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A generalized multifractal spectrum of the general Sierpinski carpets

Yongxin Gui [a,b,*,1], Wenxia Li [b,2]

[a] Department of Mathematics, Xianning College, Xianning 437100, PR China
[b] Department of Mathematics, East China Normal University, Shanghai 200062, PR China

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ABSTRACT

In this paper we study a class of subsets of the general Sierpinski carpets for which the limiting frequency of a horizontal fibre falls into a prescribed closed interval. We obtain the explicit expression for the Hausdorff dimension of these subsets in terms of the parameters of the construction and give necessary and sufficient conditions for the corresponding Hausdorff measure to be positive finite.

1. Introduction and main results

In the study of geometric properties of dynamical systems or fractal measures one is often interested in the asymptotic behavior of various local quantities associated with the underlying dynamical or geometric structure besides the classical multifractal spectrum, such as the ergodic average of a continuous function, the local entropy or the local Lyapunov exponent. This leads to the notion of more general multifractal spectra. Recently Olsen proposed a unifying multifractal framework based on the concept of the deformations of empirical measures (one can refer to [22,23] and references therein for more detail). This leads to significant extensions of already known results in multifractal analysis of local characteristics of dynamical systems and fractal measures. Indeed, Olsen obtained various multifractal spectra in the setting of self-conformal sets and self-conformal measures. As a nontrivial application, Olsen [21,24] obtained the multifractal spectrum related to frequencies of digits of N-adic digits. It is natural to ask whether these results can be extended to the setting of the general self-affine sets. There have been some papers on the Hausdorff dimensions of self-affine sets and self-affine measures [1,6,7,10,27,28]. In this paper we will investigate this problem in the setting of a special self-affine sets—the general Sierpinski carpets.

Let T be the expanding endomorphism of the 2-torus $\mathbb{T}^2=\mathbb{R}^2/\mathbb{Z}^2$ given by the matrix $\operatorname{diag}(n,m)$ where $2\leqslant m<n$ are integers. The simplest invariant sets for T have the form

$$K(T,D)=\left\{\sum_{k=1}^{\infty}\begin{pmatrix} n^{-1} & 0\\ 0 & m^{-1}\end{pmatrix}^k d_k\colon\ d_k\in D \text{ for all } k\geqslant 1\right\},$$

where $D\subseteq I\times J$ is a set of digits with $I=\{0,1,\ldots,n-1\}$ and $J=\{0,1,\ldots,m-1\}$. Alternatively, define a “representation” map $K_T:(I\times J)^{\mathbb{N}}\to\mathbb{T}^2$ by

* Corresponding author at: Department of Mathematics, East China Normal University, Shanghai 200062, PR China.
E-mail addresses: admireyou@163.com (Y. Gui), wxli@math.ecnu.edu.cn (W. Li).

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$$K_T(x) = \sum_{k=1}^{\infty} \begin{pmatrix} n^{-1} & 0 \\ 0 & m^{-1} \end{pmatrix}^k d_k, \quad x = (d_k)_{k=1}^{\infty} \in (I \times J)^{\mathbb{N}}. \tag{1}$$

Then $K(T, D) = K_T(D^{\mathbb{N}})$. So each element of $K(T, D)$ can be represented as an expansion in base $\operatorname{diag}(n^{-1}, m^{-1})$ with digits in D. The set $K(T, D)$, called as the *general Sierpinski carpets*, was first studied by C. McMullen [18] and T. Bedford [5], independently, to determine its Hausdorff and box-counting dimensions. From then on, some further problems related to the Sierpinski carpets $K(T, D)$ are proposed and considered by lots of authors. Y. Peres [25,26] studied its packing and Hausdorff measures. R. Kenyon and Y. Peres [15,16] extended the results of McMullen [18] and Bedford [5] to the compact subsets of the 2-torus corresponding to shifts of finite type or sofic shifts and to the Sierpinski sponges. Gatzouras and Lalley [11] and recently K. Barański [3] extended the construction of McMullen and Bedford to the more complicated geometric constructions, respectively. The singular spectrum was studied by King [17] for the general Sierpinski carpets, and later by Olsen [20] for the Sierpinski sponges. O.A. Nielsen [19] studied a certain subset of $K(T, D)$ by insisting that the allowed digits in the expansions occur with prescribed frequencies.

Now we describe the setting in this paper and state the main results. Let σ denote the projection of $\mathbb{R}^2$ onto its second coordinate. Throughout this paper we use $\#E$ to denote the cardinality of a finite set E. Denote $B = \sigma(D)$. To avoid triviality, we assume that $\#B \geqslant 2$. For each point $b \in B$ put $n_b = \#\{d \in D\colon \sigma(d) = b\}$. D is said to have *uniform horizontal fibres* if $n_b = n_{b'}$ for all $b, b' \in B$. For any fixed $s \in B$, let

$$\Gamma_s = \{d \in D\colon \sigma(d) = s\},$$

we will call it to be a *horizontal fibre* of D. Then $n_s = \#\Gamma_s$.

For any $x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}$ and $d \in D$, define

$$N_k(x, d) = \#\{1 \leqslant i \leqslant k\colon x_i = d\} \tag{2}$$

and

$$N_k(x, \Gamma_s) = \#\{1 \leqslant i \leqslant k\colon x_i \in \Gamma_s\}.$$

Whenever there exists the limit

$$f(x, \Gamma_s) := \lim_{k \to \infty} \frac{N_k(x, \Gamma_s)}{k} \tag{3}$$

it is called the frequency of the horizontal fibre Γ_s in the coding x. When we write the symbol $f(x, \Gamma_s)$ we are already assuming the existence of the limit in (3).

Some results related with the fiber frequencies were earlier studied by the authors in [12,13]. For a probability vector $(e_b)_{b \in B}$ let

$$\Omega = \{x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}\colon f(x, \Gamma_b) = e_b \text{ for all } b \in B\},$$

i.e., the set of elements of $D^{\mathbb{N}}$ where its entry of each element falls into each horizontal fibre Γ_b with a prescribed frequency e_b. The Hausdorff and packing dimensions of $K_T(\Omega)$ and the sufficient and necessary conditions for the corresponding Hausdorff and packing measures to be positive finite are obtained in [12]. In fact, the approach used in [12] works for a bit more general case (see (4) below) by a minor modification. Let B_j, $j = 1, \ldots, \ell$, be a partition of B, i.e., B_j's are disjoint nonempty subsets of B with union equal to B. Let

$$\Omega\big((B_j)_{1 \leqslant j \leqslant \ell}, (e_j)_{1 \leqslant j \leqslant \ell}\big) = \left\{x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}\colon f\left(x, \bigcup_{b \in B_j} \Gamma_b\right) = e_j,\ 1 \leqslant j \leqslant \ell\right\}, \tag{4}$$

where $(e_j)_{1 \leqslant j \leqslant \ell}$ is a probability vector. The Hausdorff dimension of $K_T(\Omega((B_j)_{1 \leqslant j \leqslant \ell}, (e_j)_{1 \leqslant j \leqslant \ell}))$ can be also determined explicitly.

