Image Segmentation: the Mumford–Shah functional

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Geometry of Neuroscience

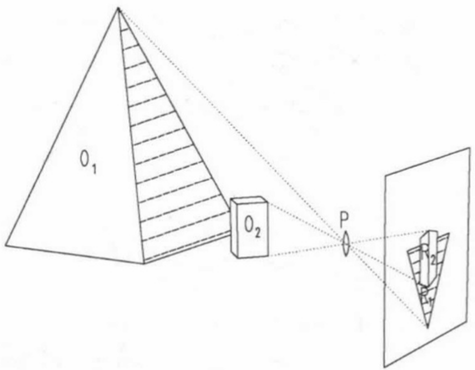
References for this lecture:

- David Mumford, Jayant Shah, *Optimal Approximations by Piecewise Smooth Functions and Associated Variational Problems*, Commun. Pure Applied Math. Vol. XLII (1989) 577–685.

• a three-dimensional scene observed by an eye or camera: at a point P intensity of light $g_1(\rho)$ coming from direction ρ

• a lens at P focuses light on a retina $\mathcal{R}$ (a surface): intensity $g(x, y)$ of light signal received by $\mathcal{R}$ at a point of coordinates (x, y); obtained from $g_1(\rho)$ through some transformation that depends on the functioning of the optical system

• the resulting function $g(x, y)$ is "an image"



- there will be discontinuities in the function $g(x, y)$: boundaries (an object in front of another, objects with a common boundary, discontinuities in illumination, in the object albedo, etc.)
- additional complications:
  - textured objects, fragmented objects (eg a canopy of leaves)
  - shadows, penumbra
  - surface markings
  - partially transparent objects
  - noisy measurements of $g(x, y)$

Segmentation Problem

• goal: compute a decomposition

$$\mathcal{R} = \mathcal{R}_1 \cup \cdots \cup \mathcal{R}_n$$

of the domain of $g(x, y)$ such that

1. the function $g(x, y)$ is smooth within each domain $\mathcal{R}_i$
2. the function $g(x, y)$ varies discontinuously (and/or very rapidly) across *most* of the boundary between different $\mathcal{R}_i$

• equivalently: problem of computing optimal approximations of a function $g(x, y)$ by piecewise smooth functions

Mathematical Approach

• what constitutes an optimal segmentation?

• a functional measuring the degree of match between a function and a segmentation, to be optimized

• $\mathcal{R}_i$ connected open subsets of a given planar domain $\mathcal{R}$, each with piecewise smooth boundary $\partial\mathcal{R}_i$

$$\Gamma = \mathcal{R} \cap \cup_i \partial\mathcal{R}_i$$

$$\mathcal{R} = \Gamma \sqcup \mathcal{R}_1 \sqcup \cdots \sqcup \mathcal{R}_n$$

• discuss three different action functionals whose minimization provides an optimal image segmentation: a functional E that depends on two parameters μ and ν and two limiting cases E_0 and E_∞ depending on ν parameter

The Mumford–Shah Functional

- f differentiable function on $\cup_i \mathcal{R}_i$, can be discontinuous across Γ
- Γ piecewise smooth arcs joined at a finite set of singular points; $|\Gamma|$ total length of the arcs in Γ
- action functional:

$$E(f, \Gamma) = \mu^2 \int_{\mathbb{R}^2} (f - g)^2 \, dx \, dy + \int_{\mathcal{R} \setminus \Gamma} \|\nabla f\|^2 \, dx \, dy + \nu \, |\Gamma|$$

  - first term: measures how good f is as an approximation of g
  - second term: f does not vary too much within each $\mathcal{R}_i$
  - third term: boundary that achieves decomposition as short as possible

- Note: need all these terms to have nontrivial minimum
  1. without first term: $f = 0$ and $\Gamma = \emptyset$ give $E = 0$
  2. without second term: $f = g$ and $\Gamma = \emptyset$ give $E = 0$
  3. without third term: f average of g on a grid of N^2 squares has limit to $E = 0$

- heuristic interpretation: a solution f of the minimization of E is a "cartoon" version of the image g where contours are drawn sharply and scene is simplified

- Question: is the minimization problem for E well posed?

