\mathbb{S}$ is the spinor bundle for one of the eight spin structures on the ordinary torus T^3 and the Dirac operator $\partial\!\!\!/$ is of the form (2.7) (up to an overall additive constant) with $\{\tau_j\}_{j=1,2,3}$ a basis.

3.3. **Noncommutative 3-tori, moduli, and mixmaster evolution.** Now we consider again the discrete dynamical system of mixmaster evolution, where the passage from one Kasner era to the next is determined by the action of the matrix $\gamma_n \in \mathrm{GL}_2(\mathbb{Z})$ of (2.11). In this noncommutative setting, in addition to the action on the exponents of the Kasner metric and the permutation of the spatial directions (here given by a permutation of the three generators of the noncommutative torus algebra) and by the corresponding action on the Dirac operator in (3.2) as in the commutative case, one also has the modulus Θ of the noncommutative torus, which is trivial in the commutative case. Thus, one can propose extensions of the mixmaster dynamics where the modulus Θ is also acted upon in the transition from one Kasner epoch to the other. We present in the following subsections examples of two possible such extensions of the mixmaster dynamics and we illustrate some of their properties. As we discuss below, these will have some interesting consequences on the properties of the noncommutative mixmaster cosmologies that differ from their classical counterparts.

3.4. **Moduli evolution by auxiliary mixmaster data.** In this scenario, we assume given a choice of mixmaster data, by which we mean a choice of a geodesic on the modular curve $X_0(2)$, or equivalently a choice of data (ω, s) up to the action of the shift T as recalled in Section 2.3. This means that we have an assigned sequence of matrices $\gamma_n \in \mathrm{GL}_2(\mathbb{Z})$, of the form (2.11). We also have a sequence s_n of elements in $\mathbb{P}^1(\mathbb{F}_2)$, which determines, in each Kasner era, which spatial direction is the dominant direction driving expansion or contraction.

Recall then that, if we write the modulus for the noncommutative 3-torus as the vector $\underline{\theta} = (\theta_1, \theta_2, \theta_3) \in \mathbb{R}^3$ of the three parameters out of which the skew-symmetric matrix Θ of (3.4) is built, then for any matrix $M \in \mathrm{GL}_3(\mathbb{Z})$ we have an action $\underline{\theta}' = M\underline{\theta}$ and we let Θ' be the skew-symmetric matrix corresponding to the new values $\underline{\theta}'$.

The fact that, in each era of the era of the mixmaster dynamics, there is one of the three spatial directions that dominates the contraction or expansion, identified by the given sequence s_n allows us then to define an action of the matrices $\gamma_n \in \mathrm{GL}_2(\mathbb{Z})$ on the parameters of the noncommutative torus, by embedding $\mathrm{GL}_2(\mathbb{Z})$ inside $\mathrm{GL}_3(\mathbb{Z})$, so that it acts as the identity on the parameter associated to the dominant direction and as γ_n on the other two. Since this action is accompanied by a permutation of the directions, at the following change of Kasner era the dominant direction will have changed and the embedding of $\mathrm{GL}_2(\mathbb{Z})$ inside $\mathrm{GL}_3(\mathbb{Z})$ used in defining the action on $\underline{\theta}$ will change accordingly.

Thus, for example, if at the nth Kasner era the first coordinate is the dominant direction, we obtain the transformation

$$\underline{\theta} = (\theta_1, \theta_2, \theta_3) \mapsto \underline{\theta}' = (\theta_1, -[u_n]\theta_2 + \theta_3, \theta_2). \tag{3.7}$$

As we discuss in the next subsection, the choice of this model for mixmaster evolution of the noncommutative torus moduli has an interesting consequence: along the sequence of Kasner epochs, not only the metric structure of the noncommutative tori undergoes a sequence of transformations analogous to the classical mixmaster dynamics, but at the same time the *smooth structure* of the noncommutative 3-torus undergoes a sequence of transitions to different *exotic structures*.

3.5. **Smooth structures and Kasner eras.** In ordinary commutative geometry, the first occurrence of exotic smooth structures, meaning examples of smooth manifolds that are homeomorphic but not diffeomorphic, occurs in dimension four. In noncommutative geometry, however, the simplest example of exotic smooth structures is known to occur already in three dimensions, for noncommutative 3-tori.

We consider here the algebra $\mathcal{A}_\Theta^\infty$ of smooth functions described in (3.6).

One can then introduce the following two equivalence relations on the modulus Θ (see [4]):

- $\Theta \sim \Theta' \Leftrightarrow \underline{\theta'} = M\,\underline{\theta}$, with $M \in \mathrm{SL}_3(\mathbb{Z})$;
- $\Theta \approx \Theta' \Leftrightarrow \underline{\theta'} = M\,\underline{\theta}$, with $M \in \mathrm{GL}_3(\mathbb{Z})$.

Then (see [4]) one has algebra isomorphisms:

$$\mathcal{A}_\Theta \simeq \mathcal{A}_{\Theta'} \quad \Leftrightarrow \quad \Theta \approx \Theta' \quad \text{while} \quad \mathcal{A}_\Theta^\infty \simeq \mathcal{A}_{\Theta'}^\infty \quad \Leftrightarrow \quad \Theta \sim \Theta'. \tag{3.8}$$

The C^*-algebra case follows from [6], [4] and [21], while the smooth subalgebras case uses [6], [7], [16].

One sees from this result that the noncommutative 3-tori have exotic smooth structures: any two tori with $\Theta \approx \Theta'$ through a matrix $\underline{\theta'} = M\underline{\theta}$ with $M \in \mathrm{GL}_3(\mathbb{Z})$ but $M \notin \mathrm{SL}_3(\mathbb{Z})$ are homeomorphic (in the sense that the algebras of continuous functions are isomorphic) but not diffeomorphic (in the sense that the algebras of smooth functions are not isomorphic).

Thus, an interesting phenomenon happens in the mixmaster dynamics, whereby the passage to each successive Kasner era, which is determined by the action of a matrix $M \in \mathrm{GL}_3(\mathbb{Z})$, which has $\det(M) = \det(\gamma_n) = -1$, changes the torus $\mathbb{T}^3_\Theta$ to a new torus $\mathbb{T}^3_{\Theta'}$ which is homeomorphic, but with a different smooth structure.

The topic of exotic smooth structures and their relevance to physics has been explored in various aspects in recent years, see for instance [1], [17]. This simple observation about the noncommutative tori shows that, when allowing noncommutativity in the space coordinates, one can more easily encounter phenomena involving exotic smoothness, such as, in this case, changes of smooth structure.

3.6. **Moduli evolution by internal mixmaster data.** We propose here another possible way to extend to the torus moduli the discrete dynamical system defining the mixmaster dynamics. In this case, one does not assume a given classical mixmaster solution, but constructs it directly in terms of the torus moduli themselves.

