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Multifractal Analysis of Lyapunov Exponent for Continued Fraction and Manneville–Pomeau Transformations and Applications to Diophantine Approximation

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Abstract: We extend some of the theory of multifractal analysis for conformal expanding systems to two new cases: The non-uniformly hyperbolic example of the Manneville–Pomeau equation and the continued fraction transformation. A common point in the analysis is the use of thermodynamic formalism for transformations with infinitely many branches.

We effect a complete multifractal analysis of the Lyapunov exponent for the continued fraction transformation and as a consequence obtain some new results on the precise exponential speed of convergence of the continued fraction algorithm. This analysis also provides new quantitative information about cuspital excursions on the modular surface.

1. Introduction

In this paper we extend some aspects of the multifractal analysis which are useful in studying problems in Diophantine approximation, in studying the behavior of geodesics on the modular surface, and in studying an important non-uniformly hyperbolic dynamical system. In particular, we first study the continued fraction (Gauss) transformation $T_1 : [0, 1] \to [0, 1]$ defined by

$$T_1 x \equiv \frac{1}{x} - \left[\frac{1}{x}\right] = \left\{\frac{1}{x}\right\},$$

for $x \neq 0$ and $T_1(0) \equiv 0$. Here $[1/x]$ denotes the integer part of $1/x$.

This map (see Fig. 1) is uniformly hyperbolic, but being naturally coded by an infinite alphabet and having infinite topological entropy, the usual theory of multifractal analysis for conformal maps [PW1,PW2] does not directly apply.

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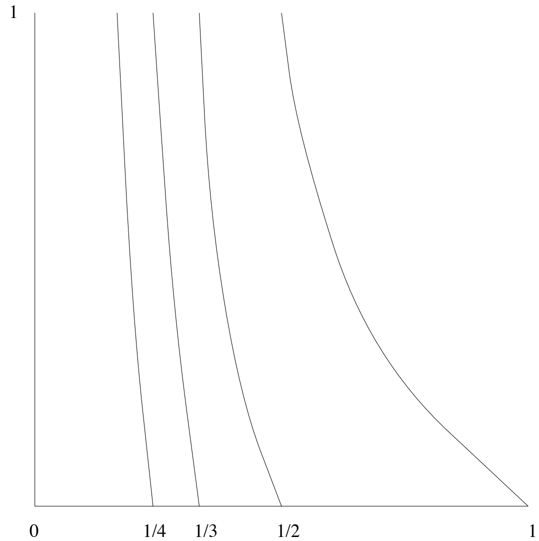

Fig. 1. Graph of the map T_1

For the continued fraction transformation, our multifractal analysis of the Lypaunov exponent will yield new detailed information about the precise *rates* of Diophantine approximation to irrational numbers. This has immediate implications for cuspital excursions of geodesics on the modular surface.

We also study the Manneville–Pomeau transformation [MP1] defined by

$$\begin{cases} T_2 : [0,1] \to [0,1] \\ T_2 x = x + x^{1+\alpha} \mod 1, \end{cases}$$

where α is a non-negative constant.

This important model (see Fig. 2a) is a non-uniformly hyperbolic transformation having the most benign type of non-hyperbolicity: an indifferent fixed point at 0, i.e., $T_2(0) = 0$ and $T_2'(0) = 1$, and exhibits *intermittent* behavior [MP1]. In general one could not expect the full force of multifractal analysis to apply for general non-uniformly hyperbolic systems. However, we show that part of the theory carries over to this, and similar, transformations.

A key aspect of our analysis is a reduction, via inducing, of this system to a countable state uniformly hyperbolic system (i.e., using the Schweiger jump transformation). In particular, we show that the Lyapunov exponent attains an interval of values, realized on dense sets with positive Hausdorff dimension. This quantifies the size of the set of points whose orbits spend a disproportionate amount of time near the indifferent fixed point.

A major tool in the multifractal analysis is the use of symbolic dynamics and thermodynamic formalism, i.e., pressure (and its derivatives), equilibrium states, etc. In the present context of infinite state subshifts of finite type, we are fortunate in having at our disposal a theory worked out by Walters [Wa1].

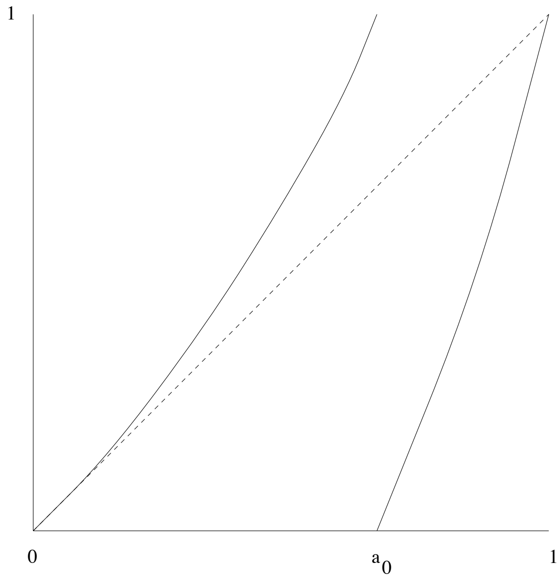

Fig. 2a. Graph of the map T_2

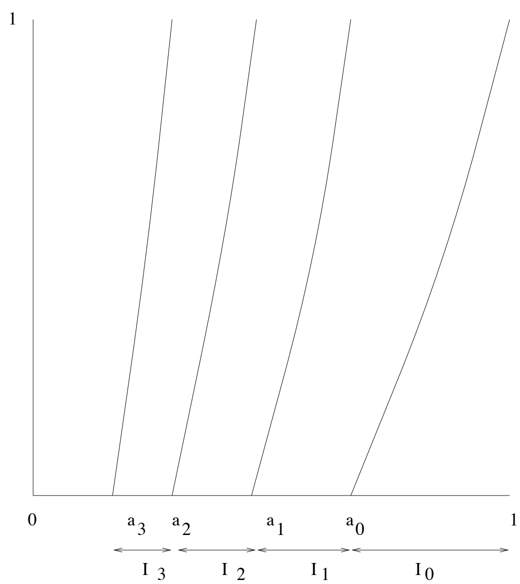

Fig. 2b. Graph of the map $\widehat{T}_2$

Finally, while we state our main results for these two model transformations, our analysis works in greater generality. Our results are valid for interval maps which have induced maps (coded by an infinite alphabet) which satisfy the EMR (expanding-Markov-Rényi) property (see Sect. 2) and potentials (for equilibrium states) which satisfy properties (W1) and (W2) (see Sect. 2). In particular, our analysis works for the family of functions, which Prellberg calls $\mathcal{C}_s$ [Pr], containing certain piecewise monotone transformations of the interval with an indifferent fixed point at the boundary. For instance, the Farey map

$$F(x) = \begin{cases} x/(1-x), & 0 \leq x < 1/2 \\ (1-x)/x, & 1/2 \leq x \leq 1 \end{cases}$$

belongs to the class $\mathcal{C}_1$.

There are other examples to which a similar sort of analysis extends, particularly in the realm of number theoretic analysis. Our analysis applies to the family of maps $T_s(x) = \{1/(s(1-x))\}$ for $0 < s < 4$ and $s \neq 4\cos^2(\pi/q)$, $q = 3, 4, \cdots$, and there is a theory of $s-$backward continued fractions based on this family of maps [GH]. Our analysis of the Manneville–Pomeau map easily extends to the complex continued fraction algorithm [Sc, §23.6.], where the underlying map is conformal and has an indifferent fixed point.

The underlying map for the Jacobi Perron Algorithm [Sc, §23.1] for continued fractions in several variables is hyperbolic with infinitely many branches, but is not conformal. In this case, a multifractal analysis based on local entropy rather than pointwise dimension should routinely follow.

Shortly after the completion of this manuscript, the authors were given a recent preprint [N] which contains an argument corresponding to Proposition 3(4).

2. Markov Maps and Inducing

Let I denote an interval of real numbers. For the maps T_1 and T_2 the interval I will be $[0, 1]$. The study of our two transformations can be reduced to studying transformations of the following general form:

Definition. *We say that a transformation $T : I \to I$ is an* **EMR** *Transformation if we may write $I = \cup_{n=0}^{\infty} I_n$ as a countable union of closed intervals (with disjoint interiors $\mathring{I}_n$), which we call* **basic subintervals***, such that*

(1) *The map T is C^2 on $\cup_{k=1}^{\infty} \mathring{I}_n$.*
(2) *Some power of T is uniformly expanding, i.e., there exists a positive integer r and $\alpha > 0$ such that $|(T^r)'(x)| \geq \alpha > 1$ for all $x \in \cup_{n=1}^{\infty} \mathring{I}_n$.*
(3) *The map T is Markov.*
(4) *The map T satisfies Renyi's condition, i.e., there exists a positive number K such that*

$$\sup_n \sup_{x,y,z \in I_n} \frac{|T''(x)|}{|T'(y)||T'(z)|} \leq K < \infty.$$

It is easy to verify that $|(T_1^2)'(x)| \geq 4$ for all $x \in \cup_{n=1}^{\infty} \mathring{I}_n$.

Proposition 1 ([Sc, Wa1, p. 148])**.** *The continued fraction transformation T_1 satisfies EMR.*

For $\alpha = 0$ the Manneville–Pomeau transformation is the usual doubling map of the circle. For $0 < \alpha < 1$, Thaler [T] constructed an finite absolutely continuous invariant measure. For $\alpha > 1$, Thaler also constructed a sigma-finite but not finite absolutely continuous invariant measure. However, we shall not consider this range.