Mumford and Shah conjectured: for all continuous g a minimum of E exists with f differentiable on each $\mathcal{R}_i$ and Γ made of $\mathcal{C}^1$-arcs joined at a finite number of singular points

The functional E_0

- restriction of E to piecewise constant functions $f|_{\mathcal{R}_i} \equiv a_i$

$$\mu^{-2} E(f, \Gamma) = \sum_i \int_{\mathcal{R}_i} (g - a_i)^2 dxdy + \nu_0 |\Gamma|$$

with $\nu_0 = \nu / \mu^2$

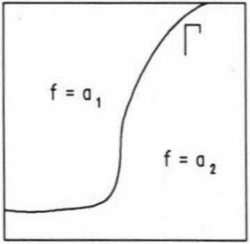
- it is minimized (as a function of the a_i) by the average

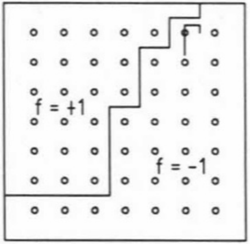
$$a_i = \mathrm{mean}_{\mathcal{R}_i}(g) = \frac{1}{A(\mathcal{R}_i)} \int_{\mathcal{R}_i} g\, dxdy$$

- so functional E_0 defined by

$$E_0(\Gamma) = \sum_i \int_{\mathcal{R}_i} (g - \mathrm{mean}_{\mathcal{R}_i}(g))^2 \, dx\, dy + \nu_0 \, |\Gamma|$$

Relation to the Ising Model (continuous/discrete segmentation)





- suppose f locally constant with only values ± 1
- assume f and g "discretized": defined on a lattice
- Γ a path made of segments of horizontal and vertical lines between pairs of adjacent lattice sites where f changes sign
- functional E_0 (seen as function of f) becomes Ising Model Energy

$$E_0(f) = \sum_{i,j} (f(i,j) - g(i,j))^2 + \nu_0 \sum_{(i,j),(i',j')} (f(i,j) - f(i',j'))^2$$

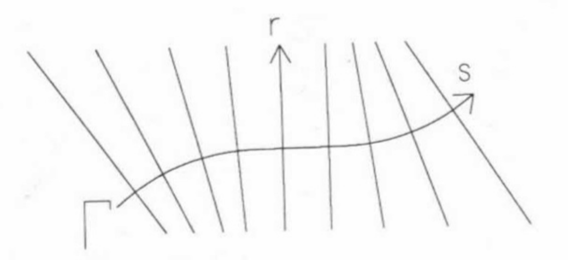
first sum on lattice site, second sum on pairs of neighboring sites

The Functional E_∞

- a functional of Γ

$$E_\infty(\Gamma) = \int_\Gamma \left(\nu_\infty - \left(\frac{\partial g}{\partial n} \right)^2 \right) ds$$

ν_∞ a constant, ds arc length on Γ, unit normal $\partial/\partial n$ along Γ



- minimizing E_∞ means finding Γ so that
  - length of Γ is as short as possible
  - variation of g in the direction normal to Γ is as large as possible

Relation of E_∞ to E

- smooth parts of Γ, curvilinear coordinates (s, r)
- take $f = g$ outside a tubular neighborhood of Γ
- set $\mu = 1/\epsilon$ and $\nu = 2\epsilon\nu_\infty$ and

$$f(r,s) = g(r,s) + \epsilon \,\mathrm{sign}(r)\, e^{-|r|/\epsilon} \frac{\partial g}{\partial r}(0,s)$$

- then

$$E(f,\Gamma) - E(g,\Gamma) = 2\epsilon E_\infty(\Gamma) + O(\epsilon^2)$$

so can think of E_∞ as a $\mu \to \infty$ limit of E

First Goal: Analyzing Variational Equation for E (Summary)

• for fixed Γ positive definite quadratic functional in f with unique minimum solution of elliptic boundary value problem on each $\mathcal{R}_i$

$$\Delta f = \mu^2 (f - g), \qquad \frac{\partial f}{\partial n}|_{\partial \mathcal{R}_i} \equiv 0$$