In this case, to define the action of the mixmaster dynamics on the modulus Θ, we recall the following equivalent description of the noncommutative 3-torus $\mathcal{A}_\Theta$, see [4]. One can view the 3-dimensional noncommutative torus $\mathcal{A}_\Theta$ as a crossed product C^*-algebra for an action of $\mathbb{Z}$ on a 2-dimensional noncommutative torus:

$$\mathcal{A}_\Theta = \mathcal{A}_{\theta_3} \rtimes_\alpha \mathbb{Z}, \tag{3.9}$$

where $\mathcal{A}_{\theta_3}$ is the 2-dimensional noncommutative torus generated by two unitaries U and V with the relation $VU = e^{2\pi i\theta_3}UV$, and the action $\alpha : \mathbb{Z} \to \mathrm{Aut}(\mathcal{A}_{\theta_3})$ is given by

$$\alpha(U) = e^{2\pi i\theta_2}U, \quad \alpha(V) = e^{-2\pi i\theta_1}V.$$

One can then change the parameters of the noncommutative 3-torus, in passing to the next Kasner era, by acting on the 2-dimensional noncommutative 2-torus

$\mathcal{A}_{\theta_3}$ by a Morita equivalence, implementing the change of parameter given by the action of the matrix γ_n on θ_3 by fractional linear transformations

$$\theta_3 \mapsto \gamma_n(\theta_3) = \frac{-[u_n]\theta_3 + 1}{\theta_3} = \frac{1}{\theta_3} - [u_n], \tag{3.10}$$

for γ_n as in (2.11), but where now the integers $k_n = [u_n]$ are the digits of the continued fraction expansion of θ_3 itself.

In the particular case where the parameter θ_3 is a quadratic irrationality, namely an irrational number that is contained in some real quadratic field embedded in $\mathbb{R}$, then the digits of the continued fraction expansion of θ_3 are eventually periodic, and there is a natural choice of the mixmaster data (ω, s), with $\omega^\pm = \{\theta_3, \theta_3'\}$, with θ_3' the Galois conjugate of θ_3. This choice corresponds to a closed geodesic in $X_0(2)$.

Notice that here we are acting by Morita equivalences of the 2-dimensional noncommutative torus $\mathcal{A}_{\theta_3}$, which are implemented by an action of $\mathrm{GL}_2(\mathbb{Z})$, while for 3-dimensional smooth noncommutative tori in the generic case, the Morita equivalences are implemented by the action of the group $SO(3,3|\mathbb{Z})$, see [19], [39] and [20] for a complete classification up to Morita equivalences.

4. The spectral action and inflation scenarios

We now consider the action functional, the *spectral action*, for the noncommutative mixmaster cosmologies.

The spectral action functional is defined as $\mathrm{Tr}(f(D/\Lambda))$, where Λ is the energy scale and f is a test function, usually a smooth approximation of a cutoff function. This is regarded as a spectral formulation of gravity in noncommutative geometry. This action functional has an asymptotic expansion at high energies (see (5.10) below). Thus, one can approach the computation of this action functional either by explicit information on the spectrum and a computation of the series defined by $\mathrm{Tr}(f(D/\Lambda))$ (as in [10], [11], [9], [31], [32]), or else through its asymptotic expansion and the computation via heat kernel methods and local expressions in curvature tensors of the various terms in the expansion, as, for instance, in [9], [12], [22], [24].

The computation of the terms in the asymptotic expansion shows (see for instance [12]) that one recovers the terms in the usual classical action for gravity, namely the Einstein–Hilbert action and the cosmological term, together with other "modified gravity" terms, such as a Weyl curvature conformal gravity term. In addition, depending on the possible introduction of a fiber over spacetime given by a finite noncommutative geometry, one obtains further terms that give a Larangian for matter minimally coupled to gravity. This can be the Lagrangian of the minimal standard model (see [14]) or of the standard model with additional right handed neutrinos with Majorana mass terms (see [12]) or models with supersymmetry (see [8]). In addition to the minimal coupling of matter to gravity one also finds terms such as a non-minimal conformal coupling of the curvature to the Higgs field.

All this shows that one can use the spectral action as an action functional either for pure gravity (on a commutative manifold) or for gravity coupled to matter on a product geometry. The main philosophy behind it is that pure gravity on a noncommutative space can manifest itself as gravity coupled to matter when seen from a commutative point of view.

The asymptotic expansion of the spectral action was also computed explicitly for truly noncommutative spaces like noncommutative tori, see [22], [24].

We work here under the assumption that the spectral action is our modified gravity model, for commutative and noncommutative geometries alike and we discuss its behavior in a mixmaster case where the underlying spatial slices are noncommutative 3-tori.

We also discuss possible slow-roll inflation scenarios derived from the spectral action and how they affect and interfere with the underlying mixmaster dynamics.

4.1. **Slow-roll inflation in anisotropic cosmologies.** In the usual isotropic Friedmann cosmologies with Lorentzian metrics of the form

$$ds^2 = -dt^2 + a(t)^2(dx^2 + dy^2 + dz^2),$$

inflation is an accelerated expansion of the universe that corresponds to the scale factor $a(t)$ satisfying $\ddot{a} > 0$. The Friedmann equation relates the scale factor to the Hubble parameter

$$\frac{\dot{a}}{a} = H,$$

and the slow-roll models of inflation are based on the relation of the latter to a scalar field ϕ with potential $V(\phi)$ via

$$H^2 = \frac{8\pi G}{3}(\frac{1}{2}\dot{\phi}^2 + V(\phi)).$$

The slow-roll condition then corresponds to the condition that $\dot{\phi}^2 << |V(\phi)|$, so that the term in $\dot{\phi}$ in the Friedmann equation becomes negligible.

In the case of anisotropic spacetimes of the form (2.1), one introduces an average scale factor

$$\mathfrak{a}(t) = (a(t)b(t)c(t))^{1/3} \tag{4.1}$$

and directional Hubble parameters

$$H_1 = \frac{\dot{a}}{a}, \quad H_2 = \frac{\dot{b}}{b}, \quad H_3 = \frac{\dot{c}}{c},$$

and an average Hubble parameter

$$H = \frac{1}{3}(H_1 + H_2 + H_3).$$

This satisfies

$$H = \frac{1}{3}(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}) = \frac{1}{3}\frac{(\dot{a}bc + \dot{b}ac + \dot{c}ab)}{(abc)^{2/3}} \cdot \frac{1}{(abc)^{1/3}} = \frac{\dot{\mathfrak{a}}}{\mathfrak{a}}.$$

Thus, we obtain the same picture as in the isotropic case, but for the average Hubble parameter and the average scale factor.