The Manneville–Pomeau map has two branches for $[0, a_0]$ and $[a_0, 1]$ where $1 = a_0 + a_0^\alpha$. There is a natural topological conjugacy of this map to the standard doubling map and has topological entropy equal to log 2. Although this map is not hyperbolic since the derivative $T_2'(0) = 1$, the induced map on $[a_0, 1]$ is hyperbolic [Sc]. More precisely, we choose the monotone decreasing sequence $a_n \to 0$ such that $T(a_{n+1}) = a_n$ and define $I_n = [a_n, a_{n-1}]$, for $n \geq 1$, and $I_0 = [a_0, 1]$. For $a_{n+1} < x < a_n$ we define $\widehat{T}_2(x) = T_2^n(x)$ (see Fig. 2B). The map $\widehat{T}_2$ is piecewise analytic on countably many intervals $\{I_n\}$ and is uniformly hyperbolic. Each point which is not an end point of an interval I_n has two pre-images under T_2.

Proposition 2 ([T, pp. 312–313, I, §2]). *For $0 < \alpha < 1$, the transformation $\widehat{T}_2$ satisfies EMR.*

We will first effect a multifractal analysis for the map $\widehat{T}_2$ and then *transfer* the multifractal analysis to the map T_2.

For an EMR transformation T we define the set $\boldsymbol{\mathcal{O}} = \cup_{k=0}^\infty T^{-k}(\cup_{k=0}^\infty \partial I_k)$, where ∂I_k denotes the two endpoints of the interval I_k. Clearly $\mathcal{O}$ is a countable subset of I. We denote by $\mathbf{I_n(x)}$ the element of the refinement $\vee_{i=0}^{n-1} T^{-i}\big(\{I_n\}\big)_{n=1}^\infty$ of the original partition containing x. Every $x \in I \setminus \mathcal{O}$ has a *unique* symbolic coding since for every $k \in \mathbb{N}$ there is a unique basic subinterval that contains $T^k(x)$.

A fundamental property of an EMR transformation is that it satisfies what is usually called the **Jacobian Estimate** [CFS, p. 171], i.e., there exists positive K such that for all $x \in I \setminus \mathcal{O}$ one has

$$0 < \frac{1}{K} \leq \sup_{n \geq 0} \sup_{y \in I_n(x)} \left| \frac{(T^n)'(x)}{(T^n)'(y)} \right| \leq K < \infty. \tag{J}$$

This property will be exploited many times in the proof of Proposition 3 and is also an essential component in the proof of Proposition 2.

For the reasons mentioned in the introduction, it is natural that for infinitely coded maps like T_1 and T_2, the class of potentials which admit unique equilibrium states is more restrictive than in the usual case. We will now discuss a class of potentials, which we denote $\boldsymbol{\mathcal{W} = \mathcal{W}(T)}$ (for Walters), for which the usual results in thermodynamic formalism are valid.

Let $T : I \to I$ be an EMR transformation and $\phi : I \to \mathbb{R}$ a function such that $\exp \phi$ is continuous and satisfies the following two properties:

(W1) There exits a constant $C > 0$ such that the sum $\sum_{Ty=x} \exp \phi(y) \leq C$, for all $x \in I$.

(W2) The function

$$C_\phi(x, x') = \sup_{n \geq 1} \sup_{y \in T^{-n}x} \sum_{i=0}^{n-1} \left| \phi(T^i y) - \phi(T^i y') \right|$$

is bounded by a constant C_ϕ and $C_\phi(x, x')$ tends to zero as $|x - x'| \to 0$.

Such potentials will be said to belong to the class $\mathcal{W}$.

The thermodynamic **Pressure** $P(\phi)$ can be defined for a continuous function ϕ via the variational principle as

$$P(\phi) = \sup_{\substack{\mu \\ T-\text{inv}}} \left(h_\mu(T) + \int_I \phi d\mu \right),$$

where $h_\mu(T)$ denotes the measure theoretic entropy of T, and the supremum is taken over all $T-$invariant Borel probability measures μ [Wa2].

Example 1. It is an important feature of allowable potentials for the continued fraction transformation T_1 that $\phi(x) \to -\infty$ as $x \to 0$, in contrast to the usual boundedness of potentials for EMR maps having only finitely many intervals. Observe also that for T_1, the non-zero constant function never satisfies (W1).

For T_1 with the piecewise analytic potential $\phi = -t \log |T_1'|$ we have that

$$\sum_{T_1 y = x} \exp \phi(y) \leq \sum_{n=1}^{\infty} \frac{1}{n^{2t}} < \infty,$$

and condition (W1) holds for $t > \frac{1}{2}$.

Condition (W2) can similarly be seen to hold on the same range [Wa1]. Let $x, x' \in X$, $y \in T_1^{-n} x$ and let y' be the corresponding point of $T_1^{-n} x'$. Two applications of the Mean Value Theorem yield that

$$|\phi(T^i y) - \phi(T^i y')| = \left| t \frac{T_1''(z_i)}{T_1'(z_i)(T_1^{n-i})'(w_i)} \right| |x - x'|,$$

where w_i and z_i lie between $T_1^i y$ and $T_1^i y'$. Using properties (2) and (4) of EMR we obtain that $|\phi(T^i y) - \phi(T^i y')| \leq Kt|x - x'|/\alpha^{n-i-1}$ and thus

$$\sum_{i=0}^{n-1} \left| \phi(T^i y) - \phi(T^i y') \right| \leq Kt|x - x'|/(\alpha - 1).$$

Condition (W2) immediately follows.

On the range $t > \frac{1}{2}$, the function $t \mapsto P(-t \log |T_1'|)$ is analytic since the function $\exp(P(-t \log |T_1'|))$ is an isolated eigenvalue for the associated transfer operator, about which we shall say more in [M1, Sect. VI]. It is also strictly convex (see Fig. 3). Moreover, as in the usual theory, one can use perturbation theory to compute the second derivative of $\exp(P(-t \log |T_1'|))$ and deduce this is strictly convex. It follows from the standard Rohlin equality that $P(-\log |T_1'|) = 0$. Mayer [M1] has also shown that the function $t \mapsto P(-t \log |T_1'|)|$ has a logarithmic singularity at $t = \frac{1}{2}$. This will explain the range of values in the statement of Theorem 1.

Example 2. For the induced Manneville–Pomeau transformation $\widehat{T}_2$ with the piecewise analytic potential $\phi = -t \log |\widehat{T}_2'|$, we have that

$$\sum_{\widehat{T}_2 y = x} \exp \phi(y) = \sum_{\widehat{T}_2 y = x} \frac{1}{|\widehat{T}_2'(y)|^t} = \sum_{n=1}^{\infty} |G_n'(x)|^t,$$

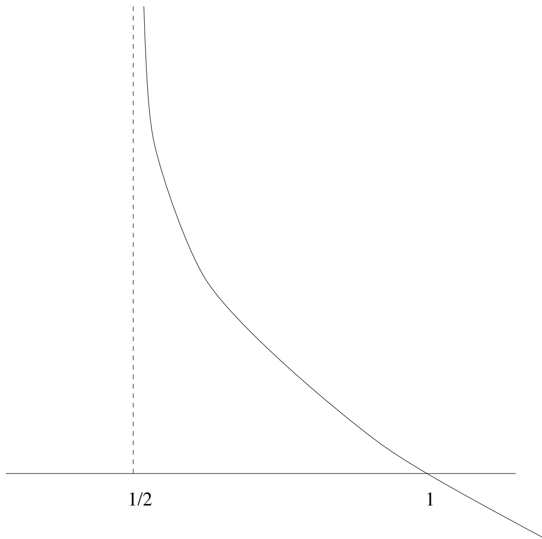

Fig. 3. Graph of $t \mapsto P(-t \log |T_1'|)$

where $G_n = F_1 F_0^{n-1}$, and F_0, F_1 denote the two branches of the inverse of T_2. Properties (W1) and (W2) for $\widehat{T_2}$ are established in [T, p. 312] and [I, Lemma 2.2] for $t < 1$.

Prellburg [Pr,PS,V] showed that the function $P(-t \log |T_2'|) = 0$ for $t > 1$, and that on the range $1/\alpha < t < 1$, the function $t \mapsto P(-t \log |T_2'|)$ is analytic and strictly convex Fig. 4). It follows from the Rohlin equality that $P(-\log |T_2'|) = 0$. Using the variational principle, it is easy to see that for all values of t we have that $P(-t \log |T_2'|) \geq 0$, since if we take μ to be the Dirac measure supported at 0, then $h_\mu(T_2) = 0$ and $\int_I \log |T_2'| d\mu = 0$. Lopes [L] has studied the precise nature of the singularity of the map $t \mapsto P(-t \log |T_1'|)$ at $t = 1$. This explains the range of values in the statement of Theorem 2.

Lyapunov exponents. Lyapunov exponents measure the exponential rate of divergence of *infinitesimally close* orbits of a smooth dynamical system. These exponents are intimately related with the global stochastic behavior of the system and are fundamental invariants of a smooth dynamical system. For a transformation $T : I \to I$ we define the **Lyapunov exponent** $\lambda(x)$ of T by

$$\lambda(x) \equiv \lim_{n \to \infty} \frac{1}{n} \log |(T^n)'(x)| = \lim_{n \to \infty} \frac{1}{n} \log \prod_{i=0}^{n-1} |T'(T^i x)|, \tag{3}$$

when the limit exists. The function $\lambda(x)$ is clearly $T-$invariant.