For distinct $s, t \in B$ (recall $B = \sigma(D)$, the projection of D onto its second coordinate) and $\beta > 0$ let

$$\Omega_{s,t,\beta} = \{x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}\colon f(x, \Gamma_s) = \beta f(x, \Gamma_t) > 0\},$$

i.e., the subset of $D^{\mathbb{N}}$ such that the frequency of the horizontal fibre Γ_s in its element x is β proportional to that of Γ_t. Take $B_1 = \{s\}$, $B_2 = \{t\}$ and $B_3 = B \setminus (B_1 \cup B_2)$. Then

$$\Omega_{s,t,\beta} = \bigcup_{y \in (0, 1/(1+\beta)]} \Omega\big((B_j)_{1 \leqslant j \leqslant 3}, (\beta y, y, 1 - y - \beta y)\big).$$

The Hausdorff dimension of $K_T(\Omega_{s,t,\beta})$ (as well as the property of its corresponding Hausdorff measure) was obtained explicitly in [13] by showing that (in fact the supremum below is reached)

$$\dim_H \Omega_{s,t,\beta} = \sup_{y\in(0,1/(1+\beta)]} \dim_H K_T\big(\Omega\big((B_j)_{1\leqslant j\leqslant 3}, (\beta y, y, 1-y-\beta y)\big)\big).$$

This makes one to expect a general result that for any set E of probability vectors

$$\dim_H K_T\Bigg(\bigcup_{(e_j)_{1\leqslant j\leqslant \ell}\in E} \Omega\big((B_j)_{1\leqslant j\leqslant \ell}, (e_j)_{1\leqslant j\leqslant \ell}\big)\Bigg) = \sup_{(e_j)_{1\leqslant j\leqslant \ell}\in E} \dim_H K_T\big(\Omega\big((B_j)_{1\leqslant j\leqslant \ell}, (e_j)_{1\leqslant j\leqslant \ell}\big)\big).$$

The analogue for the case of self-similar sets has been verified to be correct by lots of authors (e.g. in [2,8,9,21–24]). However, for the case under consideration (non-self-similar) it is hard to prove it in a unifying way. In the present paper, we will consider another individual case and show that the above assertion is correct. As one can see, some special techniques are required for an individual case.

Now for $0 < c_1 < c_2 < 1$ and a fixed $s \in B$ we consider the set

$$\Omega(c_1, c_2) = \big\{x = (x_i)_{i=1}^{\infty} \in D^{\mathbb{N}}:\ c_1 \leqslant f(x, \Gamma_s) \leqslant c_2\big\},$$

i.e., $\Omega(c_1, c_2)$ is a subset of $D^{\mathbb{N}}$ such that the frequency of horizontal fibre Γ_s in the coding x falls into the closed subinterval $[c_1, c_2]$ of $(0, 1)$. Again $\Omega(c_1, c_2)$ can be represented as an uncountable union of set of form (4)

$$\Omega(c_1, c_2) = \bigcup_{y\in[c_1,c_2]} \Omega\big((B_1, B_2), (y, 1-y)\big),$$

where $B_1 = \{s\}$ and $B_2 = B \setminus B_1$.

The purpose of this paper is to compute the Hausdorff dimension of $K_T(\Omega(c_1, c_2))$. For an arbitrary closed subinterval $[c_1, c_2]$ of $(0, 1)$, the Hausdorff dimension of $K_T(\Omega(c_1, c_2))$ may therefore be viewed as generalized multifractal spectrum in view of the multifractal framework introduced by Olsen in [22].

For any Borel subset E of $\mathbb{R}^2$, let $\dim_H E$ denote its Hausdorff dimension, and $\mathcal{H}^{\gamma}(E)$ denote its γ-dimensional Hausdorff measure. Our first main result gives an explicit formula for the Hausdorff dimension of $K_T(\Omega(c_1, c_2))$.

Theorem 1.1. *Let* $A = \frac{n_s^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} = \frac{n_s^{\log_n m}}{\sum_{b\in B} n_b^{\log_n m}}$. *For* $x \in (0, 1)$ *let*

$$h(x) = x\big(\log_m n_s^{\log_n m} - \log_m x\big) + (1-x)\Bigg(\log_m \sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1} - \log_m(1-x)\Bigg).$$

Then

$$\dim_H K_T\big(\Omega(c_1, c_2)\big) = \begin{cases} h(A) & \text{if } A \in [c_1, c_2],\\ h(c_1) & \text{if } A < c_1,\\ h(c_2) & \text{if } A > c_2.\end{cases}$$

As to the corresponding Hausdorff measure, we have the following result.

Theorem 1.2. *Let* $\gamma = \dim_H K_T(\Omega(c_1, c_2))$ *and* A *be given in Theorem* 1.1. *Then*

(I) *If* $A \in [c_1, c_2]$ *and* D *has uniform horizontal fibres then* $0 < \mathcal{H}^{\gamma}(K_T(\Omega(c_1, c_2))) < \infty$;
(II) *If* $A \notin [c_1, c_2]$ *or* D *does not have uniform horizontal fibres then* $\mathcal{H}^{\gamma}(K_T(\Omega(c_1, c_2))) = \infty$.

The rest of this paper is organized as follows. In Section 2, some basic facts and known results needed in the proof of our theorems are described. Proofs of Theorems 1.1 and 1.2 are arranged in Sections 3 and 4, respectively.

2. Preliminaries

As in [18,19,25,26], a class of approximate squares are used to calculate the various dimensions of the general Sierpinski carpets and its subsets. For each $x = (x_j)_{j=1}^{\infty} \in (I \times J)^{\mathbb{N}}$ and each positive integer k, let

$$Q_k(x) = \big\{K_T(y):\ y = (y_j)_{j=1}^{\infty} \in (I \times J)^{\mathbb{N}},\ y_j = x_j \text{ for } 1 \leqslant j \leqslant [k\log_n m] \text{ and } \sigma(y_j) = \sigma(x_j) \text{ for } [k\log_n m]+1 \leqslant j \leqslant k\big\},$$

where, as usual, $[x]$ with $x \in \mathbb{R}$ denotes the greatest integer function. The sets $Q_k(x)$ are approximate squares in $[0, 1]^2$, whose sizes have length $n^{-[k\log_n m]}$ and m^{-k}. Note that the ratio of the sizes of $Q_k(x)$ is at most n, and their diameters $\operatorname{diam} Q_k(x)$ satisfy

$$\sqrt{2}m^{-k} \leqslant \operatorname{diam} Q_k(x) \leqslant \sqrt{2}nm^{-k}.$$

So in the definition of Hausdorff measure, we can restrict attention to covers by such approximate squares since any set of diameter less than m^{-k} can be covered by a bounded number of approximate squares $Q_k(x)$. The following lemma appears in [19] in which the approximate square $Q_k(x)$ behaves as an analogue as the ball does in the classical density theorems. It is just a reformulation of the Rogers–Taylor density theorem as stated by Peres in Section 2 of [26].

Lemma 2.1. *(See [19, Lemma 4].) Suppose that δ is a positive number, that μ is a finite Borel measure in $[0,1]^2$, and that E is a subset of $(I \times J)^{\mathbb{N}}$ such that $K_T(E)$ is a Borel subset of $[0,1]^2$, and $\mu(K_T(E)) > 0$, put*

$$M(x) = \limsup_{k\to\infty}\big(k\delta + \log_m \mu\big(Q_k(x)\big)\big)$$

for each point $x \in E$.