In the case of a mixmaster cosmology (2.2), where $p_1 + p_2 + p_3 = 1$, the average scale factor is just given by $\mathfrak{a} = (t^{p_1+p_2+p_3})^{1/3} = t^{1/3}$, with $\dot{\mathfrak{a}}/\mathfrak{a} = (1/3)t^{-1}$. Thus, the Friedmann equation for the Hubble parameter in a pure mixmaster dynamics is of the form

$$\frac{\dot{\mathfrak{a}}}{\mathfrak{a}} = H = \frac{1}{3}t^{-1}, \tag{4.2}$$

or equivalently $H_1 + H_2 + H_3 = t^{-1}$.

For a mixmaster cosmology, one also has

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0,$$

since, for $a(t) = t^{p_1}$, one has $\ddot{a}/a = p_1(p_1 - 1)t^{-2}$, and similarly for the scale factors $b(t)$ and $c(t)$ so that

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = (p_1(p_1 - 1) + p_2(p_2 - 1) + p_3(p_3 - 1))t^{-2},$$

which vanishes, since we assume $p_1 + p_2 + p_3 = 1$ and $p_1^2 + p_2^2 + p_3^2 = 1$.

4.2. **The nonperturbative spectral action for 3-tori.** Since we are dealing with spectral triples that are isospectral deformations of commutative tori, we can refer to the computation of the nonperturbative spectral action obtained in [31] for 3-dimensional flat tori, using the Poisson summation formula as in [11].

Proceeding as in Theorem 8.1 of [31], for a torus T^3 with metric $a(t)^2dx^2 + b(t)^2dy^2 + c(t)^2dz^2$ and Dirac operator

$$\partial\!\!\!/_t = -i(\sigma_1 \frac{1}{a(t)}\frac{\partial}{\partial x} + \sigma_2 \frac{1}{b(t)}\frac{\partial}{\partial y} + \sigma_3 \frac{1}{c(t)}\frac{\partial}{\partial z}), \tag{4.3}$$

the spectral action is of the form

$$\mathrm{Tr}(f(\partial\!\!\!/_t^2/\Lambda^2)) = a(t)b(t)c(t)\frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2 + v^2 + w^2)\, du\, dv\, dw\, + O(\Lambda^{-k}), \tag{4.4}$$

for arbitrary $k > 0$.

Thus, for the mixmaster torus T_t^3 with Dirac operator $\partial\!\!\!/_t$ as in (2.9), one finds, independently of the spin structure,

$$\mathrm{Tr}(f(\partial\!\!\!/_t^2/\Lambda^2)) = t \cdot \frac{\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2 + v^2 + w^2)\, du\, dv\, dw\, + O(\Lambda^{-k}), \tag{4.5}$$

since in this case $p_1 + p_2 + p_3 = 1$, so that $a(t)b(t)c(t) = t^{p_1+p_2+p_3} = t$.

Notice that the factor $a(t)b(t)c(t)\Lambda^3/(4\pi^3)$ in (4.4) behaves exactly like the isotropic case, when one introduces the average scale factor $\mathfrak{a}(t) = (a(t)b(t)c(t))^{1/3}$ as discussed in the previous subsection. In terms of the average scale factor the spectral action has the form

$$\mathrm{Tr}(f(\partial\!\!\!/_t^2/\Lambda^2)) = \frac{\mathfrak{a}(t)^3\Lambda^3}{4\pi^3}\int_{\mathbb{R}^3} f(u^2 + v^2 + w^2)\, du\, dv\, dw\, + O(\Lambda^{-k}). \tag{4.6}$$

4.3. **Noncommutative mixmaster cosmologies and inflation.** We can then adapt the same analysis used in [31], [32], [9] to obtain a slow-roll potential out of a perturbation of the spectral action. To that purpose, we compute the spectral action in 4-dimensions, on a product $T^3_t \times S^1$, with a compactified direction S^1 of size β (after passing to a Wick rotation to Euclidean space and a compactification), with the product Euclidean metric. The replacement $D^2 \mapsto D^2 + \phi^2$ produces a potential for the field ϕ, which, for sufficiently small values of the parameter $x = \phi^2/\Lambda^2$, recovers the usual shape of a quartic potential for the field ϕ, conformally coupled to gravity. This gives, as in [31],

$$\mathrm{Tr}(h(D^2/\Lambda^2)) = \frac{\Lambda^4 \beta a(t)b(t)c(t)}{4\pi} \int_0^\infty u h(u)\, du + O(\Lambda^{-k}). \tag{4.7}$$

The perturbation is then computed as

$$\mathrm{Tr}(h((D^2+\phi^2)/\Lambda^2)) = \mathrm{Tr}(h(D^2/\Lambda^2)) + \frac{\Lambda^4 \beta a(t)b(t)c(t)}{4\pi} \mathcal{V}(\phi^2/\Lambda^2), \tag{4.8}$$

with $\mathcal{V}(x) = \int_0^\infty u(h(u+x) - h(u))\, du$, for $x = \phi^2/\Lambda^2$ where the last term

$$\frac{\Lambda^4 \beta a(t)b(t)c(t)}{4\pi} \mathcal{V}(\phi^2/\Lambda^2)$$

determines the slow-roll potential $V(x)$.

For a Kasner metric, this term is of the form

$$V(x) = \frac{\Lambda^4 \beta\, \mathfrak{a}(t)^3}{4\pi} \mathcal{V}(x), \tag{4.9}$$

with $\mathfrak{a}(t)$ the average scale factor. This calls for a discussion of the t-dependence of the other parameters, Λ and β, in this expression.

4.4. **Time dependence of the parameters.** We have argued in [31] and [32] that in the case of an isotropic expanding cosmology with a single scale factor $a(t)$, the energy scale associated to the cosmological timeline should behave like $\Lambda(t) \sim 1/a(t)$. The possible interpretations of a time dependence for the parameter β are less easily justified, as this parameter is an artifact of the S^1-compactification. We refer the reader to section 3.1 of [32] for a discussion of the interpretation of this parameter as an inverse temperature and its relation to the parameter Λ.

In view of the expression of the spectral action (4.6), in terms of the average scale factor $\mathfrak{a}(t) = (a(t)b(t)c(t))^{1/3}$, it seems natural to apply the same reasoning on the time dependence of the parameter Λ and β, as in [32], in terms of $\mathfrak{a}(t)$. This means, for example, that one expects the energy scale Λ, along the cosmological timeline, to behave like $\Lambda(t) \sim 1/\mathfrak{a}(t)$.