• solution f_Γ of previous elliptic problem, then E becomes function of Γ, to minimize for Γ

$$E(f_\Gamma, \Gamma)$$

• infinitesimal variation of Γ by a normal vector field $X = a(x,y)\,\frac{\partial}{\partial n}$ (vanishing in neighborhood of singular points of Γ)

• then show that

$$\frac{\delta}{\delta X} E(f_\Gamma, \Gamma) = \int_\Gamma a\,(e_+ - e_- + \nu\,\mathrm{curv}(\Gamma))\, ds$$

$$e_\pm = \mu^2 (f_\Gamma^\pm - g)^2 + \left(\frac{df_\Gamma^\pm}{ds} \right)^2$$

with $f_\Gamma^\pm$ boundary values of f_Γ, and $\mathrm{curv}(\Gamma)$ curvature (function of second derivative of curve Γ)

• then $E(f_\Gamma, \Gamma)$ is minimized by a Γ that satisfies variational equation

$$e_+ - e_- + \nu\,\mathrm{curv}(\Gamma) \equiv 0$$

Complications due to singular points of Γ:

• the minimizing function f_Γ is bounded pointwise by

$$\min_{\mathcal{R}} g \leq f_\Gamma(x,y) \leq \max_{\mathcal{R}} g$$

... but gradient need not be bounded near singular points of Γ

• if Γ made of $\mathcal{C}^2$-arcs joined at endpoints then can use theory of elliptic boundary value problems in domains with corners to handle this problem

• obtain that if minimum at Γ then singularities only

1. *triple points*: three $\mathcal{C}^2$ arcs meet at 120°
2. *crack tips*: a single $\mathcal{C}^2$ arc ends
3. *boundary points*: a $\mathcal{C}^2$ arc of Γ meets perpendicularly a smooth point of $\partial\mathcal{R}$

• further complications: minimizer Γ may have worst singularities than meeting of $\mathcal{C}^2$ arcs: *cusp singularities* at the end of arcs may also occur

More detailed discussion of the variational problem for E

- Hölder spaces $\mathcal{C}^{k,\alpha}(\Omega)$ with $k \in \mathbb{Z}_{\geq 0}$ and $0 < \alpha \leq 1$

$$\|f\|_{\mathcal{C}^{k,\alpha}} = \|f\|_{\mathcal{C}^k} + \max_{|\beta|=k} |D^\beta f|_{\mathcal{C}^{0,\alpha}}$$

$$\|f\|_{\mathcal{C}^k} = \max_{|\beta|\leq k} \sup_{x\in\Omega} |D^\beta f(x)|$$

$$\|f\|_{\mathcal{C}^{0,\alpha}} = \sup_{x\neq y\in\Omega} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

- start by assuming Γ union of $\mathcal{C}^{1,1}$ curves γ_a joined at endpoints and that $f \in \mathcal{C}^1$ on $\mathcal{R} \setminus \Gamma$, initially assume first derivative continuous up to boundary points (will weaken later)

• fix Γ: variational problem for f with variation δf

$$E(f + t\,\delta f, \Gamma) - E(f, \Gamma)$$

$$= t\left[\mu^2 \iint 2\,\delta f \cdot (f - g)\,dx\,dy + \iint 2(\nabla(\delta f) \cdot \nabla f)\,dx\,dy\right]$$

$$+ t^2\left[\mu^2 \iint (\delta f)^2\,dx\,dy + \iint \|\nabla(\delta f)\|^2\,dx\,dy\right].$$

$$\frac{\delta E}{\delta f}(f, \Gamma) = \lim_{t \to 0} \frac{E(f + t\,\delta f, \Gamma) - E(f, \Gamma)}{t}$$

$$= 2\left[\mu^2 \iint \delta f \cdot (f - g)\,dx\,dy + \iint (\nabla(\delta f) \cdot \nabla f)\,dx\,dy\right]$$

$$\frac{1}{2}\frac{\delta E}{\delta f}(f, \Gamma) = \mu^2 \iint \delta f \cdot (f - g)\,dx\,dy - \iint \delta f \cdot \nabla^2 f\,dx\,dy + \int_B \delta f \frac{\partial f}{\partial n}\,ds$$

$$= \iint \delta f(\nabla^2 f - \mu^2(f - g))\,dx\,dy + \int_B \delta f \frac{\partial f}{\partial n}\,ds,$$