(1) *If $M(x) = -\infty$ for all $x \in E$, then $\mathcal{H}^\delta(K_T(E)) = +\infty$.*
(2) *If $M(x) = +\infty$ for all $x \in E$, then $\mathcal{H}^\delta(K_T(E)) = 0$.*
(3) *If there are numbers a and b such that $a \leqslant M(x) \leqslant b$ for all $x \in E$, then $0 < \mathcal{H}^\delta(K_T(E)) < +\infty$.*

The Borel measures on $[0,1]^2$ to which the above lemmas will be applied are constructed as follows. Let $\mathbf{p} = (p_d)_{d\in D}$ be a probability vector on D, i.e., $\sum_{d\in D} p_d = 1$ with each $p_d \in [0,1]$. Then $\mathbf{p}$ determines a unique infinite product Borel probability measure, denoted by $\mu_{\mathbf{p}}$, on $D^{\mathbb{N}}$. For any finite sequence $(x_1, x_2, \dots, x_k) \in D^k$,

$$\mu_{\mathbf{p}}\big([x_1, x_2, \dots, x_k]\big) = \prod_{j=1}^{k} p_{x_j}, \tag{5}$$

where $[x_1, x_2, \dots, x_k] := \{d = (d_j)_{j=1}^\infty \in D^{\mathbb{N}}\colon\ d_j = x_j \text{ for } 1 \leqslant j \leqslant k\}$ is a cylinder set of $D^{\mathbb{N}}$ with base $(x_1, x_2, \dots, x_k)$. Let $\widetilde{\mu}_{\mathbf{p}}$ be the Borel probability measure on $K_T(D^{\mathbb{N}})$ which is the image measure of $\mu_{\mathbf{p}}$ under K_T, i.e., $\widetilde{\mu}_{\mathbf{p}}(B) = \mu_{\mathbf{p}}(K_T^{-1}B)$ for Borel set $B \subseteq \mathbb{R}^2$. From the definition of approximate square $Q_k(x)$ it follows that for any $x = (x_j)_{j=1}^\infty \in D^{\mathbb{N}}$ (cf. formula (4) in [19], also formula (4.4) in [11])

$$\widetilde{\mu}_{\mathbf{p}}\big(Q_k(x)\big) = \prod_{j=1}^{[k\log_n m]} p_{x_j} \cdot \prod_{j=[k\log_n m]+1}^{k} q_{\sigma(x_j)}, \tag{6}$$

where and throughout this paper the probability vector $(q_b)_{b\in B}$ on B, induced by $\mathbf{p} = (p_d)_{d\in D}$, is defined by

$$q_b = \sum_{d\in D\cap(I\times b)} p_d \quad \text{for each point } b \in B.$$

The following lemma shows that $K_T(\Omega(c_1, c_2))$ is of full $\widetilde{\mu}_{\mathbf{p}}$-measure for some properly selected probability vectors $\mathbf{p}$.

Lemma 2.2. $\widetilde{\mu}_{\mathbf{p}}(K_T(\Omega(c_1, c_2))) = 1$ *for each probability vector* $\mathbf{p} = (p_d)_{d\in D} \in \Sigma$ *where*

$$\Sigma = \Big\{\mathbf{p} = (p_d)_{d\in D}\colon \sum_{d\in D} p_d = 1,\ c_1 \leqslant \sum_{d\in\Gamma_s} p_d \leqslant c_2 \text{ and } p_d \in [0,1] \text{ for all } d \in D\Big\}. \tag{7}$$

Proof. For any probability vector $\mathbf{p} = (p_d)_{d\in D} \in \Sigma$, let

$$\Delta_{\mathbf{p}} = \Big\{x = (x_j)_{j=1}^\infty \in D^{\mathbb{N}}\colon\ \lim_{k\to\infty} \frac{N_k(x,d)}{k} = p_d \text{ for all } d \in D\Big\},$$

and for each point $d \in D$ let

$$\Delta_{\mathbf{p}}(d) = \Big\{x = (x_j)_{j=1}^\infty \in D^{\mathbb{N}}\colon\ \lim_{k\to\infty} \frac{N_k(x,d)}{k} = p_d\Big\}.$$

Then $K_T(\Omega(c_1, c_2)) \supset \bigcup_{\mathbf{p}\in\Sigma} K_T(\Delta_{\mathbf{p}})$ and $\Delta_{\mathbf{p}} = \bigcap_{d\in D} \Delta_{\mathbf{p}}(d)$. So it suffices to show that $\mu_{\mathbf{p}}(\Delta_{\mathbf{p}}(d)) = 1$ for each $d \in D$. Fix a $d \in D$ and define a sequence of random variables $\{X_j\}_{j=1}^\infty$ on the probability space $(D^{\mathbb{N}}, \mathcal{F}, \mu_{\mathbf{p}})$ ($\mathcal{F}$ is the Borel σ-algebra) by letting

$$X_j\big((x_k)_{k=1}^\infty\big) = \begin{cases} 1, & x_j = d, \\ 0, & x_j \neq d. \end{cases}$$

Then $X_1, X_2, \dots$ are independent and identically distributed random variables with $\mu_{\mathbf{p}}(X_1 = 1) = p_d$ and $\mu_{\mathbf{p}}(X_1 = 0) = 1 - p_d$. By Kolmogrov strong law of large numbers, we have that for $\mu_{\mathbf{p}}$-a.e. $x = (x_i)_{i=1}^\infty \in D^{\mathbb{N}}$,

$$\lim_{k\to\infty}\frac{N_k(x,d)}{k}=\lim_{k\to\infty}\frac{1}{k}\sum_{j=1}^{k}X_j(x)=\mathbb{E}(X_1)=p_d,$$

implying $\mu_{\mathbf{p}}(\Delta_{\mathbf{p}}(d))=1$. □

It will be convenient to refer to the Hausdorff dimension of a Borel probability measure μ. This is defined as the infimum of the dimensions of sets of full μ-measure, i.e., $\dim_H\mu=\inf\{\dim_H E:\ \mu(E)=1\}$. A valid way to determine $\dim_H\mu$ is the following lemma.

Lemma 2.3 *(Modification of Billingsley lemma, cf. [26]). Let $\mu_{\mathbf{p}}$ and $\widetilde{\mu}_{\mathbf{p}}$ be defined as in* (5) *and* (6) *for $\mathbf{p}=(p_d)_{d\in D}\in\Sigma$. If $\liminf_{k\to\infty}\frac{\log\widetilde{\mu}_{\mathbf{p}}(Q_k(x))}{\log m^{-k}}=\beta$ for $\mu_{\mathbf{p}}$-almost every $x\in\Omega(c_1,c_2)$, then $\dim_H\widetilde{\mu}_{\mathbf{p}}=\beta$.*

3. Proof of Theorem 1.1

In this section we will determine the Hausdorff dimension of $K_T(\Omega(c_1,c_2))$. The method is to find an appropriate probability measure $\widetilde{\mu}_{\mathbf{p}}$ supported on the $K_T(\Omega(c_1,c_2))$ in order to obtain a lower bound of the Hausdorff dimension of $K_T(\Omega(c_1,c_2))$. The key is to choose a concrete $\mathbf{p}$ such that $\dim_H\widetilde{\mu}_{\mathbf{p}}$ reaches its maximum. The estimation of the upper bound of its Hausdorff dimension will be done by Lemma 2.1.