In the usual setting of isotropic Friedmann cosmology, one relates the scale factor and the Hubble parameter to a density function,

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3} \rho(t), \tag{4.10}$$

where the form of the density function depends on the various cosmological eras: in a modern matter-dominated universe, the density function behaves, as a function of time, like $\rho(t) \sim a(t)^{-3}$, which gives an evolution of the scaling factor with power law $a(t) \sim t^{2/3}$; in a radiation-dominated universe the density function behaves like $\rho(t) \sim a(t)^{-4}$, and consequently the scaling factor evolves as $a(t) \sim t^{1/2}$; while in a vacuum-dominated universe, as in the inflation epoch, $\rho(t)$ is constant and $a(t)$ is growing exponentially. In a slow-roll model of inflation, the constant is given by the plateau value of the slow-roll potential.

The latter observation implies that, in the expression (4.9) for the potential obtained from the spectral action, assuming as above that Λ and $\mathfrak{a}$ have an inverse dependence on time, the consistency with (4.10) would then suggest that the parameter β, introduced artificially in the model as a radius of Euclidean compactification, should have a time dependence that is determined by the behavior of the density function $\beta(t) \sim \rho(t)$ (up to a multiplicative constant).

As we will see in §4.6 below, it is convenient to consider different possible t-dependences of the parameter β, as these make it possible to model transitions between different regimes in the very early universe (see also [30] for other discussions of time dependence of parameters in the very early universe in cosmological models based on the spectral action).

Notice that, in any case, the time dependence (and the dependence on β and Λ) is only in the amplitude multiplicative factor of (4.9), and it disappears entirely when one computes the slow-roll parameters, which depend only on the ratios V'/V and V''/V (see [31] and [32]), through the expressions

$$\epsilon = \frac{M_{Pl}^2}{16\pi}\left(\frac{V'}{V}\right)^2, \quad \eta = \frac{M_{Pl}^2}{8\pi}\frac{V''}{V}.$$

4.5. **The scalar field potential and the mixmaster dynamics.** In the presence of a slow-roll potential, an initial mixmaster dynamics gets altered: the coupling of the potential to the average scale factor via the (anisotropic) Friedmann equation disrupts the mixmaster oscillations. This phenomenon was already well known in the case of the classical mixmaster dynamics (see for instance chapter 8 of [35]), where it is used to argue that the potential provides a mechanism of transition from an anisotropic chaotic system to a standard big bang singularity.

We discuss here the effect of the slow-roll potential associated to the spectral action on the underlying mixmaster dynamics, and on the corresponding transformation of the torus moduli.

It is easy to see why, already in the classical case, the presence of a slow-roll potential disrupts a mixmaster dynamics. In fact, suppose given, at time t_0, an initial condition given by the Kasner metric of a Kasner epoch in a mixmaster universe. We have $a(t_0) = t_0^{p_1}$, $b(t_0) = t_0^{p_2}$, $c(t_0) = t_0^{p_3}$. Suppose that a slow-roll potential is turned on, with value near the plateaux level. Then the Friedmann equation predicts an evolution of the average scale factor by $\mathfrak{a}(t) = \mathfrak{a}(t_0)\exp(\gamma(t - t_0))$, where $\gamma = \sqrt{(8\pi G V_\infty)/3}$, with V_∞ the plateaux value of the potential. This is compatible, for example, with a solution for the individual scale factors of the form

$$\begin{aligned} a(t) &= t_0^{1/3} t^{p_1 - 1/3}\exp(\gamma(t - t_0)), \\ b(t) &= t_0^{1/3} t^{p_2 - 1/3}\exp(\gamma(t - t_0)), \\ c(t) &= t_0^{1/3} t^{p_3 - 1/3}\exp(\gamma(t - t_0)), \end{aligned} \tag{4.11}$$

where the exponential factor dominates over the underlying mixmaster dynamics.

4.6. **Damping effects.** It would be interesting to further study if one can use the presence of a slow-roll potential associated to the spectral action to model a transition from a noncommutative early universe to a commutative modern universe, through a simultaneous "override" of the mixmaster oscillations both for the metric parameters and for the noncommutative torus moduli.

In fact, according to the general philosophy about noncommutativity in the spacetime directions, one expects it to appear only close to the Planck scale (or the Planck era in the cosmological timeline), while near the unification scale (and therefore during the inflationary epoch, which is located between the unification and the electroweak epoch), the universe is already exhibiting noncommutativity only in the extra dimensions of the finite geometry that describe matter and forces, but not in the spacetime directions. Thus, one expects to find some mechanisms that damp the noncommutativity before the slow-roll inflation mechanism becomes relevant.

For example, one can conceive of the following type of toy model scenario, where one obtains in two steps a transition from a mixmaster dynamics with noncommutative tori in the very early universe to a slow-roll inflationary cosmology with ordinary commutative tori. The first step is a deformation of noncommutative tori to commutative tori, obtained via a deformation of the Kasner exponents, which disrupts the mixmaster dynamics but leaves the Friedmann equation unaltered, and then a second step, in a purely commutative setting, where a deformation of the parameter β affects a second transition from a regime based on the Friedmann equation for a Kasner metric to the Friedmann equation for an inflationary cosmology with slow-roll potential determined by the spectral action.

4.7. **Deforming the Kasner parameters.** Suppose that the dependence of the exponents $p_i(u)$ of (2.3) on a parameter u is modified from a pure mixmaster dynamics, where u varies over the discrete set u_n determined by the corresponding dynamical system, to a continuous deformation with $u > 0$, with corresponding matrices

$$\gamma(u) = \begin{pmatrix} -u & 1 \\ 1 & 0 \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R}),$$

replacing the sequence of matrices $\gamma_n \in \mathrm{GL}_2(\mathbb{Z})$ of (2.11). The action of γ_n on the torus moduli, in either (3.7) or (3.10), then extends to a similarly defined action of the matrices $\gamma(u) \in \mathrm{GL}_2(\mathbb{R})$, which is no longer by isomorphisms of $\mathcal{A}_\Theta$ (respectively, Morita equivalences of $\mathcal{A}_{\theta_3}$) but by transformations that deform $\mathcal{A}_\Theta$ (respectively, $\mathcal{A}_{\theta_3}$) to a family of non-isomorphic $\mathcal{A}_{\Theta(u)}$ (respectively, non-Morita equivalent $\mathcal{A}_{\theta_3(u)}$).