Last by integration by parts and applying Green's theorem, with B whole boundary of $\mathcal{R} \smallsetminus \Gamma$ given by $\partial\mathcal{R}$ and each side $\gamma_a^{\pm}$ of Γ

• so resulting variational equation from imposing vanishing of variation for all test functions δf

$$\nabla^2 f = \mu^2 (f - g) \quad \text{and} \quad \frac{\partial f}{\partial n} = 0 \text{ on } \partial \mathcal{R} \cup_a \gamma_a^{\pm}$$

• the operator $\mu^2 - \nabla^2$ is positive-definite self-adjoint, has a Green function $K(x, y; u, v)$ that is $\mathcal{C}^\infty$ outside diagonal $(x, y) = (u, v)$ with singularity

$$K(x, y; u, v) \sim \frac{1}{2\pi} \log(\mu \sqrt{(x-u)^2 + (y-u)^2})$$

• unique solution f on each $\mathcal{R}_i$ constructed by convolution with the Green function

$$f(x, y) = \mu^2 \int_{(u,v)\in\mathcal{R}_i} K(x, y; u, v) g(u, v) \, du dv$$

• Note: in the absence of singularities and no boundaries, K would be Green function on all plane $\mathbb{R}^2$ given by Fourier transform of

$$L(\xi,\eta) = \frac{1}{\mu^2+\xi^2+\eta^2} \quad \text{(massive propagator)}$$

evaluated at $(x-u, y-v)$, given by

$$K(x,y;u,v) = \frac{1}{2\pi} K_0(\mu\sqrt{(x-u)^2+(y-v)^2})$$

with K_0 *modified Bessel function of the second kind*, solution of

$$K_0''(r) + \frac{1}{r}K_0'(r) - K_0(r) = 0$$

with asymptotic behavior

$$K_0(r) \sim \log(\frac{1}{r}), \text{ for } r \to 0, \quad K_0(r) \sim \sqrt{\frac{\pi}{2r}}e^{-r}, \text{ for } r \to \infty$$

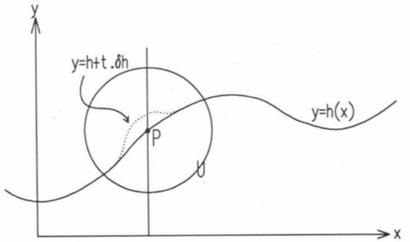
- Variation of $E(f_\Gamma, \Gamma)$ with respect to Γ

- if move Γ near a simple point (not a point where several arcs meet) the point is on one arc $\gamma_a \in \mathcal{C}^{1,1}$

- the $\mathcal{C}^{1,1}$ regularity property of γ_a is used to ensure it can be written locally as the graph of a function, either $y = h(x)$ or $x = h(y)$ (implicit function theorem); then can deform the path by deforming the function

$$\gamma_a(t) = \{y = h(x) + t\,\delta h(x)\}$$

variation $\delta h(x) \equiv 0$ outside small neighborhood $\mathcal{U}$ of point, so new curve does not meet other arcs γ_b outside endpoints



• Note: varying Γ forces f to vary too because $f \in \mathcal{C}^1$ of $\mathcal{R} \smallsetminus \Gamma$ and discontinuous across Γ

• $\mathcal{U}^+ = \{(x,y) \,:\, y > h(x)\} \cap \mathcal{U}$ and $\mathcal{U}^-\{(x,y) \,:\, y > h(x)\} \cap \mathcal{U}$ and $f^\pm = f|_{\mathcal{U}^\pm}$ extend both $f^\pm$ to all $\mathcal{U}$ with a $\mathcal{C}^1$ extension $\tilde{f}^\pm$

$$f^t(x,y) = \begin{cases} f(x,y) & (x,y) \notin \mathcal{U} \\ \tilde{f}^+(x,y) & (x,y) \in \mathcal{U}, \text{ above } \gamma_a(t) \\ \tilde{f}^-(x,y) & (x,y) \in \mathcal{U}, \text{ below } \gamma_a(t) \end{cases}$$