Proposition 3.1. *Let Σ be defined as in Lemma* 2.2. *Then for each $\mathbf{p}=(p_d)_{d\in D}\in\Sigma$,*

$$\dim_H\widetilde{\mu}_{\mathbf{p}}=-\log_n m\sum_{d\in D}p_d\log_m p_d-(1-\log_n m)\sum_{b\in B}q_b\log_m q_b,$$

where let us recall that $q_b=\sum_{d\in D\cap(I\times b)}p_d$ for $b\in B=\sigma(D)$.

Proof. For any $\mathbf{p}=(p_d)_{d\in D}\in\Sigma$, let $\mu_{\mathbf{p}}$ and $\widetilde{\mu}_{\mathbf{p}}$ be the Borel probability measure on $\Omega(c_1,c_2)$ and $K_T(\Omega(c_1,c_2))$ respectively as above. For any point $x=(x_j)_{j=1}^{\infty}\in\Omega(c_1,c_2)$ and any integer $k\in\mathbb{N}$, taking logarithm in (6), we have

$$\log_m\widetilde{\mu}_{\mathbf{p}}\big(Q_k(x)\big)=\sum_{j=1}^{[k\log_n m]}\log_m p_{x_j}+\sum_{j=[k\log_n m]+1}^{k}\log_m q_{\sigma(x_j)}.$$

By Ergodic theorem (or Kolmogrov strong law of large numbers) we have for $\mu_{\mathbf{p}}$-a.e. $x=(x_j)_{j=1}^{\infty}\in D^{\mathbb{N}}$,

$$\lim_{k\to\infty}\frac{\log\widetilde{\mu}_p(Q_k(x))}{\log m^{-k}}=-\log_n m\sum_{d\in D}p_d\log_m p_d-(1-\log_n m)\sum_{b\in B}q_b\log_m q_b.$$

The desired result is then obtained by Lemma 2.3. □

Our next target is to maximize $\dim_H\widetilde{\mu}_{\mathbf{p}}$ for $\mathbf{p}=(p_d)_{d\in D}\in\Sigma$. To do it, we need the following simple observation.

Lemma 3.2. *Let $W(x)=\log_m x-\log_m n_s^{\log_n m}-\log_m(1-x)+\log_m\sum_{d\in D\setminus\Gamma_s}n_{\sigma(d)}^{\log_n m-1}$ with $x\in(0,1)$. Then $W(x)$ is strictly increasing on $(0,1)$ and $W(A)=0$ where $A=\frac{n_s^{\log_n m}}{\sum_{d\in D}n_{\sigma(d)}^{\log_n m-1}}$.*

Proposition 3.3. *Let $A=\frac{n_s^{\log_n m}}{\sum_{d\in D}n_{\sigma(d)}^{\log_n m-1}}$ and Σ be given by* (7). *Let $h(x)$ be defined as in Theorem* 1.1. *For $\mathbf{p}=(p_d)_{d\in D}\in\Sigma$ let*

$$g(\mathbf{p})=\dim_H\widetilde{\mu}_{\mathbf{p}}=-\log_n m\sum_{d\in D}p_d\log_m p_d-(1-\log_n m)\sum_{b\in B}q_b\log_m q_b. \tag{8}$$

There exists a unique probability vector $\mathbf{p}^*=(p_d^*)_{d\in D}\in\Sigma$ such that

$$g\big(\mathbf{p}^*\big)=\max_{\mathbf{p}\in\Sigma}g(\mathbf{p}).$$

Furthermore, $\mathbf{p}^*$ is an interior point of Σ, and precisely

(I) *If* $A \in [c_1, c_2]$, $\mathbf{p}^* = (p_d^*)_{d\in D}$ *is determined by*

$$p_d^* = \frac{n_{\sigma(d)}^{\log_n m-1}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} \quad \text{for } d \in D, \tag{9}$$

and at this moment $g(\mathbf{p}^*) = h(A) = \log_m \sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}$.

(II) *If* $A < c_1$, $\mathbf{p}^* = (p_d^*)_{d\in D}$ *is determined by*

$$\begin{cases} p_d^* = \dfrac{c_1}{n_s} \quad \text{for } d \in \Gamma_s, \\ p_d^* = \dfrac{n_{\sigma(d)}^{\log_n m-1}(1-c_1)}{\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}} \quad \text{for } d \in D\setminus\Gamma_s, \end{cases} \tag{10}$$

and at this moment

$$g(\mathbf{p}^*) = h(c_1) = c_1\left(\log_m n_s^{\log_n m} - \log_m c_1\right) + (1-c_1)\left(\log_m \sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1} - \log_m(1-c_1)\right).$$

(III) *If* $A > c_2$, $\mathbf{p}^* = (p_d^*)_{d\in D}$ *is determined by*

$$\begin{cases} p_d^* = \dfrac{c_2}{n_s} \quad \text{for } d \in \Gamma_s, \\ p_d^* = \dfrac{n_{\sigma(d)}^{\log_n m-1}(1-c_2)}{\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}} \quad \text{for } d \in D\setminus\Gamma_s, \end{cases} \tag{11}$$

and at this moment

$$g(\mathbf{p}^*) = h(c_2) = c_2\left(\log_m n_s^{\log_n m} - \log_m c_2\right) + (1-c_2)\left(\log_m \sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1} - \log_m(1-c_2)\right).$$

Proof. Note that $g(\mathbf{p})$ is a strictly concave function of a probability vector $\mathbf{p}$. In fact the first summand of $g(\mathbf{p})$ is strictly concave and the second is concave. However, Σ is convex and its constraint inequalities and its constraint equality are all linear. By a well-known property of strictly convex programming, there exists a unique probability vector $\mathbf{p}^* = (p_d^*)_{d\in D}$ in Σ such that $g(\mathbf{p})$ attains its maximum at $\mathbf{p} = \mathbf{p}^*$.