We can check the effect of such a deformation of the Kasner parameters in (2.3), with $u = u(t)$, on the Friedmann equation. We find

$$\begin{aligned} \frac{\dot a}{a} &= \frac{-u}{1+u+u^2}\, t^{-1} + \frac{\dot u\,(u^2-1)}{(1+u+u^2)^2}\, \log t \\ \frac{\dot b}{b} &= \frac{1+u}{1+u+u^2}\, t^{-1} - \frac{\dot u\, u\,(u+2)}{(1+u+u^2)^2}\, \log t \\ \frac{\dot c}{c} &= \frac{u(1+u)}{1+u+u^2}\, t^{-1} + \frac{\dot u\,(2u+1)}{(1+u+u^2)^2}\, \log t, \end{aligned}$$

so that the Hubble parameter

$$H = \frac{\dot{\mathfrak{a}}}{\mathfrak{a}} = \frac{1}{3}(\frac{\dot{a}}{a} + \frac{\dot{b}}{b} + \frac{\dot{c}}{c}) = \frac{1}{3}t^{-1}$$

remains the same, along the deformation, as the Hubble parameter for the initial mixmaster dynamics (4.2). In fact, the first terms in the above expression for the logarithmic derivatives of the scale factors add up to t^{-1}, while the second terms add up to zero.

Thus, through a deformation $u = u(t)$ of this sort, which modifies the Kasner exponents, one can disrupt the mixmaster dynamics and deform the underlying noncommutative 3-torus $\mathcal{A}_\Theta$ to a commutative torus, through the corresponding action on the torus moduli of the transformations $\gamma(u) \in \mathrm{GL}_2(\mathbb{R})$ as above, while maintaining the Friedmann equation unaffected.

4.8. **Transition to slow-roll inflation.** As a second step, after interrupting the mixmaster oscillation and damping the noncommutativity as described above, we want to obtain a transition from the Friedmann equation for a Kasner metric to the Friedmann equation for an inflationary cosmology. This can also be obtained through a deformation, this time of the parameter β.

In fact, for a fixed plateaux value $\mathcal{V}_\infty$ of the slow-roll potential $\mathcal{V}(x)$ of (4.9), we consider the expression

$$\frac{\Lambda^4(t)\beta(t)\,\mathfrak{a}(t)^3}{4\pi}\mathcal{V}_\infty.$$

Here we may assume, as discussed in §4.4, that $\Lambda(t) \sim \mathfrak{a}(t)^{-1}$, which leaves us with

$$\frac{\Lambda(t)\beta(t)}{4\pi}\mathcal{V}_\infty \sim \frac{\beta(t)}{\mathfrak{a}(t)}\frac{\mathcal{V}_\infty}{4\pi}.$$

A deformation from an initial phase with

$$\beta(t) \sim \frac{4\pi}{3\mathcal{V}_\infty}\frac{\mathfrak{a}(t)}{t}$$

to a later phase with $\beta(t) \sim \mathfrak{a}(t)$ would then give the desired transition from the Hubble parameter of a Kasner metric to the one of a slow-roll inflationary cosmology.

The two steps described in §4.7 and in this subsection are only a toy model, since we do not provide a viable physical mechanism that produces the desired deformations affecting the transition to commutativity and to the slow-roll inflation scenario. We only exhibit a geometric model of how such transitions may be possible.

5. Other aspects of the model

In this final section we outline briefly other aspects of the model of mixmaster dynamics on noncommutative 3-tori, in relation to recently developed methods for explicit computations of the spectral action [10]; to some existing computations, in the commutative case, of the modified gravity terms in the asymptotic expansion of the spectral action for a Kasner metric [36]; to the role of the diophantine condition in the asymptotic expansion of the spectral action for noncommutative tori [22], [24]; and to the effect of coupling gravity to matter [9]. We also mention other physical models in which mixmaster dynamics play an important role and for which the model presented in this paper may have some relevance.

5.1. **The 4-dimensional Dirac operator for Euclidean Kasner metrics.** We now look at the 4-dimensional Euclidean version of the Kasner metric, of the form $dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$, and in particular at the case where $a(t) = t^{p_1}$, $b(t) = t^{p_2}$, $c(t) = t^{p_3}$, with $p_1 + p_2 + p_3 = 1 = p_1^2 + p_2^2 + p_3^2$ as above. We adapt some of the results obtained in [10] for the isotropic Euclidean Robertson–Walker metrics to this non-isotropic case.

One can consider the Euclidean version of the anisotropic metric (2.2), of the form

$$dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2. \tag{5.1}$$

One can proceed as in [10], and construct the corresponding 4-dimensional Dirac operator (with summation over repeated indices)

$$\not{D} = \gamma^r e_r^\mu \frac{\partial}{\partial x^\mu} + \frac{1}{4}\gamma^s \omega_{srl} \gamma^{rl}, \tag{5.2}$$

where γ^r are the gamma matrices, and the spin connection ω_{srl} now has nonzero terms

$$\omega_{101} = \frac{\dot{a}}{a}, \quad \omega_{202} = \frac{\dot{b}}{b}, \quad \omega_{303} = \frac{\dot{c}}{c}.$$

Thus, one obtains an operator of the form

$$\not{D} = \gamma^0 (\frac{\partial}{\partial t} + \frac{3}{2}\frac{\dot{\mathfrak{a}}}{\mathfrak{a}}) + D, \tag{5.3}$$

with $\mathfrak{a}$ the average scale factor as in (4.1), and where the operator D is given by

$$D = \gamma^1 \frac{1}{a}\frac{\partial}{\partial x} + \gamma^2 \frac{1}{b}\frac{\partial}{\partial y} + \gamma^3 \frac{1}{c}\frac{\partial}{\partial z},$$

so that, as in the case of [10], one has $\gamma^0 D = -\not{\partial}_{T_t^3} \oplus \not{\partial}_{T_t^3}$, since $\gamma^0\gamma^j = i\sigma_j$, with a sign difference due to our use of $-i\sigma_j$ instead of $i\sigma_j$ in (2.6). We also have, as in [10], that $\gamma^0 D = -D\gamma^0$ so that $D^2 = (\gamma^0 D)^2$.

It is convenient here to rename the operator $\not{\partial}_{T_t^3}$ using the following notation

$$\not{\partial}_{a,b,c} := -i(\sigma_1 \frac{1}{a}\frac{\partial}{\partial x} + \sigma_2 \frac{1}{b}\frac{\partial}{\partial y} + \sigma_3 \frac{1}{c}\frac{\partial}{\partial z}). \tag{5.4}$$

One can write the square of the Dirac operator $\not{D}^2$ as

$$-(\frac{\partial}{\partial t} + \frac{3}{2}\frac{\dot{\mathfrak{a}}}{\mathfrak{a}})^2 + \not{\partial}_{a,b,c}^2 - \not{\partial}_{\frac{a^2}{\dot{a}},\frac{b^2}{\dot{b}},\frac{c^2}{\dot{c}}}, \tag{5.5}$$

since the cross term contributes only a term of the form $\gamma^0 \frac{\partial}{\partial t} D + D\gamma^0 \frac{\partial}{\partial t}$ which gives

$$-i(\sigma_1 \frac{\partial}{\partial t}(\frac{1}{a})\frac{\partial}{\partial x} + \sigma_2 \frac{\partial}{\partial t}(\frac{1}{b})\frac{\partial}{\partial y} + \sigma_3 \frac{\partial}{\partial t}(\frac{1}{c})\frac{\partial}{\partial z}),$$

and, as above, $\gamma^0 D = -\not{\partial}_{a,b,c} \oplus \not{\partial}_{a,b,c}$.