There is a natural decomposition of the interval I by level sets of the Lyapunov exponent $L_\beta = \{x \in I : \lambda(x) = \beta\}$,

$$I = \bigcup_{-\infty < \beta < \infty} L_\beta \cup \{x \in I \mid \lambda(x) \text{ does not exist}\}.$$

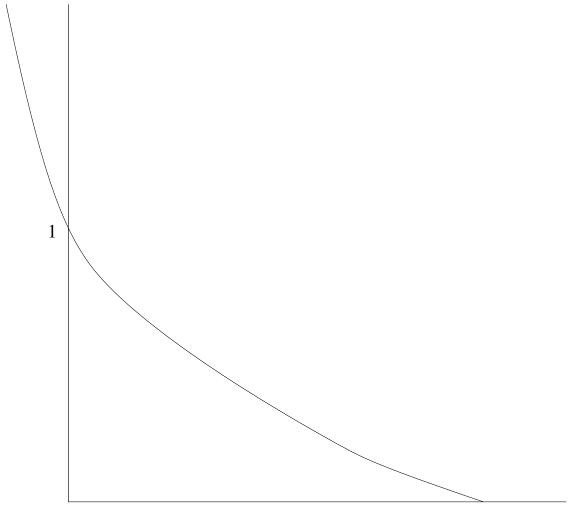

Fig. 4. Graph of $t \mapsto P(-t \log |T_2'|)$

To study this complicated decomposition we introduce the **Lyapunov spectrum** by considering the level sets of the Lyapunov exponent and by defining

$$g(\beta) = \dim_H(L_\beta),$$

where $\dim_H(L_\beta)$ denotes the Hausdorff dimension of L_β.

In the next section we relate the Lyapunov spectrum to a related spectrum for local dimension and prove several remarkable properties about it.

3. The Multifractal Analysis of Equilibrium States

The general concept of a multifractal analysis for a dynamical system concerns a detailed study of the exceptional behavior of asymptotically defined dynamical quantities such as pointwise dimension, Lyapunov exponent, local entropy, Birkhoff average, etc. In many examples with hyperbolic structure these quantities are constant almost everywhere, with respect to an appropriate ergodic measure.

We consider two notions of local or pointwise dimension with respect to an invariant measure. The **pointwise dimension** of a Borel probability measure μ defined on I is defined by

$$d_\mu(x) = \lim_{r \to 0} \frac{\log \mu(B(x, r))}{\log r}, \tag{1}$$

when the limit exists. Here $B(x, r)$ denotes the ball of radius r centered at the point x.

The **Markov pointwise dimension** of a $T-$invariant Borel probability measure μ is defined on I, defined by

$$\delta_\mu(x) \equiv \lim_{n \to \infty} \frac{\log \mu(I_n(x))}{-\log \ell(I_n(x))},$$

when the limit exists. Here the intervals $\{I_n\}$ are those in the definition of **ERM** transformation and $\ell(I_n(x))$ denotes the length of the interval $I_n(x)$.

In [PW1,PW2] the authors establish deep relationships between these two notions of local dimension for equilibrium states for conformal expanding maps. By definition conformal expanding maps are local homeomorphisms, which the maps we consider in this paper are not. Not surprisingly, some of relationships that are valid at *every* point in certain sets are no longer true, and this is a major obstacle in generalizing the usual multifractal theory to these more general transformations.

There are natural decompositions of the interval I by level sets $K_\alpha = \{x : d_\mu(x) = \alpha\}$,

$$I = \bigcup_{-\infty<\alpha<\infty} K_\alpha \cup \{x \in I \mid d_\nu(x) \text{ does not exist}\},$$

and by the level sets $K_\alpha^M = \{x : \delta_\mu(x) = \alpha\}$,

$$I = \bigcup_{-\infty<\alpha<\infty} K_\alpha^M \cup \{x \in I \mid \delta_\nu(x) \text{ does not exist}\}.$$

Since we are mostly interested in effecting a multifractal analysis of the Lyapunov exponent, we only analyze the latter decomposition. To study this decomposition we define the **Markov dimension spectrum**

$$f_\mu(\alpha) = \dim_H(K_\alpha^M),$$

where $\dim_H(K_\alpha^M)$ denotes the Hausdorff dimension of the set K_α^M. This is similar to the Lyapunov spectrum we defined in Sect. 2. The parts of the multifractal analysis which we establish are that under suitable hypotheses, the function $f_\mu(\alpha)$ is real analytic and strictly convex (on a suitable interval) and is given in terms of thermodynamic formalism.

More precisely, let $T : I \to I$ be an EMR transformation, and let $\phi \in \mathcal{W}$. The **equilibrium state** μ is a T-invariant probability measure μ such that there exists a positive C such that

$$\frac{1}{C} \le \frac{\mu(I_n(x))}{\exp(-nP(\phi) + \sum_{j=0}^{n-1} \phi(T^j y))} \le C, \tag{2}$$

for all $x \in I$ and $y \in I_n(x)$.

We assume that ϕ is not cohomologous to $\log|T'|$ [PW1]. Let ψ be the positive function such that $\log\psi = \phi - P(\phi)$. Clearly $\psi \in \mathcal{W}$, the pressure $P(\log\psi) = 0$, and μ is also the equilibrium state for $\log\psi$. For these potentials it follows from the Jacobian estimate (J) that the Markov pointwise dimension satisfies

$$\delta_\mu(x) \equiv \lim_{n\to\infty} \frac{\log\mu(I_n(x))}{-\log\ell(I_n(x))} = \lim_{n\to\infty} \frac{\log\prod_{i=0}^{n-1}\psi(T^i x)}{-\log|(T^n)'(x)|},$$

when the limits exist.

The following proposition establishes relationships between the two notions of local dimension for equilibrium states and the Lyapunov exponent.

Proposition 3. *Let $T : I \to I$ be an EMR transformation and let μ be the equilibrium state corresponding to the potential $\phi \in \mathcal{W}$. Let $\overline{\phi}(x)$ denote the Birkhoff average $\lim_{n\to\infty} 1/n \sum_{i=0}^{n-1} \phi(T^k(x))$ at x and consider $x \in I \setminus \mathcal{O}$.*

(1) *Suppose that* $\delta_\mu(x)$ *exists. Then* $\overline{d}_\mu(x) \le \delta_\mu(x)$. *If* $\overline{\phi}(x)$ *also exists, then* $\underline{d}_\mu(x) \ge \delta_\mu(x)$. *In this case*

$$d_\mu(x) = \delta_\mu(x) = \frac{P(\phi) - \overline{\phi}(x)}{\lambda(x)} = \frac{-\overline{\log\psi}(x)}{\lambda(x)},$$

where $\lambda(x)$ *denotes the Lyapunov exponent at* x.

(2) *Suppose that* $d_\mu(x)$ *exists. Then* $\underline{\delta}_\mu(x) \ge d_\mu(x)$. *If* $\overline{\phi}(x)$ *also exists, then* $\overline{\delta}_\mu(x) \le d_\mu(x)$. *In this case* $\delta_\mu(x) = d_\mu(x)$.

(3) *If* $\exp\phi$ *is uniformly bounded away from zero, then the pointwise dimension* $d_\mu(x) = \gamma$ *if and only if the Markov pointwise dimension* $\delta_\mu(x) = \gamma$.

(4) *If* $T = T_1$ *or* $T = T_2$ *and* $\log\psi = \phi - P(\phi)$ *is bounded away from zero, then the Markov pointwise dimension* $\delta_\mu(x) = \gamma$ *implies that the pointwise dimension* $d_\mu(x) = \gamma$.

Remark. We caution that the hypothesis in (3) does not hold for the important family of potentials $\phi = \phi_s = -s\log|T_1'|$ since $\exp\phi_s = x^{2s}$. As noted in Example 1 of Sect. 2, every potential $\phi \in \mathcal{W}$ for the continued fraction map $\overline{T}_1$ must approach 0 as $x \to 0$ and thus (3) will not hold for *any* such potential for this map. However, Property (4) is satisfied (at least for s close to 1). This is an important difference from the multifractal analysis of the *classical* conformal expanding maps where we consider Hölder continuous potentials and where this condition is always satisfied.

Proof of Proposition 3. The Jacobian estimate (J) allows us to estimate the lengths (ℓ) of the intervals $I_n(x)$ using the derivative of T [PW1], i.e., there exist positive constants C_1 and C_2 such that for all $x \in I \setminus \mathcal{O}$ and all $n \in \mathbb{N}$, we have

$$C_1 \le \frac{\ell(I_n(x))}{|(T^n)'(x)|^{-1}} \le C_2. \tag{4}$$

Suppose that the Markov pointwise dimension $\delta_\mu(x)$ exists at a point $x \in I\setminus\mathcal{O}$. Given $r > 0$ there exists a unique $n = n(r)$ such that $C_1|(T^n)'(x)|^{-1} < r \le C_1|(T^{n-1})'(x)|^{-1}$. It immediately follows from (4) that

$$\frac{\log\mu(B(x,r))}{\log r} \ge \frac{\log\mu(B(x, C_1|(T^{n-1})'(x)|^{-1})}{\log r} \ge \frac{\log\mu(I_{n-1}(x))}{\log r} \tag{5}$$

$$\ge \frac{\log\mu(I_{n-1}(x))}{\log(C_1|(T^n)'(x)|^{-1})} = \frac{\log\mu(I_{n-1}(x))}{\log\mu(I_n(x))}\,\frac{\log\mu(I_n(x))}{\log(C_1|(T^n)'(x)|^{-1})}. \tag{6}$$

It follows from the definition of equilibrium state that

$$\frac{\mu(I_{n-1}(x))}{\mu(I_n(x))} \asymp \frac{\exp(-(n-1)P(\phi) + \sum_{j=0}^{n-2}\phi(T^jx))}{\exp(-nP(\phi) + \sum_{j=0}^{n-1}\phi(T^jx))},$$

and thus if $\overline{\phi}(x)$ exists, then

$$\lim_{n\to\infty}\frac{\log\mu(I_{n-1}(x))}{\log\mu(I_n(x))} = 1.$$

We obtain that if $\delta_\mu(x)$ exists, then $\underline{d}_\mu(x) \ge \delta_\mu(x)$. This proves the first part of (1).