• then compute explicitly variation $E(f^t, \Gamma(t)) - E(f, \Gamma)$ where $\Gamma(t) = \gamma_a(t) \cup_{b \neq a} \gamma_b$

$$
\begin{aligned}
E(f^t, \Gamma(t)) - E(f, \Gamma) = \mu^2 \int\int_U \left[(f^t - g)^2 - (f - g)^2 \right] dx\, dy \\
+ \int\int_{U-\Gamma(t)} \|\nabla f^t\|^2 \, dx\, dy - \int\int_{U-\Gamma} \|\nabla f\|^2 \, dx\, dy \\
+ \nu \left[|\gamma_\alpha(t)| - |\gamma_\alpha| \right] \\
= \mu^2 \int \left(\int_{h(x)}^{h(x)+t\,\delta h(x)} \left[(f^- - g)^2 - (f^+ - g)^2 \right] dy \right) dx \\
+ \int \left(\int_{h(x)}^{h(x)+t\,\delta h(x)} \left[\|\nabla f^-\|^2 - \|\nabla f^+\|^2 \right] dy \right) dx \\
+ \nu \int \left[\sqrt{1 + (h + t\,\delta h)'^2} - \sqrt{1 + h'^2} \right] dx;
\end{aligned}
$$

- so get the variational equation for the path variation $\delta\gamma$

$$
\begin{aligned}
\frac{\delta E}{\delta \gamma} = \mu^2 \int \left[(f^- - g)^2 - (f^+ - g)^2 \right] \Big|_{y=h(x)} \delta h \, dx \\
+ \int \left[\|\nabla f^-\|^2 - \|\nabla f^+\|^2 \right] \Big|_{y=h(x)} \delta h \, dx \\
+ \nu \int \frac{h'}{\sqrt{1 + h'^2}} (\delta h)' \, dx.
\end{aligned}
$$

- curvature: since $\gamma_a \in \mathcal{C}^{1,1}$ well defined curvature almost everywhere

$$\mathrm{curv}(\gamma_a)(x, h(x)) = \frac{h''(x)}{(1 + h'(x)^2)^{3/2}}$$

- integrating by parts in the last term of the variational equation

$$\frac{\delta E}{\delta \gamma} = \int_{\gamma_a} \left[\left(\mu^2 (f^- - g)^2 + \|\nabla f^-\|^2 \right) - \left(\mu^2 (f^+ - g)^2 + \|\nabla f^+\|^2 \right) \right] \cdot \frac{\delta h}{\sqrt{1 + h'^2}}\, ds$$
$$- \nu\, \mathrm{curv}(\gamma_a)$$

so along each γ_a

$$(\mu^2 (f^+ - g)^2 + \|\nabla f^+\|^2) - (\mu^2 (f^- - g)^2 + \|\nabla f^-\|^2) + \nu\, \mathrm{curv}(\gamma_a) = 0$$

- energy density $e(f; x, y) = \mu^2 (f(x,y) - g(x,y))^2 + \|\nabla f(x,y)\|^2$

$$e(f^+) - e(f^-) + \nu\, \mathrm{curv}(\gamma_a) = 0 \quad \text{on } \gamma_a$$

for fixed $f^\pm$ second order ODE for $h(x)$

Special points of Γ: more complicated analysis of the variational problem (restrictions at these points imposed by stationary condition for the functional E)

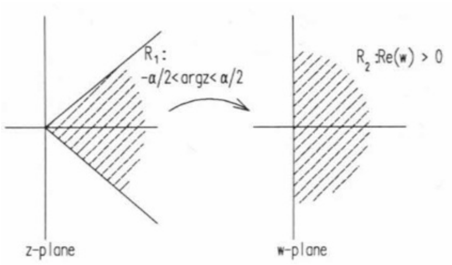
- Cases:

1. points where Γ meets $\partial\mathcal{R}$
2. corners where two γ_a arcs meet
3. vertices where three or more γ_a meet
4. crack-tips where a γ_a ends without meeting another arc