Thus, one can reduce the spectral problem for this operator to an infinite family of one-dimensional problems, similarly to what happens in [10]. Here one uses the fact that the two operators $\partial\!\!\!/_{a,b,c}$ and $\partial\!\!\!/_{\frac{a^2}{\dot a},\frac{b^2}{\dot b},\frac{c^2}{\dot c}}$ are simultaneously diagonalized with spectra

$$\text{Spec}(\partial\!\!\!/_{a,b,c}) = \{\pm 2\pi \| (\frac{k+\mathfrak{s}_1/2}{a}, \frac{m+\mathfrak{s}_2/2}{b}, \frac{n+\mathfrak{s}_3/2}{c}) \| \,|\, (k,m,n)\in\mathbb{Z}^3\}, \tag{5.6}$$

$$\text{Spec}(\partial\!\!\!/_{\frac{a^2}{\dot a},\frac{b^2}{\dot b},\frac{c^2}{\dot c}}) = \{\pm 2\pi \| (\frac{\dot a(k+\mathfrak{s}_1/2)}{a^2}, \frac{\dot b(m+\mathfrak{s}_2/2)}{b^2}, \frac{\dot c(n+\mathfrak{s}_3/2)}{c^2}) \| \,|\, (k,m,n)\in\mathbb{Z}^3\}. \tag{5.7}$$

We obtain in this way the one-dimensional problems

$$-(\frac{\partial}{\partial t} + \frac{3}{2}\frac{\dot{\mathfrak{a}}}{\mathfrak{a}})^2 + \lambda^2_{a,b,c} - \lambda_{\frac{a^2}{\dot a},\frac{b^2}{\dot b},\frac{c^2}{\dot c}}, \tag{5.8}$$

where we write $\lambda_{a,b,c}$ for the elements in the spectrum (5.6) and $\lambda_{\frac{a^2}{\dot a},\frac{b^2}{\dot b},\frac{c^2}{\dot c}}$ for those in the spectrum (5.7). This family (5.8) of one-dimensional problems is then, in principle, suitable for a computation of the spectral action based on the Feynman–Kac formula, as described in [10].

In the mixmaster case, one has spectra

$$\lambda^2_{a,b,c} = t^{-2p_1}(k+\frac{\mathfrak{s}_1}{2})^2 + t^{-2p_2}(m+\frac{\mathfrak{s}_2}{2})^2 + t^{-2p_3}(n+\frac{\mathfrak{s}_3}{2})^2$$

$$\lambda^2_{\frac{a^2}{\dot a},\frac{b^2}{\dot b},\frac{c^2}{\dot c}} = p_1^2 t^{-2(p_1+1)}(k+\frac{\mathfrak{s}_1}{2})^2 + p_2^2 t^{-2(p_2+1)}(m+\frac{\mathfrak{s}_2}{2})^2 + p_3^2 t^{-2(p_3+1)}(n+\frac{\mathfrak{s}_3}{2})^2,$$

since $a(t)=t^{p_1}$, $b(t)=t^{p_2}$, $c(t)=t^{p_3}$, with $\dot a/a^2 = p_1 t^{p_1-1}t^{-2p_1} = p_1 t^{-(p_1+1)}$ etc, while the first term in the operators (5.8) of the one-dimensional problems takes the form

$$-(\frac{\partial}{\partial t} + \frac{1}{2t})^2,$$

since $\mathfrak{a} = t^{1/3}$ and $\dot{\mathfrak{a}}/\mathfrak{a} = t^{-1}/3$.

5.2. **Asymptotic expansion and the Kasner metrics.** In the asymptotic expansion of the spectral action over a commutative or almost-commutative geometry, one finds gravitational terms of the form (see [12])

$$\int \left(\frac{1}{2\kappa_0^2}R + \frac{\alpha_0}{2}C_{\mu\nu\rho\sigma}C^{\mu\nu\rho\sigma} + \tau_0 R^*R^* - \xi_0 R|H|^2 \right) dv,$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl curvature, R^*R^* is the form representative of the Pontrjagin class, which integrates to the Euler characteristic, and H is the Higgs field, with the coefficients expressed in terms of the momenta of the test function f in the spectral action and the Yukawa coupling matrix for the matter part of the model.

An analysis of the gravitational terms in the asymptotic expansion of the spectral action in the case of a Kasner metric (and more generally for Bianchi V models) was given in [36]. In that paper, the authors focus on the modified gravity term given by the Weyl curvature that appears in the asymptotic expansion of the spectral action, neglecting the term with the conformal coupling of the Higgs field to gravity. They obtain modified Einstein equations of the form

$$R^{\mu\nu} - \frac{1}{2}g^{\mu\nu}R - \alpha_0\kappa_0^2\left(2C^{\mu\lambda\nu\kappa}_{;\lambda;\kappa} - C^{\mu\lambda\nu\kappa}R_{\lambda\kappa}\right) = \kappa_0^2 T^{\mu\nu}_{matter}.$$

They compute this explicitly in the case of Kasner metrics and other anisotropic Bianchi models, and they find that for such models the obtained equations indeed differ from the ordinary Einstein–Hilbert case, hence distinguishing the modified gravity model given by the spectral action from ordinary GR.

In the mixmaster case, with ordinary commutative tori T^3_t, one can see that their computation gives a correction term in the above equation with respect to the usual Einstein–Hilbert case of the form

$$\frac{-4\alpha_0\kappa_0^2}{3t^4}\sum_i p_i\Big(p_1p_2p_3 + p_{i+1}\Big((p_i-p_{i+1})^2 - p_ip_{i+1})$$
$$+(p_i-1)\Big(\frac{1}{2}p_{i+1}(p_{i+1}-1)+\frac{1}{2}p_{i+2}(p_{i+2}-1)-p_i(p_i-1)\Big)$$
$$+(p_i^2+2-3p_i+(1-p_i)(p_i-p_{i+1}-p_{i+2})\Big)(2p_i-p_{i+1}-p_{i+2})\Big).$$

As observed in [36], this contribution is relevant in the early universe and becomes negligible for later times.

5.3. **The asymptotic expansion and the diophantine condition.** In the case where the ordinary tori T^3_t in the mixmaster model are replaced by noncommutative tori $\mathbb{T}^3_\Theta$, dealing with the asymptotic expansion of the spectral action becomes more delicate.