We note that by replacing the potential ϕ by $\log \psi$, we obtain the following expression:

$$\frac{\mu(I_{n-1}(x))}{\mu(I_n(x))} \asymp \frac{\prod_{j=0}^{n-2} \psi(T^j x)}{\prod_{j=0}^{n-1} \psi(T^j x)} = \frac{1}{\psi(T^{n-1}x)},$$

and if we assume that ψ (or $\exp\phi$) is uniformly bounded away from zero, then clearly the quotient $\mu(I_{n-1}(x))/\mu(I_n(x))$ will be uniformly bounded for all x and all n, and thus for all x we have

$$\lim_{n\to\infty} \frac{\log \mu(I_{n-1}(x))}{\log \mu(I_n(x))} = 1.$$

Under this uniform boundedness away from zero assumption, we also obtain that $\underline{d}_\mu(x) \geq \delta_\mu(x)$ contributing to the proof of (3).

Next given $r > 0$ there exists a unique $n = n(r)$ such that $C_2|(T^n)'(x)|^{-1} < r \leq C_2|(T^{n-1})'(x)|^{-1}$. It immediately follows from (4) that

$$\begin{aligned}\frac{\log \mu(B(x,r))}{\log r} &\leq \frac{\log \mu(B(x, C_2|(T^n)'(x)|^{-1})}{\log r} \\ &\leq \frac{\log \mu(I_n(x))}{\log r} \leq \frac{\log \mu(I_n(x))}{\log(C_2|(T^n)'(x)|^{-1})}.\end{aligned}$$

We obtain that $\overline{d}_\mu(x) \leq \delta_\mu(x)$ and hence $d_\mu(x) = \delta_\mu(x)$.

Now suppose that the pointwise dimension $d_\mu(x)$ exists at a point $x \in I \setminus \mathcal{O}$. Equation (4) immediately implies that

$$\frac{\log \mu(I_n(x))}{\log |(T^n)'(x)|^{-1}} \geq \frac{\log \mu(B(x, C_2|(T^n)'(x)|}{\log(|(T^n)'(x)|^{-1})},$$

and we obtain that $\underline{\delta}_\mu(x) \geq d_\mu(x)$.

Finally, choose an increasing sequence of positive integers $\{n_k\}$ such that

$$\lim_{n_k\to\infty} \frac{\log \mu(I_{n_k}(x))}{\log(C_1|(T^{n_k})'(x)|^{-1})} = \overline{\delta}_\mu(x),$$

and for each n_k choose some $r_k > 0$ such that

$$C_1|(T^{n_k})'(x)|^{-1} < r_k \leq C_1|(T^{n_k-1})'(x)|^{-1}.$$

From (5) and (6) we have that

$$\frac{\log \mu(B(x,r_k))}{\log r_k} \geq \frac{\log \mu(I_{n_k-1}(x))}{\log \mu(I_{n_k}(x))} \frac{\log \mu(I_{n_k}(x))}{\log(C_1|(T^{n_k})'(x)|^{-1})}.$$

Again, under the assumption that $\overline{\phi}(x)$ exists we showed that

$$\lim_{n\to\infty} \log \mu(I_{n_k-1}(x))/\log \mu(I_{n_k}(x)) = 1,$$

and thus we obtain that $d_\mu(x) \geq \overline{\delta}_\mu(x)$. We conclude that $\delta_\mu(x) = d_\mu(x)$ completing the proof of (2). Finally, as above, we obtain the same estimate if we assume that ψ is uniformly bounded away from zero on I.

Part (3) now easily follows from parts (1) and (2).

For the final part, let $\delta_\mu(x) = \gamma$. It suffices to show that for $T = T_1$ or $T = T_2$ then

$$\frac{\log |T'(T^n x)|}{\sum_{i=0}^{n-1} \log \psi(T^i x)} \longrightarrow 0 \text{ as } n \to \infty,$$

which, using $\delta_\mu(x) = \gamma$ easily implies that

$$\frac{\log \psi(T^n x)}{\sum_{i=0}^{n-1} \log \psi(T^i x)} \longrightarrow 0 \text{ as } n \to \infty,$$

from which the conclusion easily follows.

We shall concentrate on the case of the continued fraction transformation T_1; the case of T_2 being similar.

Fix $\epsilon > 0$. If $\inf_{x \in I} |\log \psi(x)| = \delta > 0$, then choose $n_0 \in \mathbb{N}$ such that $\log(n_0 + 1)^2/(\delta n_0) \leq \epsilon$. If $1/(k+1) \leq T_1^n(x) < 1/k$ (where $k \in \mathbb{N}$), then $k^2 \leq |T_1'(T_1^n x)| < (k+1)^2$. If $k \geq n_0$, then provided $n \geq n_0$, we can estimate

$$\frac{\log |T'(T^n x)|}{\sum_{i=0}^{n-1} \log \psi(T^i x)} \leq \frac{\log(n_0+1)^2}{\delta n_0} \leq \epsilon.$$

If $k < n_0$ then we can still bound

$$\frac{\log |T'(T^n x)|}{\sum_{i=0}^{n-1} \log \psi(T^i x)} \leq \frac{\log(n_0+1)^2}{\delta n} \leq \epsilon,$$

provided n is sufficiently large. □

The essential feature of the proof of (4) above is the need for polynomial bounds on the derivative of the transformation. The polynomial bounds on the derivative for T_2 may be found in [I,Pr].

We define the two parameter family of functions $\phi_{q,t} = -t \log |T'| + q \log \psi$ in $\mathcal{W}$. Define the function $t(q)$ by requiring that $P(\phi_{q,t(q)}) = 0$ and let μ_q be the equilibrium state for $\phi_{q,t(q)}$.

Definition. *We say that a triple (T, ϕ, μ) satisfies a* **multifractal analysis**[1] *if*

(1) *The Markov pointwise dimension $\delta_\mu(x)$ exists for μ-almost every $x \in I$. Moreover, $\delta_\mu(x) = \delta_\mu \equiv h_\mu(T)/\int_I \log |T'|\, d\mu$ for μ-almost every $x \in I$.*
(2) *The function $t(q)$ is the Legendre transform of the dimension spectrum, i.e., we have that $f_\mu(\alpha(q)) = t(q) + q\alpha(q)$, where*

$$\alpha(q) = -t'(q) = -\int_I \log \psi \, d\mu_q \Big/ \int_I \log |T'| d\mu_q.$$

In particular, $t(q)$ is smooth and strictly convex on some interval (q_{min}, q_{max}).

[1] This multifractal analysis should properly be called a Markov multifractal analysis although this is not standard terminology. The term multifractal analysis should refer to an analysis effected using the pointwise dimension. However, in order not to introduce non-standard terminology, we will refer to this analysis as a multifractal analysis.

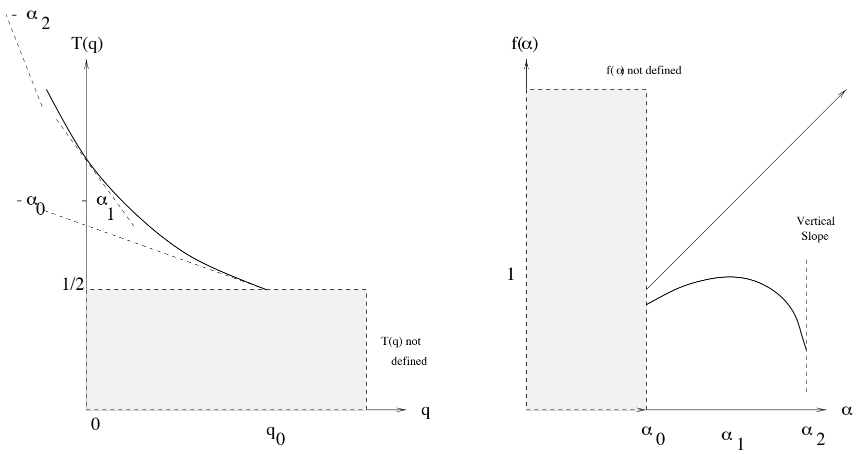

Fig. 5. Multifractal analysis for the continued fraction transformation

An immediate consequence is that if (T, ϕ, μ) satisfies a multifractal analysis, then the dimension spectrum $f_\mu(\alpha)$ is smooth and strictly convex on an interval, and hence the Markov pointwise dimension $\delta_\mu(x)$ attains the interval of values $(\alpha(q_{\max}), \alpha(q_{\min}))$, where each value is attained on an uncountable dense set which supports an equilibrium state.

We note that this is only a partial multifractal analysis in the sense of [PW1,PW2], in that a complete multifractal analysis also establishes analogous results for the Rényi spectrum of dimensions for μ and then establishes a Legendre transform relation between the dimension spectrum and the Rényi spectrum. Here we only extend those aspects of the theory which we use in our applications. One technical complication in extending the entire theory is that the equilibrium state μ may not be included in the family of equilibrium states μ_q, where in the usual theory $\mu_1 = \mu$. In other words, the interval $(\alpha(q_{\max}), \alpha(q_{\min}))$ may not contain d_μ. This happens for the continued fraction transformation T_1 with potential $\phi = -s \log |T_1'|$, $s > \frac{1}{2}$ (see Corollary 5).

Henceforth for T_1 we shall only consider potentials ϕ which are elements of $\mathcal{W}(T_1)$, and for T_2 we shall only consider potentials ϕ which are elements of $\mathcal{W}(\widehat{T}_2)$. For these classes of potentials we establish a multifractal analysis for the pointwise dimension of the associated equilibrium state. For applications to Diophantine approximation, we only require a multifractal analysis for the very special class of potentials of the form $-s \log |T'|$.