Next we show that $\mathbf{p}^* \in \operatorname{int}(\Sigma)$, i.e., $p_d^* \neq 0$ for all $d \in D$. Let

$$Z_1(\mathbf{p}) = -\log_n m \sum_{d\in D} p_d \log_m p_d \quad \text{and} \quad Z_2(\mathbf{p}) = (\log_n m - 1)\sum_{b\in B} q_b \log_m q_b.$$

Then $g(\mathbf{p}) = Z_1(\mathbf{p}) + Z_2(\mathbf{p})$. Suppose $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma \setminus \operatorname{int}(\Sigma)$. Let $D_1 = \{d \in D\colon\ p_d^* = 0\}$ and $D_2 = D \setminus D_1$. Then both D_1 and D_2 are nonempty. Take $\widetilde{\mathbf{p}} = (\widetilde{p}_d)_{d\in D} \in \operatorname{int}(\Sigma)$. Let $\mathbf{p}_t = t\widetilde{\mathbf{p}} + (1-t)\mathbf{p}^* = (t\widetilde{p}_d + (1-t)p_d^*)_{d\in D}$, $t \in [0,1]$. Then $\mathbf{p}_t \in \operatorname{int}(\Sigma)$ for $t \in (0,1]$ and $\mathbf{p}_0 = \mathbf{p}^*$. Note that

$$\begin{aligned} Z_1'(\mathbf{p}_t) = \frac{d}{dt} Z_1(\mathbf{p}_t) &= -\log_n m \frac{d}{dt}\left(\sum_{d\in D}\left(t\widetilde{p}_d + (1-t)p_d^*\right)\log_m\left(t\widetilde{p}_d + (1-t)p_d^*\right)\right) \\ &= -\log_n m \sum_{d\in D}\left(\widetilde{p}_d - p_d^*\right)\log_m\left(t\widetilde{p}_d + (1-t)p_d^*\right) \\ &= -\log_n m\left(\sum_{d\in D_1}\widetilde{p}_d \log_m(t\widetilde{p}_d) + \sum_{d\in D_2}\left(\widetilde{p}_d - p_d^*\right)\log_m\left(t\widetilde{p}_d + (1-t)p_d^*\right)\right). \end{aligned}$$

Thus we have $\lim_{t\to 0+} Z_1'(\mathbf{p}_t) = +\infty$. The same argument shows that $\lim_{t\to 0+} Z_2'(\mathbf{p}_t) = +\infty$ if $q_b^* = 0$ for some $b \in B$, or equals to a finite real number. Therefore, $\lim_{t\to 0+} g'(\mathbf{p}_t) = +\infty$. Note that $\lim_{t\to 0+} g(\mathbf{p}_t) = g(\mathbf{p}^*)$. Thus, $g(\mathbf{p}_t) > g(\mathbf{p}^*) = \max_{\mathbf{p}\in\Sigma} g(\mathbf{p})$ when t is small enough, leading a contradiction. Therefore $\mathbf{p}^*$ is an interior point of Σ.

Note that $g(\mathbf{p})$ is strictly concave on the convex set Σ for which constraint equality and constraint inequalities are all linear, and $\mathbf{p}^*$ is an interior point of Σ (it implies that $p_d^* \neq 0$ for all $d \in D$). Thus the admissible solution which satisfies Kuhn–Turker conditions on Σ is just the unique maximum point. Consider the generalized Largrange function

$$L(\mathbf{p},\lambda,\lambda_1,\lambda_2) = -\log_n m \sum_{d\in D} p_d \log_m p_d - (1-\log_n m)\sum_{b\in B} q_b \log_m q_b + \lambda\left(\sum_{d\in D} p_d - 1\right) + \lambda_1\left(\sum_{d\in\Gamma_s} p_d - c_1\right) + \lambda_2\left(c_2 - \sum_{d\in\Gamma_s} p_d\right).$$

Kuhn–Turker conditions mean

$$\begin{cases} \dfrac{\partial L(\mathbf{p},\lambda,\lambda_1,\lambda_2)}{\partial \mathbf{p}_d} = 0, & d\in D, \\ \lambda_1\left(\sum_{d\in\Gamma_s} p_d - c_1\right) = 0, \\ \lambda_2\left(c_2 - \sum_{d\in\Gamma_s} p_d\right) = 0, \\ \lambda_1 \geqslant 0, \quad \lambda_2 \geqslant 0. \end{cases}$$

For more information on Kuhn–Turker conditions or generalized Largrange function the readers can refer to the books [4,14].

In our setting Kuhn–Turker conditions and constraint conditions on Σ can be written as

$$\begin{cases} -(\log p_d + 1)\log_n m - (1-\log_n m)(\log q_{\sigma(d)} + 1) + \lambda \log m = 0, & d\in D\setminus\Gamma_s, \\ -(\log p_d + 1)\log_n m - (1-\log_n m)(\log q_{\sigma(d)} + 1) + (\lambda+\lambda_1-\lambda_2)\log m = 0, & d\in\Gamma_s, \\ \lambda_1\left(\sum_{d\in\Gamma_s} p_d - c_1\right) = 0, \\ \lambda_2\left(c_2 - \sum_{d\in\Gamma_s} p_d\right) = 0, \\ \lambda_1 \geqslant 0, \quad \lambda_2 \geqslant 0, \\ \sum_{d\in D} p_d = 1, \\ c_1 \leqslant \sum_{d\in\Gamma_s} p_d \leqslant c_2, \\ 0 < p_d < 1, \quad d\in D. \end{cases} \tag{12}$$

To solve the system (12) we consider the following three cases.

Case 1. $A = \frac{n_s^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} \in [c_1, c_2]$.

(a) If both $\lambda_1 = 0$ and $\lambda_2 = 0$, then system (12) determines the unique solution

$$p_d = \frac{n_{\sigma(d)}^{\log_n m-1}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}}, \quad d\in D.$$

(b) If $\lambda_1 \neq 0$ and $\lambda_2 = 0$, then system (12) has no solution. In fact, the equalities (we ignore the constraint inequalities) in (12) determine the unique solution

$$\begin{cases} p_d = \dfrac{c_1}{n_s}, & d\in\Gamma_s, \\ p_d = \dfrac{n_{\sigma(d)}^{\log_n m-1}(1-c_1)}{\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}}, & d\in D\setminus\Gamma_s, \\ \lambda_1 = \log_m c_1 - \log_m n_s^{\log_n m} - \log_m(1-c_1) + \log_m \sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}. \end{cases}$$

Note that $\lambda_1 = W(c_1)$ where $W(x)$ is defined as in Lemma 3.2. However, Lemma 3.2 tells that $\lambda_1 = W(c_1) \leqslant W(A) = 0$ since $A\in[c_1,c_2]$.

(c) If $\lambda_1 = 0$ and $\lambda_2 \neq 0$, then the system (12) has no solution again. For this case, the equalities in (12) determine the unique solution

$$\begin{cases} p_d = \dfrac{c_2}{n_s}, \quad d \in \Gamma_s, \\ p_d = \dfrac{n_{\sigma(d)}^{\log_n m-1}(1-c_2)}{\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}}, \quad d \in D \setminus \Gamma_s, \\ \lambda_2 = -\log_m c_2 + \log_m n_s^{\log_n m} + \log_m(1-c_2) - \log_m \displaystyle\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}. \end{cases}$$

Note that $\lambda_2 = -W(c_2)$ which implies that $\lambda_2 \leqslant -W(A) = 0$ by Lemma 3.2.

Thus (9) is proved. A direction computation shows $g(\mathbf{p}^*) = h(A)$.

Case 2. $A = \frac{n_s^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} < c_1$.

As discussed in Case 1, in this case the system (12) has a unique solution shown as in (10), correspondingly $\lambda_1 = W(c_1) > W(A) = 0$ and $\lambda_2 = 0$ (see (b) above). A straightforward calculation shows that $g(\mathbf{p}^*) = h(c_1)$.

Case 3. $A = \frac{n_s^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} > c_2$.