The perturbative expansion of the spectral action for noncommutative tori was computed recently in [22], [24]. In these calculations, one considers the spectral action

$$\text{(5.9)} \qquad \mathrm{Tr}(f(\partial\!\!\!/_A^2/\Lambda^2)), \quad \text{where} \quad \partial\!\!\!/_A = \partial\!\!\!/ + \tilde{A}, \quad \text{with} \quad \tilde{A} = A + \epsilon JAJ^{-1},$$

where ϵ is the commutation sign $J\partial\!\!\!/ = \epsilon\partial\!\!\!/ J$ of the real structure involution J and the Dirac operator and A is a self-adjoint one form $A = \sum_i a_i[\partial\!\!\!/, b_i]$, with a_i and b_i in $\mathcal{A}_\Theta$. While, as we recalled in the previous sections, the Dirac operator $\partial\!\!\!/$ on $\mathbb{T}_\Theta$ is inherited via isospectral deformation from a commutative torus, the operator $\partial\!\!\!/_A$, twisted with the inner fluctuations $\tilde{A}$ as above, depends on the noncommutative torus itself, and the results of [22], [24] show that this dependence manifests itself in an explicit dependence of the spectral action $\mathrm{Tr}(f(\partial\!\!\!/_A^2/\Lambda^2))$ on refined number-theoretic information on the modulus Θ of the noncommutative torus. The computation of the spectral action in [22] is through the asymptotic expansion for large Λ, in the form

$$\text{(5.10)} \qquad \mathrm{Tr}(f(\partial\!\!\!/_A^2/\Lambda^2)) = \sum_{k\in \dim \mathrm{Spec}^+} f_k\,\Lambda^k \,⨍ |\partial\!\!\!/_A|^{-k} + f(0)\,\zeta_{\partial\!\!\!/_A}(0) + O(\Lambda^{-1}),$$

where the sum is over points in the strictly positive part of the dimension spectrum and the integration $⨍|\partial\!\!\!/_A|^{-k}$ can be expressed as a residue of the zeta function of $\partial\!\!\!/_A$ at the pole $s = k$.

In particular, a diophantine condition on Θ plays a crucial role in the results of [22], [24]. This “badly approximable” condition is formulated as follows. A vector $\underline{v}$ in $\mathbb{R}^n$ is δ-diophantine, for some $\delta > 0$, if there exists a $C > 0$ so that

$$\text{(5.11)} \qquad |\underline{v}\cdot\underline{q} - 2\pi k| \geq C\,|\underline{q}|^{-\delta}, \quad \forall q \in \mathbb{Z}^n\backslash\{0\}, \ \ \forall k \in \mathbb{Z}.$$

The set of diophantine vectors is the union over δ of the sets of δ-diophantine vectors. A matrix Θ is diophantine if there is a vector $\underline{v} \in \mathbb{Z}^n$ such that $\Theta\underline{v}$ is diophantine. Almost every matrix is diophantine (with respect to the Lebesgue measure).

The positive dimension spectrum for an n-dimensional noncommutative torus $\mathbb{T}^n_\Theta$ consists of the points $\{1, 2, \ldots, n\}$, which are all simple poles, under the assumption that the matrix $\frac{1}{2\pi}\Theta$ is diophantine. Under this same diophantine assumption, the residues at the points of the dimension spectrum are computed explicitly in [22]. The top n-th term is the same as for the unperturbed Dirac operator $\not{\partial}$; the $(n-k)$ terms with k odd vanish, and the $(n-2)$ term also vanishes.

Thus, in the case of a 3-torus, the only nonvanishing term agrees with the unperturbed case, so that the resulting perturbative spectral action under the diophantine condition does not differ in form from the ordinary torus case. By comparison, the case of a 4-dimensional torus exhibits a more interesting phenomenon whereby the perturbative spectral action in the case satisfying the diophantine condition has the form

$$8\pi^2 f_4 \Lambda^4 - \frac{4\pi^2}{3} f(0)\tau(F^{\mu\nu}F_{\mu\nu}) + O(\Lambda^{-2}),$$

where $F^{\mu\nu}$ is while for the commutative case one would have $\not{\partial}_A = \not{\partial}$, since the inner fluctuations would all be equivalent to zero in that case, so the expansion under the diophantine condition is different in this case.

The transformation (3.7) of the parameters $\underline{\theta}$ in the mixmaster dynamics via the matrix (2.11) preserves the diophantine condition, hence in a phase of pure mixmaster dynamics, the asymptotic expansion of the spectral action of the noncommutative tori given in [22], [24] remains valid. However, in the presence of a damping effect that destroys the mixmaster evolution and transitions to an isotropic and commutative torus geometry, the parameters $\underline{\theta}$ are deformed to zero along a transformation that no longer preserves the diophantine condition. In fact, along such a deformation, the parameters $\underline{\theta}$ will hit infinitely many values that do and do not satisfy the Diophantine condition and the properties of the asymptotic expansion of the spectral action may vary accordingly in a seemingly chaotic manner, until the final stage of commutative isotropic tori is reached. Even though for 3-tori, unlike the case of 4-tori, the unperturbed commutative spectral action ends up looking the same, one should still investigate what happens in the intermediate stages of the evolution of the parameter $\underline{\theta}$ in between an initial anisotropic mixmaster noncommutative case satisfying the diophantine condition and a final isotropic commutative case, especially at all the intermediate values of $\underline{\theta}$ that do not satisfy the diophantine condition.

5.4. **Coupling to matter.** In addition to pure gravity, described by the spectral action on the mixmaster tori (commutative or noncommutative), one can have a nontrivial coupling to matter, by taking the product of the spectral triple with a finite noncommutative geometry F, which specifies the matter content of the model, as in [12] and [14].

In terms of the asymptotic expansion of the spectral action, we can follow the computations of [24]. We now consider the example of a product geometry of a noncommutative torus $\mathbb{T}^n_\Theta$ and a finite geometry of the form

$$\mathcal{A}_F = M_q(\mathbb{C}), \quad \mathcal{H}_F = M_q(\mathbb{C}), \quad D_F = 0, \quad J_F = J_q, \tag{5.12}$$

where $J_q(L(T)) = L(T^*)$, for T an element in the algebra and $L(T)$ its representation on the Hilbert space.