Theorem 1. *A multifractal analysis holds for the continued fraction transformation T_1 in the range of q such that $t(q) > \frac{1}{2}$ (see Fig. 5).*

Theorem 2. *A multifractal analysis holds for the Manneville–Pomeau transformation $\widehat{T}_2$ for $1/\alpha < q < 1$ (see Fig. 6).*

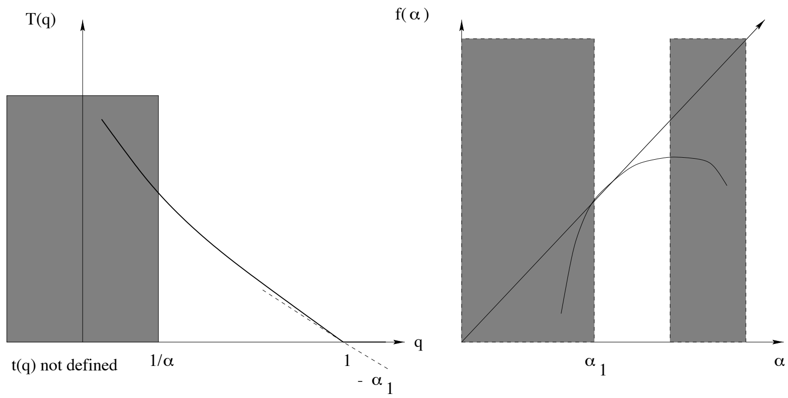

Fig. 6. Multifractal analysis for the Manneville–Pomeau transformation

If the pointwise dimension (or Lyapunov exponent) exists at a point x for an EMR transformation T, then the same limit exists for the induced map $\widehat{T}$. In particular, the estimates for $\widehat{T}$ provide a lower bound on the dimensions of the set of values with the same limit for T.

However, the converse need not necessarily be true. It is plausible that there exist uncountably many points (comprising a set of positive Hausdorff dimension) for which the limit defining the pointwise dimension or Lyapunov exponent for T does not exist, but the subsequential limit, which corresponds to the pointwise dimension or Lyapunov exponent for $\widehat{T}$ does exist.

The following is an immediate consequence of Proposition 3 and Theorem 1, and allows us to relate the Lyapunov spectrum to the Markov dimension spectrum for a special class of equilibrium states (see [We]).

Corollary 1. *Let $T : I \to I$ be an EMR transformation. If $\phi(x) = -s \log |T'|$, then $\lambda(x) = P(-s \log |T'|)/(\delta_{\mu_s}(x) - s)$, where μ_s is the equilibrium state for $-s \log |T'|$. In the case $s = 0$ we obtain that except on a countable set, $\lambda(x) = h_{TOP}(T)/\delta_{\mu_{MAX}}(x)$, where $h_{TOP}(T)$ denotes the topological entropy of the map T and μ_{MAX} denotes the measure of maximal entropy* [2]. *Since countable sets have zero Hausdorff dimension, we have that*

$$f_{\mu_s}(\alpha) = g\left(\frac{P(-s \log |T'|)}{\alpha - s}\right).$$

The Lyapunov spectrum $g(\beta)$ is smooth and strictly convex on an interval.

We will see in Sect. 4 that the Lyapunov exponent for the continued fraction transformation measures the precise exponential rate of rational approximation for the continued fraction algorithm.

4. Continued Fractions, Diophantine Approximation, and Cuspital Excursions on the Modular Surface

For a wealth of classical results about continued fractions we recommend the superb books [C], [HW] and [K]. The books [B,CFS] contain an excellent introduction to the

[2] We remind the reader that $h_{TOP}(T_1) = \infty$ and thus care must be taken in the allowable range of s.

dynamics of the continued fraction transformation and the connection with Diophantine approximation.

Every irrational number $0 < x < 1$ has a continued fraction expansion of the form

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} = [a_1, a_2, a_3, \ldots],$$

where $a_1, a_2, \cdots$ are positive integers. For every positive integer n define the n-th approximant p_n/q_n to be the rational number

$$\frac{p_n}{q_n} = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ldots + \cfrac{1}{a_n}}}}.$$

There is a simple recursive relationship between p_n, q_n and a_n:

$$\begin{aligned} &p_0 = 0, \quad && p_{-1} = 1, \quad && p_n = a_n p_{n-1} + p_{n-2}, \quad && k = 1, 2, \ldots, \\ &q_0 = 1, && q_{-1} = 0, && q_n = a_n q_{n-1} + q_{n-2}, && k = 1, 2, \cdots. \end{aligned} \tag{7}$$

The continued fraction transformation can be considered as a simple *algorithm* for associating to irrational numbers $0 < x < 1$ a sequence of rational numbers p_n/q_n. It is well known that the approximants p_n/q_n satisfy

$$\frac{1}{2q_{n+1}^2} < \frac{1}{q_n(q_n + q_{n+1})} \le \left| x - \frac{p_n}{q_n} \right| \le \frac{1}{q_n(a_{n+1}q_n + q_{n-1})} = \frac{1}{q_n q_{n+1}} < \frac{1}{q_n^2}. \tag{8}$$

There is an intimate connection between the numbers $a_1, a_2, \cdots$ and the continued fractions transformation T_1. Given $0 < x < 1$ we can write

$$\begin{aligned} x &= \frac{1}{\frac{1}{x}} = \frac{1}{\left[\frac{1}{x}\right] + \left\{\frac{1}{x}\right\}} = \frac{1}{a_1 + T_1 x} = \frac{1}{a_1 + \frac{1}{\frac{1}{T_1 x}}} = \frac{1}{a_1 + \frac{1}{\left[\frac{1}{T_1 x}\right] + \left\{\frac{1}{T_1 x}\right\}}} \\ &= \frac{1}{a_1 + \frac{1}{a_2 + \left\{\dfrac{1}{T_1 x}\right\}}} = \frac{1}{a_1 + \frac{1}{a_2 + T_1^2 x}} = \cdots. \end{aligned}$$

Thus $a_1 = [1/x], a_2 = [1/T_1 x], \cdots, a_k = \left[1/T_1^{k-1} x\right]$. Alternatively, if $x = [a_1, a_2, \cdots]$, then $T_1^n(x) = [a_{n+1}, a_{n+2}, a_{n+3}, \cdots]$. From this relation one immediately sees a close connection between the distribution of the values of a_k and the ergodic properties of the map T_1. It also easily follows from the recursion that $p_n(x) = q_{n-1}(T_1 x)$. To see this

$$\begin{aligned} \frac{p_n(x)}{q_n(x)} &= [a_1, \cdots, a_n] = \frac{1}{a_1 + [a_2, \cdots, a_n]} = \frac{1}{a_1 + [p_{n-1}(T_1 x)/q_{n-1}(T_1 x)]} \\ &= \frac{q_{n-1}(T_1 x)}{a_1 q_{n-1}(T_1 x) + p_{n-1}(T_1 x)}. \end{aligned} \tag{9}$$

Since all fractions are irreducible, the result immediately follows.

There is an absolutely continuous T_1-invariant probability measure μ_G on [0, 1], usually called the **Gauss measure**, defined by

$$\mu_G(B) = \frac{1}{\log 2}\int_B \frac{1}{(1+x)}dx,$$

where B is a Borel subset of I. Clearly

$$\frac{1}{2\log 2}\ell(B) \leq \mu_G(B) \leq \frac{1}{\log 2}\ell(B),$$

where ℓ denotes Lebesgue measure. The map T_1 is ergodic with respect to μ_G [K].

For $x \in I \setminus \mathcal{O}$ the set $I_n(x)$ consists of all points $0 \leq y \leq 1$ whose n^{th} approximant is the same as for x. Dynamically this is equivalent to $T^k x, T^k y \in [1/(a_k(x)+1), 1/a_k(x)]$ for $1 \leq k \leq n$. An easy calculation shows that $\ell(I_n(x)) = 1/(q_n(x)(q_n(x)+q_{n-1}(x)))$. Applying (4) and (8) we obtain

$$\lambda(x) = -\lim_{n\to\infty}\frac{1}{n}\log \ell(I_n(x)) = 2\lim_{n\to\infty}\frac{1}{n}\log q_n(x) \equiv 2q(x), \tag{10}$$

when the limits exist, where $q(x)$ denotes the exponential growth rate of the sequence $\{q_n(x)\}$. Moreover, when one or the other limit exists, the other limit must also exist.

An immediate consequence of (8) is that

$$\lambda(x) = -\lim_{n\to\infty}\frac{1}{n}\log\left|x - \frac{p_n(x)}{q_n(x)}\right| \tag{11}$$

when one or the other limit exists.

It also follows from the simple estimate on the density of the Gauss measure μ_G that

$$\lambda(x) = -\lim_{n\to\infty}\frac{1}{n}\log \mu_G(I_n(x)) = 2q(x) = -\lim_{n\to\infty}\frac{1}{n}\log\left|x - \frac{p_n(x)}{q_n(x)}\right|$$

when the limit exists. Again, when one or the other limit exists the other limit must also exist.

Since T_1 is ergodic with respect to μ_G, it follows from the Birkhoff ergodic theorem applied to the function $\log|T_1' x|$ and a simple calculation [B] that for μ_G-almost all $x \in I$,

$$\lambda(x) = \lambda_0 \equiv \frac{\pi^2}{6\log 2} = 2.37314\cdots \quad \text{and} \quad q(x) = q_0 \equiv \frac{\pi^2}{12\log 2} = 1.18657\cdots. \tag{12}$$

Thus the Lyapunov exponent of the continued fraction transformation measures both the precise exponential speed of approximation of a number by its approximants and the exponential growth rate of the sequence $\{q_n(x)\}$. For μ_G-almost all $x \in I$, $|x - p_n(x)/q_n(x)| \asymp \exp(-n\pi^2/6\log 2)$ and $q_n(x) \asymp \exp(n\pi^2/12\log 2)$.