• Problem with corners: can have a function $\nabla^2 f = 0$ in open region, $\frac{\partial f}{\partial n} = 0$ on boundary, with *singularity* at corner point

$$f(z) = \Im(w) = r^{\pi/\alpha} \sin(\frac{\pi}{\alpha}\theta)$$

for $z = re^{i\theta}$ and $w = z^{\pi/\alpha}$ (conformal map that flattens corner)



then

$$\frac{\partial f}{\partial r} = \frac{\pi}{\alpha} r^{\frac{\pi}{\alpha}-1} \sin(\frac{\pi}{\alpha}\theta) \to \infty \quad \text{when } r \to 0, \quad \text{if } \alpha > \pi$$

Elliptic boundary value problems on domains with corners (Kondratiev)

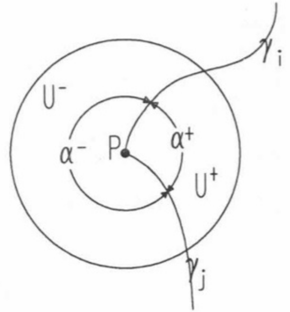
• previous example is typical behavior of solutions of elliptic boundary value problems in domains with corners: solutions f satisfy

- f bounded everywhere
- f is $\mathcal{C}^1$ at corners with angle $0 < \alpha < \pi$
- at corners with $\pi < \alpha \leq 2\pi$ (case 2π is crack-tip)

$$f = c r^{\pi/\alpha} \sin(\frac{\pi}{\alpha}(\theta - \theta_0)) + \tilde{f}$$

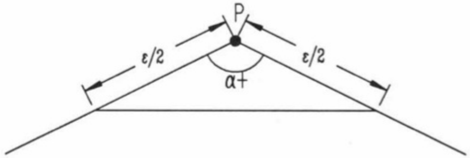
with $\tilde{f} \in \mathcal{C}^1$

• use this to show that Mumford–Shah minimizers cannot have kinks (two-arcs corners with angle $\neq \pi$)



Divide neighbor into sectors with angle larger/smaller than π; take smooth cutoff function η_U on a ball near corner

• cut the corner at small distance, shrinking U^+ enlarging U^-

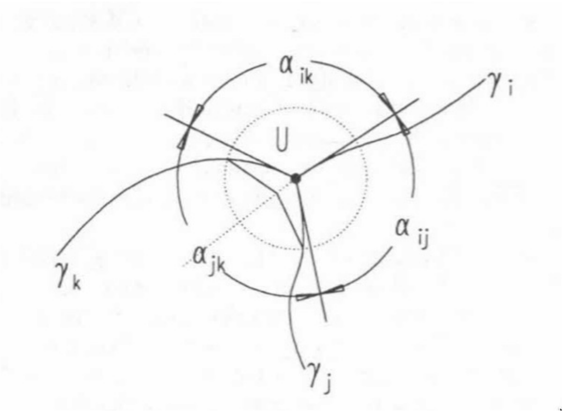


• new f on smaller U^+ by restriction; new f on larger U^- extended by cutoff function $f^-(0) + \eta_U(f^- - f^-(0))$

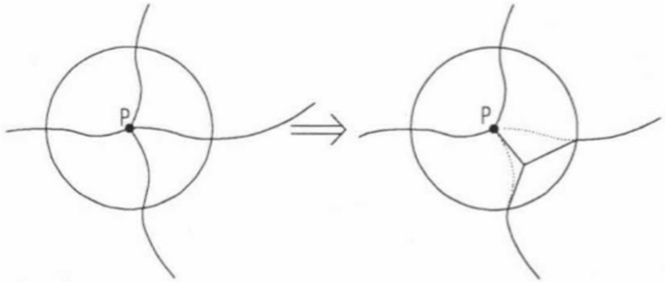
• measure corresponding change in $E(f, \Gamma)$: find that if $\alpha \neq \pi$ the functional $E(f, \Gamma)$ decreases when cutting corner as above, so original kink path cannot be a minimizer

• similar argument shows a minimizer Γ will meet $\partial\mathcal{R}$ orthogonally