As discussed in Case 1, in this case the system (12) has a unique solution shown as in (11), correspondingly $\lambda_2 = -W(c_2) > -W(A) = 0$ and $\lambda_1 = 0$ (see (c) above). A straightforward calculation shows that $g(\mathbf{p}^*) = h(c_2)$. $\square$

By Lemmas 2.2 and 2.3, Propositions 3.1 and 3.3 we actually have obtained that $g(\mathbf{p}^*)$ (it equals to $h(A)$, $h(c_1)$ or $h(c_2)$ in different cases, see Proposition 3.3) is the lower bound of $\dim_H K_T(\Omega(c_1,c_2))$, i.e., $\dim_H K_T(\Omega(c_1,c_2)) \geqslant g(\mathbf{p}^*) = \max_{\mathbf{p}\in\Sigma} g(\mathbf{p})$. To finish the proof of Theorem 1.1 we need to show that $g(\mathbf{p}^*)$ is also the upper bound of $\dim_H K_T(\Omega(c_1,c_2))$. It will be done by means of Lemma 2.1.

For any $x = (x_i)_{i=1}^\infty \in \Omega(c_1,c_2)$ and any positive integer k, denote

$$S_k(x) = \sum_{d\in D\setminus\Gamma_s} N_k(x,d)\log_m n_{\sigma(d)}.$$

In the following the probability vector $\mathbf{p}^* = (p_d^*)_{d\in D} \in \Sigma$ on D is determined as in Proposition 3.3 (recall it satisfies $g(\mathbf{p}^*) = \max_{\mathbf{p}\in\Sigma} g(\mathbf{p})$). Taking logarithm in (6) we have that for any $x = (x_i)_{i=1}^\infty \in \Omega(c_1,c_2)$,

$$\begin{aligned}
\log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) &= \sum_{j=1}^{[k\log_n m]} \log_m p_{x_j}^* + \sum_{j=[k\log_n m]+1}^{k} \log_m q_{\sigma(x_j)}^* \\
&= \sum_{d\in\Gamma_s} N_{[k\log_n m]}(x,d)\log_m p_d^* + \sum_{d\in D\setminus\Gamma_s} N_{[k\log_n m]}(x,d)\log_m p_d^* \\
&\quad + \sum_{d\in\Gamma_s}\big(N_k(x,d) - N_{[k\log_n m]}(x,d)\big)\log_m q_{\sigma(d)}^* + \sum_{d\in D\setminus\Gamma_s}\big(N_k(x,d) - N_{[k\log_n m]}(x,d)\big)\log_m q_{\sigma(d)}^* \\
&= \sum_{d\in\Gamma_s} N_k(x,d)\log_m q_{\sigma(d)}^* + \sum_{d\in\Gamma_s} N_{[k\log_n m]}(x,d)\big(\log_m p_d^* - \log_m q_{\sigma(d)}^*\big) \\
&\quad + \sum_{d\in D\setminus\Gamma_s} N_k(x,d)\log_m q_{\sigma(d)}^* + \sum_{d\in D\setminus\Gamma_s} N_{[k\log_n m]}(x,d)\big(\log_m p_d^* - \log_m q_{\sigma(d)}^*\big).
\end{aligned} \tag{13}$$

When $A = n_s^{\log_n m}/\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1} \in [c_1,c_2]$, $p_d^* = n_{\sigma(d)}^{\log_n m-1}/\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}$ for all $d \in D$ and

$$g\big(\mathbf{p}^*\big) = h(A) = \log_m \sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}$$

(see Proposition 3.3(I)). Thus (13) reduces to

$$\begin{aligned}
\log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) &= \sum_{d\in\Gamma_s} N_k(x,d)\log_m \frac{n_s^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} - \sum_{d\in\Gamma_s} N_{[k\log_n m]}(x,d)\log_m n_s \\
&\quad + \sum_{d\in D\setminus\Gamma_s} N_k(x,d)\log_m \frac{n_{\sigma(d)}^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} - \sum_{d\in D\setminus\Gamma_s} N_{[k\log_n m]}(x,d)\log_m n_{\sigma(d)} \\
&= N_k(x,\Gamma_s)\log_m \frac{n_s^{\log_n m}}{\sum_{d\in D} n_{\sigma(d)}^{\log_n m-1}} - N_{[k\log_n m]}(x,\Gamma_s)\log_m n_s
\end{aligned}$$

$$- \big(k - N_k(x, \Gamma_s)\big) \log_m \sum_{d \in D} n_{\sigma(d)}^{\log_n m - 1} + S_k(x) \log_n m - S_{[k \log_n m]}(x).$$

Therefore, for any $x = (x_i)_{i=1}^{\infty} \in \Omega(c_1, c_2)$,

$$\begin{aligned}
\limsup_{k \to \infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) &= f(x, \Gamma_s) \log_m \frac{n_s^{\log_n m}}{\sum_{d \in D} n_{\sigma(d)}^{\log_n m - 1}} - \log_n m f(x, \Gamma_s) \log_m n_s \\
&\quad - \big(1 - f(x, \Gamma_s)\big) \log_m \sum_{d \in D} n_{\sigma(d)}^{\log_n m - 1} + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right) \\
&= -g(\mathbf{p}^*) + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right).
\end{aligned}$$

When $A = n_s^{\log_n m} / \sum_{d \in D} n_{\sigma(d)}^{\log_n m - 1} < c_1$, $\mathbf{p}^* = (p_d^*)_{d \in D}$ is given by (10), i.e., $p_d^* = \frac{c_1}{n_s}$ for $d \in \Gamma_s$, $p_d^* = \frac{n_{\sigma(d)}^{\log_n m - 1}(1 - c_1)}{\sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1}}$ for $d \in D \setminus \Gamma_s$ and $g(\mathbf{p}^*) = h(c_1)$ (see Proposition 3.3(II)). Thus (13) reduces to

$$\begin{aligned}
\log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) &= \sum_{d \in \Gamma_s} N_k(x, d) \log_m c_1 - \sum_{d \in \Gamma_s} N_{[k \log_n m]}(x, d) \log_m n_s \\
&\quad + \sum_{d \in D \setminus \Gamma_s} N_k(x, d) \log_m \frac{n_{\sigma(d)}^{\log_n m}(1 - c_1)}{\sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1}} - \sum_{d \in D \setminus \Gamma_s} N_{[k \log_n m]}(x, d) \log_m n_{\sigma(d)} \\
&= N_k(x, \Gamma_s) \log_m c_1 - N_{[k \log_n m]}(x, \Gamma_s) \log_m n_s \\
&\quad + \big(k - N_k(x, \Gamma_s)\big) \left(\log_m(1 - c_1) - \log_m \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} \right) + S_k(x) \log_n m - S_{[k \log_n m]}(x).
\end{aligned}$$