Our aim now is to understand all the possible values which the Lyapunov exponent attains on the exceptional set of zero measure, as well understanding the distribution and structure of the sets of points where the exceptional values are realized.

By studying periodic points of T_1 one can easily find points x such that $\lambda(x) \neq \lambda_0$. The fixed points of T_1 correspond to numbers of the form $x = [a, a, a, \cdots]$. The Lyapunov exponent at a fixed point x is $-2\log x$, and since fixed points exist arbitrarily close to 0 ($[2a, 2a, 2a, \cdots] = \sqrt{a^2+1} - a$), it follows that $\lambda(x)$ attains arbitrarily large values. On the other hand, it is easy to see that the minimum value that $\lambda(x)$ can attain is $2\log(\gamma) = 0.962424\cdots$, where $\gamma = (1+\sqrt{5})/2$ is the Golden Mean. This follows from the fundamental recursion (7) since $q_n = a_n q_{n-1} + q_{n-2} \geq q_{n-1} + q_{n-2}$, and thus $q_n(x) \geq c\gamma^n$ for all x and all n, where c is a constant which is determined by initial conditions. This value of $\lambda(x)$ is attained, for example, at the fixed point $x = \gamma - 1 = [1, 1, 1, \cdots]$. This value of $\lambda(x)$ is also attained at any number x whose continued fraction expansion consists of all 1's from some point on, i.e., $x = [a_1, \cdots, a_n, 1, 1, 1, \cdots]$. Such numbers are sometimes called *noble* numbers. More generally, this value will be realized by precisely those numbers whose continued fraction expansions have a proportion of 1s which increases to 100 percent. The set of such numbers is dense and uncountable.

We know that $\lambda(x)$ can realize the values γ and λ_0. In the case of γ we can find a periodic orbit x_∞ such that $\lambda(x_\infty) = \gamma$. We claim that the Lyapunov exponent for T_1 possesses an intermediate value property: any intermediate value can also be realized as a Lyapunov exponent.

Lemma 1. *Given any value* $2\log\gamma < \xi < \lambda_0$*, there exists a point* $x \in I$ *such that* $\lambda(x) = \xi$.

Proof. Fix any value $2\log\gamma < \xi < \lambda_0$. We first note that since the T_1-invariant measures are convex, then

$$[2\log\gamma, \lambda_0] \cap \left\{ \int \log|T_1'| d\mu : \mu \text{ is } T_1 - \text{invariant} \right\} = [2\log\gamma, \lambda_0].$$

Moreover, since the periodic point measures are weak star dense in the T_1-invariant measures, we can choose a sequence of periodic orbits $T_1^{N_n} x_n = x_n$, $n \geq 1$, such that the associated Lyapunov exponent $\lambda(x_n)$ satisfies $|\lambda(x_n) - \xi| < 1/n$.

We can write each periodic orbit in terms of its continued fraction expansion, i.e. $x_n = [a_0(x_n), a_1(x_n), \cdots, a_{N_n-1}(x_n)]$. We can choose an increasing sequence n_k inductively such that $(1/n_k)\sum_{i=1}^{k-1} n_i N_i \to 0$.

Finally, we define the point $x \in I$ having the continued fraction expansion

$$x = [\underbrace{a_0(x_1), \cdots, a_{N_1-1}(x_1)}_{n_1 \text{ times}}, \underbrace{a_0(x_2), \cdots, a_{N_2-1}(x_2)}_{n_2 \text{ times}}, \cdots],$$

i.e., we concatenate the repeated block in the continued fraction expansion of x_1 (n_1 times), followed by the repeated block in the continued fraction expansion of x_2 (n_2 times), etc. By construction we have that $\lambda(x) = \xi$. □

To study the distribution of values of $\lambda(x)$ more precisely, we define the (T_1−invariant) level sets of the Lyapunov exponent

$$\Lambda_\alpha = \{x \in I : \lambda(x) = \alpha\}.$$

These sets (along with the set on which $\lambda(x)$ does not exist) provide a decomposition of the interval. From (10) and (11) we see that for $x \in \Lambda_\alpha$, $|x - p_n(x)/q_n(x)| \asymp \exp(-n\alpha)$ and $q_n(x) \asymp \exp(n\alpha/2)$.

The following proposition on the distribution of values of λ and the precise Hausdorff dimension of the level sets are easy consequences of Lemma 1, Theorem 1, and Proposition 3 applied to the function $\phi = -s\log|T_1'|$ for $s > 1/2$:

Proposition 4. *Let $T_1 : I \to I$ be the continued fraction transformation.*

(1) *The Lyapunov exponent $\lambda(x)$ attains the interval of values*

$$[2\log\gamma, \infty) = [2\log\gamma, \lambda_0) \cup [\lambda_0, \infty).$$

(2) *For $\alpha \in [\lambda_0, \infty)$ the value $\alpha = \int_I \log|T_1'|d\mu_s$ is attained by the Lyapunov exponent on a set of (positive) Hausdorff dimension $h_{\mu_s}(T_1)/\int_I \log|T_1'|d\mu_s = h_{\mu_s}(T_1)/\alpha$. This level set is uncountable and also dense in I.*

Proof. Lemma 1 implies that $\lambda(x)$ attains the interval of values $[\gamma, \lambda_0]$.

Consider the family of potentials $\phi^s = -s\log|T_1'|$, $s > 1/2$ and let μ_s be the corresponding family of equilibrium states. To prove Proposition 4, we shall only require (1) in our definition of multifractal analysis, that

$$\delta_{\mu_s}(x) = \delta_s \equiv \frac{h_{\mu_s}(T_1)}{\int_I \log|T_1'|d\mu_s} = s + \frac{P(-s\log|T_1'|)}{\int_I \log|T_1'|d\mu_s} > 0$$

for μ_s-almost all $x \in I$. Let $K^M_{\delta_s} = \{x \in I : \delta_{\mu_s}(x) = \delta_s\}$. From Proposition 3 we see that $\overline{d}_{\mu_s}(x) \le \delta_s$ for all $x \in K^M_{\delta_s}$ and $\underline{d}_{\mu_s}(x) \ge \delta_s$ for μ_s-almost all $x \in K^M_{\delta_s}$. By standard arguments in dimension theory [PW1, pp. 253–254] it follows that $\dim_{\mathrm{H}} K_{\delta_s} = \delta_s$ [PW1, pp. 253–254]. Proposition 1 and the Variational Principle immediately imply that if $\delta_{\mu_s}(x) = \delta_s$ then $\lambda(x) = \int_I \log|T_1'|d\mu_s$, and thus $\lambda(x) = \int_I \log|T_1'|d\mu_s$ for μ_s-almost all $x \in I$. Thus $\dim_{\mathrm{H}} \Lambda_{\int_I \log|T_1'|d\mu_s} = \delta_s > 0$.

Since $\mu_s(K^M_{d_s}) = 1$ and $\mu_s(\Lambda_{\int_I \log|T_1'|d\mu_s}) = 1$, and equilibrium states are positive on open sets, we have that each of the sets $K^M_{d_s}$ and $\Lambda_{\int_I \log|T_1'|d\mu_s}$ are dense in I.

Recall that $\mu_1 = \mu_G$ and thus $\int_I \log|T_1'|d\mu_1 = \lambda_0$. It immediately follows from the first derivative formula for pressure [R] that

$$\int_I \log|T_1'|d\mu_s = -\frac{d}{ds}P(-s\log|T_1'|).$$

Since $\lim_{s\searrow 1/2} P(-s\log|T_1'|) = \infty$ and $P(-s\log|T_1'|)$ is smooth and strictly convex on $(1/2, \infty)$, it follows that $\lim_{s\searrow 1/2}(d/ds)P(-s\log|T_1'|) = -\infty$. Thus

$$\lim_{s\searrow 1/2} \int_I \log|T_1'|d\mu_s = \infty.$$

The map $s \mapsto \int_I \log|T_1'|d\mu_s$ is smooth on $(1/2, \infty)$, by analytic perturbation theory since μ_s corresponds to an isolated maximal eigenvalue for a transfer operator [M1], and this implies that the Lyapunov exponent λ attains all values between λ_0 and ∞, each on an uncountable dense set of positive Hausdorff dimension. □

Proposition 4 immediately implies the following two number theoretic corollaries.

Corollary 2. *The asymptotic quantity* $\lim_{n\to\infty}(1/n)\log|x - p_n(x)/q_n(x)|$ *attains the interval of values* $[\lambda_0, \infty)$ *and each value in this interval is attained on an uncountable* *dense set of positive Hausdorff dimension. There is an explicit formula for the Hausdorff dimension of each level set in* $[\lambda_0, \infty)$*:*

$$\textit{For } \alpha = \int_I \log |T_1'| d\mu_s,$$
$$\dim_H \left\{ x \in [0,1] : \lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| = \alpha \right\} = h_{\mu_s}(T_1)/\alpha.$$

It easily follows from this formula that the Hausdorff dimension of the level sets vary smoothly.

Corollary 3. *The asymptotic quantity* $\lim_{n\to\infty}(1/n)\log q_n(x)$ *attains the interval of values* $[\frac{1}{2}\lambda_0, \infty)$ *and each value in this interval is attained on an uncountable dense set of positive Hausdorff dimension. There is an explicit formula for the Hausdorff dimension of each level set, which in particular shows that the Hausdorff dimension of the level sets vary smoothly in* α.