It is also convenient to write the spectral geometry for the noncommutative torus in the form $\mathcal{H} = \mathcal{H}_\tau$, the GNS representation for the tracial state $\tau(a) = a_0$ on $\mathcal{A}_\Theta^\infty$, so that, τ being faithful, this is the Hilbert space completion of $\mathcal{A}_\Theta^\infty$ in the inner product $\langle a, b\rangle = \tau(a^* b)$. Then the real structure is $J_0 : \mathcal{H}_\tau \to \mathcal{H}_\tau$, $J_0(h) = h^*$, with $J_0^{-1} = J_0$ and, for $L(a)$ an element of $\mathcal{A}_\Theta^\infty$ acting by left multiplication on $\mathcal{H}_\tau$,

$$J_0 L(a) J_0^{-1} h = J_0 L(a) J_0 h = J_0 L(a) h^* = h L(a^*) \equiv R(a^*) h,$$

where $R(a)$ is the action of a by right multiplication. Then the spectral geometry from the noncommutative torus can be written as

$$\mathcal{A} = \mathcal{A}_\Theta^\infty, \quad \mathcal{H} = \mathcal{H}_\tau \otimes \mathbb{C}^{2^{\lfloor n/2 \rfloor}}, \quad \partial\!\!\!/ = i\delta_\mu \otimes \gamma^\mu, \quad J = J_0 \otimes C_0,$$

where $\mathbb{C}^{2^{\lfloor n/2 \rfloor}}$ is an irreducible representation of $\mathbb{C}l^{(+)}(\mathbb{R}^n)$, the γ^μ are the gamma matrices implementing the representation, and C_0 is the antiunitary operator on $\mathbb{C}^{2^{\lfloor n/2 \rfloor}}$ implementing the charge conjugation. The δ_μ are the basic derivations on the noncommutative torus, acting as

$$\delta_\mu \left(\sum_r a_r u^r \right) \equiv i \sum_r r_\mu a_r u^r,$$

on elements of $\mathcal{H}_\tau$ written as $a = \sum_r a_r u^r$, where $r \in \mathbb{Z}^n$, and $\{a_r\} \in \mathcal{S}(\mathbb{Z}^n)$, and u^r is a Weyl element in $\mathcal{A}_\Theta^\infty$ defined by

$$u^r := \exp \left[\pi i \sum_{j<k} r_j \theta_{jk} r_k \right] u_1^{r_1} u_2^{r_2} \ldots u_n^{r_n}.$$

The sign of the spectral triple is $1 \otimes \gamma^{n+1}$ for n even. One has the relations $J\mathcal{D} = \epsilon \mathcal{D} J$, and $J^2 = \epsilon' = (-1)^{\lfloor n/2 \rfloor}$. The first implies that $C_0 \gamma^\mu = -\epsilon \gamma^\mu C_0$.

As shown in [41], this way of writing the spectral triple structure on the noncommutative torus is equivalent to the one we used above by isospectral deformation.

The product geometry then has algebra $\mathcal{A}_\Theta^\infty \otimes M_q(\mathbb{C})$, Hilbert space $\mathcal{H} \otimes M_q(\mathbb{C}) = \mathcal{H}_\tau \otimes \mathbb{C}^{2^{\lfloor n/2 \rfloor}} \otimes M_q(\mathbb{C})$, Dirac operator $D = i\delta_\mu \otimes \gamma^\mu \otimes \mathrm{id}_q$ and real structure $J = J_0 \otimes C_0 \otimes J_q$.

Then, following the same computation of [24], one finds that the presence of the finite spectral triple F of (5.12) alters the perturbative expansion of the spectral action by a factor of q^2 in the leading order term. For example, in the case of a 4-dimensional torus, under the assumption that the diophantine condition on Θ holds, one finds

$$8\pi^2 q^2 f_4 \Lambda^4 - \frac{4\pi^2}{3} f(0) \tau(F^{\mu\nu} F_{\mu\nu}) + O(\Lambda^{-2}).$$

In the case of the commutative mixmaster tori T_t^3, one can also proceed as in [9] to compute the effect on the gravitational part of the spectral action of the matter sector, which as in [9] also delivers an overall multiplicative factor q^2, equal to the rank of the finite geometry.

Thus, in the case of the mixmaster tori, switching back to the same notation used in the previous sections, we find that the spectral action, for this case with the finite geometry F of (5.12),

$$\mathrm{Tr}(f(D_t^2/\Lambda^2)) \sim \frac{q^2 \mathfrak{a}(t)^3 \Lambda^3}{4\pi^3} \int_{\mathbb{R}^3} f(u^2 + v^2 + w^2)\, du\, dv\, dw\,. \tag{5.13}$$

Correspondingly, when one computes the inflation potential as in (4.8), one finds that, as in the case of [9], the resulting slow roll potential is $q^2V(x)$, with $V(x)$ the inflation potential in the absence of the finite geometry. Since only the amplitude of the potential is affected, the slow-roll parameters do not change, but as argued in [32] and [9], the amplitudes for the power spectra for density perturbations and gravitational waves (scalar and tensor perturbations) detect the different scaling factors in the slow-roll potentials and can therefore, in principle, detect the presence of the matter sector through the rank of the finite geometry.

5.5. **Relation to other physical models.** We mention here briefly other recent cosmological models where the kind of noncommutative deformation of the mixmaster dynamics we described above may turn out to be useful.

A first possible context is Hořava–Lifschitz gravity. It was recently shown in [3] that the mixmaster universe provides a mini-superspace truncation of the field equations of Hořava–Lifschitz gravity. The latter is a recently introduced higher derivatives modified gravity theory, applicable in the ultraviolet regime [25]. The Lagrangian density for this theory is derived from a superpotential that contains a Chern–Simons gravitational term and a 3-dimensional Einstein–Hilbert term. The resulting expression has terms of the form $\alpha R+\beta+\gamma\mathcal{C}_{ij}\mathcal{C}^{ij}+\delta\mathcal{C}_{ij}R^{ij}+\epsilon R_{ij}R^{ij}+\zeta R^2$, where the first two terms give rise to the usual Einstein–Hilbert action and the remaining higher derivative terms contain the Cotton tensor $\mathcal{C}^{ij}$ (see (2.16) of [3]), which vanishes if the 3-dimensional spatial sections are conformally flat, and the curvature tensors. The action is invariant under coordinate transformations of the spatial sections.

Another such context is loop quantum cosmology. It was recently shown in [40] that, in the setting of loop quantum cosmology, one finds an oscillatory behavior of mixmaster type as one approaches the singularity as a simplified system from the equations of motion in the Hamiltonian formulation (see §2.3–2.8 in [40]) and that, moreover, on the resulting reduced phase space one can also introduce a scalar field with an inflation slow-roll potential (see §2.7 and chapter 4 of [40]). The presence of the scalar field has a damping effect on the mixmaster oscillations in this model (see Appendix A of [40]).

Acknowledgment. The first author was supported for this project by a Caltech Summer Undergraduate Research Fellowship and by a Richter Scholarship, provided by the Richter Memorial Fund. The second author is supported by NSF grants DMS-0901221 and DMS-1007207.

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