Therefore, for any $x = (x_i)_{i=1}^{\infty} \in \Omega(c_1, c_2)$,

$$\begin{aligned}
&\limsup_{k \to \infty} \frac{1}{k} \log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) \\
&= f(x, \Gamma_s) \log_m c_1 - \log_n m f(x, \Gamma_s) \log_m n_s + \Big(1 - f(x, \Gamma_s)\Big) \left(\log_m(1 - c_1) - \log_m \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} \right) \\
&\quad + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right) \\
&= \log_m(1 - c_1) - \log_m \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} + f(x, \Gamma_s) \left(\log_m c_1 - \log_m n_s^{\log_n m} - \log_m(1 - c_1) + \log_m \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} \right) \\
&\quad + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right) \\
&\geqslant \log_m(1 - c_1) - \log_m \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} + c_1 \left(\log_m c_1 - \log_m n_s^{\log_n m} - \log_m(1 - c_1) + \log_m \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} \right) \\
&\quad + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right) \\
&= -h(c_1) + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right) \\
&= -g\big(\mathbf{p}^*\big) + \log_n m \limsup_{k \to \infty} \left(\frac{S_k(x)}{k} - \frac{S_{[k \log_n m]}(x)}{k \log_n m} \right),
\end{aligned}$$

where the above inequality follows from the fact that in this case $\log c_1 - \log n_s^{\log_n m} - \log(1 - c_1) + \log \sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1} = W(c_1) > 0$ by Lemma 3.2.

When $A = n_s^{\log_n m} / \sum_{d \in D} n_{\sigma(d)}^{\log_n m - 1} > c_2$, $\mathbf{p}^* = (p_d^*)_{d \in D}$ is given by (11), i.e., $p_d^* = \frac{c_2}{n_s}$ for $d \in \Gamma_s$, $p_d^* = \frac{n_{\sigma(d)}^{\log_n m - 1}(1 - c_2)}{\sum_{d \in D \setminus \Gamma_s} n_{\sigma(d)}^{\log_n m - 1}}$ for $d \in D \setminus \Gamma_s$ and $g(\mathbf{p}^*) = h(c_2)$ (see Proposition 3.3(III)). Thus (13) reduces to

$$\begin{aligned}\log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) &= \sum_{d\in\Gamma_s} N_k(x,d)\log_m c_2 - \sum_{d\in\Gamma_s} N_{[k\log_n m]}(x,d)\log_m n_s \\ &\quad + \sum_{d\in D\setminus\Gamma_s} N_k(x,d)\log_m \frac{n_{\sigma(d)}^{\log_n m}(1-c_2)}{\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}} - \sum_{d\in D\setminus\Gamma_s} N_{[k\log_n m]}(x,d)\log_m n_{\sigma(d)} \\ &= N_k(x,\Gamma_s)\log_m c_2 - N_{[k\log_n m]}(x,\Gamma_s)\log_m n_s \\ &\quad + \big(k - N_k(x,\Gamma_s)\big)\Bigg(\log_m(1-c_2) - \log_m \sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}\Bigg) + S_k(x)\log_n m - S_{[k\log_n m]}(x).\end{aligned}$$

Therefore, for any $x=(x_i)_{i=1}^\infty \in \Omega(c_1,c_2)$,

$$\begin{aligned}&\limsup_{k\to\infty}\frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) \\ &\quad = f(x,\Gamma_s)\log_m c_2 - \log_n m f(x,\Gamma_s)\log_m n_s + \big(1-f(x,\Gamma_s)\big)\Bigg(\log_m(1-c_2) - \log_m\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}\Bigg) \\ &\quad\quad + \log_n m \limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg) \\ &\quad = \log_m(1-c_2) - \log_m\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1} + f(x,\Gamma_s)\Bigg(\log c_2 - \log n_s^{\log_n m} - \log(1-c_2) + \log\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}\Bigg) \\ &\quad\quad + \log_n m \limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg) \\ &\quad \geqslant \log_m(1-c_2) - \log_m\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1} + c_2\Bigg(\log c_2 - \log n_s^{\log_n m} - \log(1-c_2) + \log\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1}\Bigg) \\ &\quad\quad + \log_n m \limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg) \\ &\quad = -h(c_2) + \log_n m \limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg) \\ &\quad = -g\big(\mathbf{p}^*\big) + \log_n m \limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg),\end{aligned}$$

where above inequality follows from the fact that in this case $\log c_2 - \log n_s^{\log_n m} - \log(1-c_2) + \log\sum_{d\in D\setminus\Gamma_s} n_{\sigma(d)}^{\log_n m-1} = W(c_2) < 0$ by Lemma 3.2.

In a word, in all cases we have that for any $x=(x_j)_{j=1}^\infty \in \Omega(c_1,c_2)$,

$$\limsup_{k\to\infty}\frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) \geqslant -g\big(\mathbf{p}^*\big) + \log_n m \limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg).$$

Now we will show that for every point $x=(x_j)_{j=1}^\infty \in \Omega(c_1,c_2)$,

$$\limsup_{k\to\infty}\bigg(\frac{S_k(x)}{k} - \frac{S_{[k\log_n m]}(x)}{k\log_n m}\bigg) \geqslant 0. \tag{14}$$

This essentially can be derived from Lemma 4.1 in [15]. For every point $x=(x_j)_{j=1}^\infty \in \Omega(c_1,c_2)$ and any $k\in\mathbb{N}$, from the definition of $S_k(x)$, it is obviously that

$$\sup_k \big|S_{k+1}(x) - S_k(x)\big| < \infty. \tag{15}$$

For a fixed $x=(x_j)_{j=1}^\infty \in \Omega(c_1,c_2)$, let $T(k)=S_k(x)$. We extend T to $[1,+\infty)$ by piecewise linear interpolation. Then T is a Lipschitz function by (15). Now define $Y:[0,\infty)\to\mathbb{R}$ by

$$Y(z) = e^{-z}T\big(e^z\big).$$

We claim that $Y(z)$ is bounded and uniformly continuous on $[0,\infty)$. In fact,

$$\big|Y(z)\big| \leqslant \big|Y(0)\big|e^{-z} + \big|Y(z) - Y(0)e^{-z}\big| \leqslant \big|T(1)\big| + e^{-z}\big|T\big(e^z\big) - T(1)\big| \leqslant \big|T(1)\big| + \operatorname{Lip} T,$$

and for any $\delta > 0$,

$$\begin{aligned}\left|Y(z+\delta) - Y(z)\right| &= \left|e^{-(z+\delta)}T\left(e^{z+\delta}\right) - e^{-z}T\left(e^{z}\right)\right| \leqslant e^{-(z+\delta)}\left|T\left(e^{z+\delta}\right) - T\left(e^{z}\right)\right| + \left|Y(z)\right|\left(1-e^{-\delta}\right) \\ &\leqslant \left(1-e^{-\delta}\right)\operatorname{Lip} T + \left(1-e^{-\delta}\right)\left(\left|T(1)\right| + \operatorname{Lip} T\right).\end{aligned}$$

Now for any $v > -\log\log_n m$,

$$\begin{aligned}\left|\int_{-\log\log_n m}^{v} \left(Y(z) - Y(z+\log\log_n m)\right) dz\right| &= \left|\int_{-\log\log_n m}^{v} Y(z)\,dz - \int_{-\log\log_n m}^{v} Y(z+\log\log_n m)\,dz\right| \\ &= \left|\int_{-\log\log_n m}^{v} Y(z)\,dz - \int_{0}^{v+\log\log_n m} Y(z)\,dz\right| \\ &= \left|\int_{0}^{-\log\log_n m} Y(z)\,dz + \int_{v}^{v+\log\log_n m} Y(z)\,dz\right| \\ &\leqslant \left|\int_{0}^{-\log\log_n m} Y(z)\,dz\right| + \left|\int_{v}^{v+\log\log_n m} Y(z)\,dz\right| < +\infty,\end{aligned}$$

since Y is bounded on $[0, +\infty)$. Therefore,

$$\limsup_{z\to+\infty}\left(Y(z) - Y(z+\log\log_n m)\right) \geqslant 0.$$

Otherwise $|\int_{-\log\log_n m}^{v}(Y(z) - Y(z+\log\log_n m))\,dz| \to \infty$ as $v \to \infty$.