We have seen that $p_n(x) = q_{n-1}(T_1 x)$ and this easily implies that

$$p(x) \equiv \lim_{n\to\infty} \frac{1}{n} \log p_n(x) = \lim_{n\to\infty} \frac{1}{n} \log q_{n-1}(T_1 x) = q(T_1 x).$$

It follows from (12) that $p(x) = q_0 = \pi^2/12\log 2$ for μ_G-almost all $x \in I$. The next corollary, which analyzes the exceptional set, follows immediately.

Corollary 4. *The asymptotic quantity* $\lim_{n\to\infty}(1/n)\log p_n(x)$ *attains the interval of values* $[\frac{1}{2}\lambda_0, \infty)$ *and each value in this interval is attained on an uncountable dense set of positive Hausdorff dimension. There is an explicit formula for the Hausdorff dimension of each level set, which in particular shows that the Hausdorff dimension of the level sets vary smoothly.*

Remark. Consider the function

$$\tau(x) = -\lim_{n\to\infty} \frac{\log \left| x - \frac{p_n(x)}{q_n(x)} \right|}{\log q_n(x)} = -\lim_{n\to\infty} \frac{1}{n} \log \left| x - \frac{p_n(x)}{q_n(x)} \right| \frac{1}{\frac{1}{n}\log q_n(x)},$$

when the limits exist. It immediately follows from (10) and (11) that for μ_G-almost all x the function $\tau(x) = \tau_0 \equiv \lambda_0/q_0 = 2$. More precisely, if $q(x) = \lim_{n\to\infty}(1/n)\log q_n(x)$ exists at a point $x \in I \setminus \mathcal{O}$ then $\tau(x) = 2$.

We now consider the set of points for which $\lambda(x)$ does not exist. An example is the Liouville number $x = \sum_{k=1}^{\infty} 10^{-k!}$. It is easy to see that $|x - p_n(x)/q_n(x)| = \sum_{k=n+1}^{\infty} 10^{-k!}$ and thus $10^{-(n+1)!} < |x - p_n(x)/q_n(x)| < 2 \times 10^{-(n+1)!}$. It follows that $\lambda(x) = \infty$.

It is also easy to construct numbers for which the limit defining $\lambda(x)$ does not exist and is not infinite. The construction uses the trick in Lemma 2. Consider the number x with continued fraction expansion

$$x = [\underbrace{1, \cdots, 1}_{n_1 \text{ times}}, \underbrace{2, \cdots, 2}_{m_1 \text{times}}, \underbrace{1, \cdots, 1}_{n_2 \text{times}}, \underbrace{2, \cdots, 2}_{m_2 \text{times}}, \cdots],$$

with each n_i and m_i being much larger than the sum of all the proceeding choices. A routine argument shows that for suitable choices of n_i and m_i, the Lyapunov exponent $\lambda(x)$ does not exist.

A straightforward extension of an argument by Shereshevsky [Sh] gives that the set of points for which $\lambda(x)$ does not exist has positive Hausdorff dimension. A natural problem is to compute the *precise* Hausdorff dimension. In [BaS] the authors show that for a conformal expanding map (they assume that their map is a local homeomorphism), the Hausdorff dimension of the set of points where the Lyapunov exponent does not exist is *maximal*, i.e., the same dimension as the limit set (repeller). The essential hypothesis for their result is a smooth map which possesses a sequence μ_k of ergodic invariant measures such that $\lim_{k\to\infty} \dim_{\mathrm{H}}(\mu_k) = \dim_{\mathrm{H}}(J)$, where J is the limit set and $\dim_{\mathrm{H}}(\mu) \equiv \inf\{\dim_{\mathrm{H}}(U), \mu(U) = 1\}$ [Pe, p. 42] is the Hausdorff dimension of the measure μ. As will be noted after the proof of Theorems 1 and 2, the existence of such sequences of measures for the maps T_1 and T_2 follow immediately from the proof of Theorems 1 and 2. Thus a straightforward extension of the proof of Barreira and Schmeling proves the following result.

Theorem 3. *The set of points $x \in I$ for which $\lambda(x)$ does not exist has Hausdorff dimension equal to 1.*

There are classical results on Hausdorff dimension and Diophantine approximation, due to Jarnik, to which our results can be viewed as complimentary. Recall that the continued fraction approximants p_n/q_n of a number x all satisfy the Diophantine condition

$$\left|x - \frac{p}{q}\right| \le \frac{1}{q^2}.$$

Let us now consider the set of numbers which admit a *faster approximation* by rational numbers. For $\tau > 2$ let $\mathcal{F}_\tau$ denote the set of $\tau-$well approximable numbers, i.e., those that satisfy

$$\mathcal{F}_\tau = \left\{x \in I : \left|x - \frac{p}{q}\right| \le \frac{1}{q^\tau} \quad \text{infinitely often}\right\}.$$

Here *infinitely often* means that there are infinitely many distinct rational p/q which satisfy the relation. Legendre showed that if p/q is any rational approximation to an irrational number x satisfying $|x - p/q| \le 1/2q^2$, then p/q must be an approximant for x. Thus the rationals p/q in $\mathcal{F}_\tau$ are all approximants. It is easy to show that this set has zero measure for each $\tau > 2$. Jarnik [J] computed the Hausdorff dimension of $\mathcal{F}_\tau$ and showed that $\dim_H(\mathcal{F}_\tau) = 2/\tau$. While Jarnik explicitly computes the Hausdorff dimension of the set of numbers x such the approximation error $|x - p/q|$ is bounded above by $q^{-\tau}$ for infinitely many approximants p_n/q_n, Corollary 2 is a statement about the Hausdorff dimension of sets of numbers x such that the error $|x - p_n/q_n|$ admits *precise* asymptotic limiting behavior. Corollary 2 also quantifies the precise speed of convergence of the continued fraction algorithm on exceptional sets. Although our results are related, they do not seem to be obtainable from each other.

Application to Geodesics on the Modular Surface. The map T_1 is also closely related to the symbolic description of the geodesic flow $\phi_t : PSL(2,\mathbb{R})/PSL(2,\mathbb{Z}) \to PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ defined on the unit tangent bundle of the modular surface $M = \mathbb{H}^2/PSL(2,\mathbb{Z})$ by $\phi_t(g)PSL(2,\mathbb{Z}) = gg_tPSL(2,\mathbb{Z})$ with $g_t = \mathrm{diag}(\exp(t), \exp(-t))$ and $\mathbb{H}^2 = \{x + iy : y > 0\}$. There is an interesting connection between Corollary 1

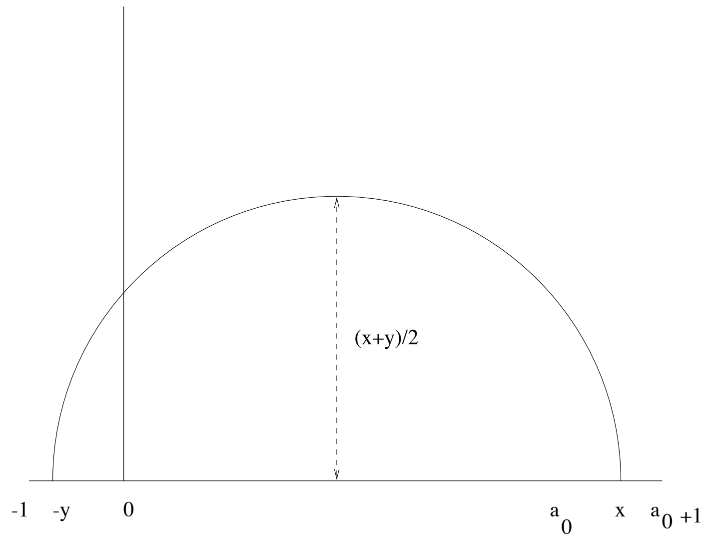

Fig. 7. Continued fractions and geodesic excursions

and such geodesic flows. The Lyapunov exponent quantifies the proportion of time that a geodesic spends in excursions on cuspital excursion [St, Su].

Given a geodesic γ on the modular surface we can consider a lift $\hat{\gamma}$ to the Poincaré upper half-plane $\mathbb{H}^2$. Such geodesics on $\mathbb{H}^2$ correspond to Euclidean semi-circles which meet the real line perpendicularly. In particular, we can choose our lift so that the end-points satisfy and $1 < x \equiv \hat{\gamma}(\infty) < \infty$ and $-1 < y \equiv \hat{\gamma}(-\infty) < 0$. Let us consider the continued fraction expansion $x = [a_0(x), a_1(x), a_2(x), \dots]$ then we see that the Euclidean height of the arc is equal to $(x-y)/2$ and lies between $a_0/2$ and $a_0/2+1$. In particular, the hyperbolic distance of the cuspital excursion can be estimated by $\log a_0(x)$ (see Fig. 7).

Fix $p \in M$. Given a unit tangent vector $v \in T_1M = PSL(2,\mathbb{R})/PSL(2,\mathbb{Z})$ at p, we let $\gamma : \mathbb{R} \to M$ be the unique unit speed geodesic with $\dot{\gamma}(0) = v$.