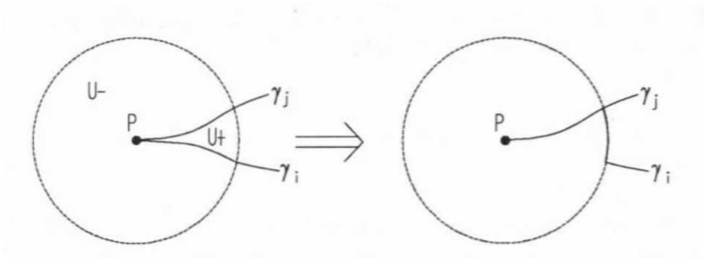
• also similar argument shows triple points must have angles $2\pi/3$ otherwise can cut a sector and lower value of $E(f, \Gamma)$



• also if Γ has points where four or more arcs meet not a minimizer: can separate into triple points and lower the value of $E(f, \Gamma)$



- can also eliminate cusp corners and lower value of $E(f, \Gamma)$

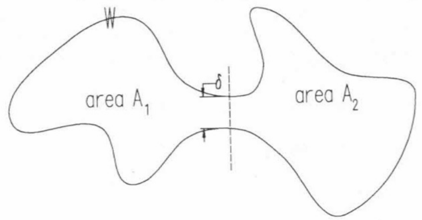


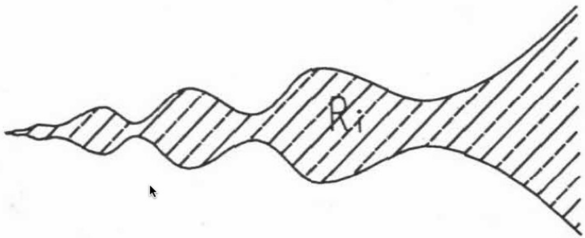
Summary of Further Results

- f_Γ minimizing $E(f,\Gamma)$ for fixed Γ, estimate proximity to locally constant average of g on each $\mathcal{R}_i$
- isoperimetric constant: measures *smallest necks* in each component $\mathcal{R}_i$

$$h(W) = \inf_\gamma \left\{ \frac{|\gamma|}{\min(|W_1|, |W_2|)} : \begin{array}{c} \gamma \text{ is a curve dividing } W \\ \text{into 2 disjoint open} \\ \text{sets } W_1 \text{ and } W_2 \end{array} \right\}$$

$|\gamma| =$ length, $|W_i| =$ area and take $\lambda_\Gamma = \min_i h(\mathcal{R}_i)$





domain with isoperimetric constant $h(\mathcal{R}_i) = 0$

- small μ limit: prove estimate

$$\mu^2 \left(E_0(\Gamma) - \frac{4\mu^2}{\lambda_\Gamma^2 + 4\mu^2} \|g\|^2_{0,2,\mathcal{R}} \right) \leq E(f_\Gamma, \Gamma) \leq \mu^2 E_0(\Gamma)$$

Existence of E_0-minimizers

if $\mathcal{R}$ rectangle, g continuous on $\mathcal{R} \cup \partial\mathcal{R}$, for paths Γ of $\mathcal{C}^{1,1}$ arcs meeting at endpoints and locally constant functions f on $\mathcal{R} \smallsetminus \Gamma$ there is a minimum (f, Γ) of

$$E_0(f, \Gamma) = \int_{\mathcal{R}} (f - g)^2 + \nu_0 \, \mathrm{length}(\Gamma)$$

Method of Proof: Geometric Measure Theory

Main Idea: first show existence of "weak solution" with a "very singular" Γ, then show that week solution must in fact be sufficiently regular as required by original problem

Weak solutions: Caccioppoli sets (measurable and with characteristic function of bounded variation), topological boundary may have infinite Hausdorff measure in dim=1 but a "reduced boundary" is 1-rectifiable

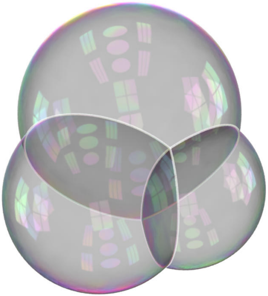
What is Geometric Measure Theory?