By letting $z = \log t$, this gives

$$\limsup_{t\to+\infty}\left(\frac{T(t)}{t} - \frac{T(t\log_n m)}{t\log_n m}\right) \geqslant 0.$$

Note that

$$\begin{aligned}\frac{T(t)}{t} - \frac{T(t\log_n m)}{t\log_n m} &= \left(\frac{T(t) - T([t])}{t} - \frac{T(t\log_n m) - T([t\log_n m])}{t\log_n m}\right) + \frac{T([t])}{[t]}\left(\frac{[t]}{t} - 1\right) \\ &\quad + \left(\frac{S_{[[t]\log_n m]}(x)}{[t]\log_n m} - \frac{S_{[t\log_n m]}(x)}{t\log_n m}\right) + \left(\frac{S_{[t]}(x)}{[t]} - \frac{S_{[[t]\log_n m]}(x)}{[t]\log_n m}\right),\end{aligned} \tag{16}$$

where, as before, $[t]$ with $t \in \mathbb{R}$ denotes the greatest integer function. However, the first three terms in the right side of (16) tend to zero as $t \to +\infty$ by the facts that both functions $|T(t) - T([t])|$ and $Y(z)$ are bounded, and $Y(z)$ is uniformly continuous. Hence (14) holds. Therefore, for every $x = (x_j)_{j=1}^{\infty} \in \Omega(c_1, c_2)$ we have

$$\limsup_{k\to\infty}\frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}\left(Q_k(x)\right) \geqslant -g\left(\mathbf{p}^*\right),$$

which leads to

$$\limsup_{k\to\infty}\left(k\delta + \log_m \widetilde{\mu}_{\mathbf{p}^*}\left(Q_k(x)\right)\right) = \limsup_{k\to\infty} k\left(\delta + \frac{1}{k}\log_m \widetilde{\mu}_{\mathbf{p}^*}\left(Q_k(x)\right)\right) = +\infty,$$

for any $\delta > g(\mathbf{p}^*)$. Now Lemmas 2.1(2) and 2.2 imply that $\dim_H K_T(\Omega(c_1, c_2)) \leqslant g(\mathbf{p}^*)$.

4. Proof of Theorem 1.2

When the conditions in Theorem 1.2(I) hold, the corresponding probability vector $\mathbf{p}^* = (p_d^*)_{d\in D}$ is given by $p_d^* = \frac{1}{\#D}$ for all $D = d \in D$ by (9). Then $A = \frac{1}{\#B}$ and

$$\gamma = \dim_H K_T\left(\Omega(c_1, c_2)\right) = h(A) = \log_m \#D + (\log_n m - 1)\log_m \frac{\#D}{\#B}$$

by Theorem 1.1. Then by (6)

$$k\gamma + \log_m \widetilde{\mu}_{\mathbf{p}^*}\big(Q_k(x)\big) = k\left(\log_m \#D + (\log_n m - 1)\log_m \frac{\#D}{\#B}\right) + [k\log_n m]\log_m \frac{1}{\#D} + \big(k - [k\log_n m]\big)\log_m \frac{1}{\#B}$$
$$= \big(k\log_n m - [k\log_n m]\big)\log_m \frac{\#D}{\#B}$$

for every $x = (x_j)_{j=1}^{\infty} \in \Omega(c_1, c_2)$ and all $k \in \mathbb{N}$. Then (I) is justified by Lemmas 2.1(3) and 2.2.

To prove (II) we need a result obtained by O.A. Nielsen in [19]. For any probability vector $\mathbf{p} = (p_d)_{d\in D}$, let

$$\Delta_{\mathbf{p}} = \left\{x = (x_j)_{j=1}^{\infty} \in D^{\mathbb{N}}\colon \lim_{k\to\infty} \frac{N_k(x,d)}{k} = p_d \text{ for all } d \in D\right\},$$

where $N_k(x,d)$ is defined as (2). The vector $\mathbf{p} = (p_d)_{d\in D}$ is said to be *uniformly distributed* on D if $p_d = \frac{1}{\#D}$ for all $d \in D$. O.A. Nielsen (cf. [19, Theorems 1 and 3]) proved that

(a) $\dim_H K_T(\Delta_{\mathbf{p}}) = -\log_n m \sum_{d\in D} p_d \log_m p_d - (1 - \log_n m)\sum_{b\in B} q_b \log_m q_b$, where K_T is the representation map defined by (1);

(b) If $\mathbf{p}$ is not uniformly distributed on D or if D does not have uniform horizontal fibres then

$$\mathcal{H}^{\delta}\big(K_T(\Delta_{\mathbf{p}})\big) = \infty,$$

where $\delta = \dim_H K_T(\Delta_{\mathbf{p}})$.

Suppose that $\mathbf{p}^* = (p_d^*)_{d\in D}$ is the probability vector determined as in Proposition 3.3. Then

$$K_T\big(\Omega(c_1,c_2)\big) \supset K_T\big(\Delta_{\mathbf{p}^*}\big) \quad \text{and} \quad \dim_H K_T\big(\Omega(c_1,c_2)\big) = \dim_H K_T\big(\Delta_{\mathbf{p}^*}\big) = g\big(\mathbf{p}^*\big)$$

by Theorem 1.1, Proposition 3.3, (8) and (a). Thus it follows from (b) that $\mathcal{H}^{\gamma}(K_T(\Omega(c_1,c_2))) = \infty$ when D does not have uniform horizontal fibres.

In the following, suppose that D has uniform horizontal fibres. What we need to do is to check that $\mathbf{p}^* = (p_d^*)_{d\in D}$ is not uniformly distributed on D if $A = n_s^{\log_n m} / \sum_{d\in D} n_{\sigma(d)}^{\log_n m - 1} = 1/\#B \notin [c_1, c_2]$ (recall that B is the projection of D onto its second coordinate and that each horizontal fibre has same number of elements when D has uniform horizontal fibres).

Case 1. $A = 1/\#B < c_1$. Then for each $d \in \Gamma_s$ we have $p_d^* = \frac{c_1}{n_s} > \frac{1}{\#Bn_s} = \frac{1}{\#D}$ by (10).

Case 2. $A = 1/\#B > c_2$. Then for each $d \in \Gamma_s$ we have $p_d^* = \frac{c_2}{n_s} < \frac{1}{\#Bn_s} = \frac{1}{\#D}$ by (11).

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