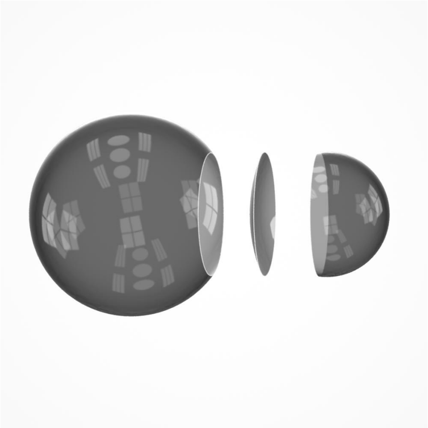
• use of measure theory methods to study geometric objects (curves, surfaces) that are highly non-smooth

• historically developed to study the *Plateau Problem* (the geometry of soap bubble film: area minimizing surfaces with given boundary curves)

• Introduction to Geometric Measure Theory:

- Frank Morgan, *Geometric measure theory: A beginner's guide* (Fourth ed.), Academic Press, 2009.
- Frederick J. Almgren, Jr., *Plateau's Problem: An Invitation to Varifold Geometry, Revised Edition*, American Mathematical Society, 2001.
- Herbert Federer, *Colloquium lectures on geometric measure theory*, Bull. Amer. Math. Soc. 84 (1978), 291–338





Plateau Problem (images by John M. Sullivan)
in “Plateau’s Problem” by Frederick J. Almgren, Jr

The large μ case

• When $\mu \to \infty$ the effect of Γ on the energy is localized into a narrow strip around Γ

• first using Green's theorem reduce E to an integration only along Γ, in terms of solutions g_μ and f_Γ of

$$\Delta g_\mu = \mu^2 (g_\mu - g), \qquad \Delta f_\Gamma = \mu^2 (f_\Gamma - g)$$

$$\frac{\partial g_\mu}{\partial n}|_{\partial \mathcal{R}} \equiv 0, \qquad \frac{\partial f_\Gamma}{\partial n}|_{\partial \mathcal{R} \cup \Gamma} \equiv 0$$

the functional E satisfies

$$E(f_\Gamma, \Gamma) = E(g_\mu, \emptyset) + \int_\Gamma \left(\nu - \frac{\partial g_\mu}{\partial n} (f_\Gamma^+ - f_\Gamma^-) \right) ds$$

with $f_\Gamma^\pm$ boundary values of f_Γ along two sides of Γ and n-vector points to $+$ side of Γ

• when μ is large prove

$$f_\Gamma^+ - f_\Gamma^- = \frac{2}{\mu}\frac{\partial g_\mu}{\partial n} + O(\frac{1}{\mu^2})$$

and uniformly in $\mathcal{R}$ have $f_\Gamma - g = O(\mu^{-1})$

• this gives, in terms of E_∞ functional (with $\nu_\infty = \mu\nu/2$)

$$E(f_\Gamma, \Gamma) = E(g_\mu, \emptyset) + \frac{2}{\mu}E_\infty(\Gamma) + O(\frac{\log\mu}{\mu^2})$$

• first variation of $E(f_\Gamma, \Gamma)$ converges for large μ to first variation of $E_\infty(\Gamma)$

• explicitly compute vanishing first variation equation for $E_\infty(\Gamma)$: find second order differential equation for Γ

• H_g = matrix of second derivatives of g; t_Γ and n_Γ unit tangent and normal vector; variational equation for $E_\infty(\Gamma)$:

$$(n_\Gamma \cdot \nabla g) \cdot \Delta g + (t_\Gamma \cdot \nabla g) \cdot (t_\Gamma \cdot H_g \cdot n_\Gamma) + \mathrm{curv}(\Gamma) \cdot \left[\tfrac{1}{2}\nu_\infty + \tfrac{1}{2}(n_\Gamma \cdot \nabla g)^2 - (t_\Gamma \cdot \nabla g)^2 \right] = 0.$$

• this equation can be interpreted as the geodesic equation in a Lorentzian metric: space-like solutions locally minimizing E_∞ and time-like solutions locally maximizing it; general solutions flip between these two types through cusps singularities at the transition

A lot of more recent results on Mumford-Shah minimizers and segmentation: rich current area of research; a large number of papers available on the topic

Suggested References:

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- Leah Bar et al. *Mumford and Shah Model and its Applications to Image Segmentation and Image Restoration*, in "Handbook of Mathematical Methods in Imaging", Springer 2011, 1095–1157