We denote by s_n, $n \geq 1$, the times $s > 0$ at which $\gamma_s(v)$ maximizes $d(\gamma_s(v), p)$ on each successive excursion into the cusp (i.e., $s_1 < s_2 < s_3 < \dots$ are local maxima for $d(\gamma_s(v), p)$), where d denotes the hyperbolic distance on M. By the above observations we see that these heights can be estimated with the function $-\log a_n(x)$. Unfortunately, this function does *not* satisfy condition (W1), so that we need to consider instead functions of the form $\phi(x) \equiv -\beta \log a_n(x)$, for $\beta > 1$. Moreover, s_n can be estimated by $\log|(T_1^n)'(x)|$. We can therefore interpret the quantity $A_n(v) \equiv \sum_{i=0}^{n-1} \log a_i(x)/\log|(T_1^n)'(x)| \sim \sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p)/s_n$ as an estimate on the average height of the first n geodesic excursions, compared with the time required for the excursions. For $\beta > 1$ let us denote

$$\Lambda_\alpha \equiv \left\{ v : \lim_{n\to\infty} (1/s_n) \sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p) = \frac{\alpha}{\beta} \right\} = \left\{ v : \lim_{n\to\infty} A_n(v) = \frac{\alpha}{\beta} \right\}.$$

Theorem 1 implies the following result.

Theorem 4. *There exists an interval of values $(\alpha_{min}, \alpha_{max})$ such that for α in this interval the set Λ_α is an uncountable dense set of positive Hausdorff dimension, and the Hausdorff dimension $dim_H(\Lambda_\alpha)$ varies smoothly (analytically).*

This can be compared with the results of Sullivan [Su], Melián-Pestana [MP2], and Stratmann [St], which are of a somewhat complementary nature. In our notation these authors' results relate to the subsequence $t_m = s_{n_m}$ of successive *farthest* geodesic excursions (i.e., $d(\gamma_{t_1}(v), p) \le d(\gamma_{t_2}(v), p) \le \dots$). Sullivan [Su] shows that a typical geodesic extends a distance at most $\log t$ into a geodesic at time t. More precisely, Sullivan shows that for almost every unit tangent vector v at p,

$$\lim_{m\to\infty} \frac{d(\gamma_{t_m}(v), p)}{\log t_m} = 1.$$

Stratmann and Melián-Pestana compute the Hausdorff dimension of the sets

$$\Pi_\alpha = \{v : \lim_{m\to\infty} d(\gamma_{t_m}(v), p)/t_m = \alpha\}.$$

Remark. As one would imagine, a similar study can be made of the behaviour of geodesics on other surfaces with cusps, generalizing those for the modular surface. In this case, one needs to use the general analysis described in [BoS]. There is also an analogous notion of Diophantine for more general Fuchsian groups [Pa], to which our results would naturally apply.

Remark. We do not yet have a multifractal analysis of Birkhoff sums for T_1. With such machinery one could make similar statements about $\sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p)/n$ as we can for $\sum_{i=0}^{n-1} d(\gamma_{s_i}(v), p)/s_n$.

5. Thermodynamic Formalism for Infinite State Systems

The following proposition contains useful formulas for the derivatives of pressure.

Proposition 5. *Let $T\ I \to I$ be an EMR transformation. Let f, g and h be functions on I such that for sufficiently small ϵ_1, ϵ_2 the family of functions $f + \epsilon_1 g + \epsilon_2 h$ satisfy (W1), (W2), and $P(f + \epsilon_1 g + \epsilon_2 h) > -\infty$. Then the function $(\epsilon_1, \epsilon_2) \mapsto P(f + \epsilon_1 g + \epsilon_2 h)$ is analytic, convex (in each variable), strictly convex if f is not cohomologous to a constant, and satisfies the following derivative formulas:*

$$\left.\frac{d}{d\epsilon_1}\right|_{\epsilon=0} P(f + \epsilon_1 g) = \int_I g\, d\mu_f,$$

and

$$\left.\frac{\partial^2 P(f + \epsilon_1 g + \epsilon_2 h)}{\partial\epsilon_1 \partial\epsilon_2}\right|_{\epsilon_1=\epsilon_2=0} \equiv Q_f(g, h),$$

where Q_f is the bilinear form on $C^\alpha(I, \mathbb{R})$ defined by

$$Q_f(g, h) = \sum_{k=0}^{\infty} \left(\int_I g \cdot\ (h \circ T^k)\, d\mu_f - \int_I g\, d\mu_f \int_I h\, d\mu_f \right),$$

and μ_f is the equilibrium state for f. Also $Q_f(g, g) \ge 0$ for all g and $Q_f(g, g) > 0$ if and only if f is not cohomologous to a constant function.

Proof. The proof is very similar to the proof in the *usual setting* where the interval map is piecewise smooth and expanding on finitely many intervals [R]. There is an additional potential complication which needs to be addressed in that there may be an infinite sum, rather than just a finite sum, for the transfer operator. This occurs for T_1 and $\widehat{T_2}$. This complication is *handled* with conditions (W1) and (W2) which ensure that the infinite sum is well defined and can be treated in the usual ways.

The key to the proof is the study of the transfer operator $L_{\phi_{q,t}} : C^0(I) \to C^0(I)$ given by

$$L_{\phi_{q,t}} k(x) = \sum_{Ty=x} \exp(\phi_{q,t}(y)) k(y),$$

whose maximal eigenvalue is $\exp(P(\phi_{q,t}))$. Moreover, to obtain an isolated eigenvalue in the appropriate range, we study $L_{\phi_{q,t}} : BV(I) \to BV(I)$ acting on the space BV of functions of bounded variation. Prellburg [Pr] has shown that for $\widehat{T_2}$, the spectrum of this operator consists of the closed unit ball, plus at most a countable number of isolated eigenvalues of modulus strictly greater than 1. Thus when the quantity $P(\phi_{q,t}) > -\infty$, its exponential is the maximal positive isolated eigenvalue for $L_{\phi_{q,t}}$. The result follows by analytic perturbation theory. The convexity is a direct consequence of the second derivative formula. □

The next proposition is an immediate consequence of Proposition 5 and the implicit function theorem [PW1].

Proposition 6. *Let $T\ I \to I$ be an EMR transformation. Assume that for a range of values (q, t) the family of functions $\phi_{q,t}$, satisfy (W1), (W2), and $P(\phi_{q,t}) > -\infty$. Then the function $(q, t) \mapsto P(\phi_{q,t})$ is analytic and convex (in each variable). Furthermore, for the one parameter family of potentials $\phi_q = \phi_{q,t(q)}$, where $t = t(q)$ is defined by requiring that $P(\phi_{q,t(q)}) = 0$, we have that*

(1) *The function $t(q)$ is real analytic and convex. It is strictly convex if* $\log \psi$ *is not cohomologous to* $-\log |T'|$.
(2) *The derivative*

$$t'(q) = -\frac{\int_I \log \psi \, d\mu_q}{\int_I \log |T'| \, d\mu_q},$$

where μ_q is the equilibrium state for ϕ_q.
(3) *The second derivative satisfies*

$$t''(q) = \frac{t'(q)^2 \left(\frac{\partial^2 P(\phi_{q,r})}{\partial r^2}\right) - 2t'(q) \left(\frac{\partial^2 P(\phi_{q,r})}{\partial q \, \partial r}\right) + \left(\frac{\partial^2 P(\phi_{q,r})}{\partial q^2}\right)}{\left(\frac{\partial P(\phi_{q,r})}{\partial r}\right)},$$

evaluated at $(q, r) = (q, -t(q))$.

The following useful corollary follows easily from Proposition 2. Here we collect many results which we will require in Sect. 7.

Corollary 5. *In the special case $\phi = -s \log |T'|$ we have that the function $\phi_q = -(t(q) + qs) \log |T'| - q P(s \log |T'|)$ and that*

(1) *The function $t(q)$ is defined implicitly by*

$$P(-(t(q) + qs) \log |T'|) = q P(-s \log |T'|).$$

(2) *The derivative*

$$t'(q) = -\frac{h_{\nu_q}(T)}{\int_I \log|T'|d\nu_q},$$

where ν_q is the equilibrium state for ϕ_q.

(3) *We have the special values $t(0) = 1$, $t(1) = 0$, and $t'(0) = -1$.*

(4) *For the continued fraction transformation T_1 the expressions in (1)-(3) are well defined (satisfy W1, W2, and $P \neq -\infty$) provided that $t(q) > 1/2$ and $s > 1/2$.*

(5) *For the induced Manneville–Pomeau transformation $\widehat{T}_2$ the expressions in (1)-(3) are well defined (satisfy W1, W2, and $P \neq -\infty$) provided that $t(q) < 1$ [Pr].*

Example 1. The transfer operator for the continued fraction transformation T_1 for the family $\phi_{q,t}$, can be written explicitly as

$$L_{\phi_{q,t}}k(x) = \sum_{n=1}^{\infty} \psi\left(\frac{1}{x+n}\right)^q \left(\frac{1}{x+n}\right)^{2t} k\left(\frac{1}{x+n}\right).$$

For the potentials $\phi_{q,t}$ (for $t > 1/2$), conditions (W1) and (W2) apply.

In the special case that ϕ is real analytic, the operator $L_{\phi_{q,t}}$ preserves the smaller space of analytic functions and is compact. In particular, we can waive the assumption that $P(\phi_{q,t}) > -\infty$.

Example 2. The transfer operator for the induced Manneville–Pomeau transformation $\widehat{T}_2$ for the family $\phi_{q,t}$, can be written explicitly as

$$L_{\phi_{q,t}}k(x) = \sum_{\widehat{T}_2 y = x} \frac{\psi^q(x)}{|\widehat{T}_2'(x)|^t} k(x) = \sum_{n=1}^{\infty} \psi(G_n(x))^q |G_n'(x)|^t k(G_n(x)),$$

where $G_n = F_1 F_0^{n-1}$, and F_0, F_1 denote the two branches of the inverse of $\widehat{T}_2$.

In the case of the Manneville–Pomeau transformation, the induced transformation $\widehat{T}_2 : I \to I$ is a smooth map on each of the intervals $[a_n, a_{n+1}]$ [Pr].

For the potentials $\phi_{q,t}$ (for $t > 1$), conditions (W1) and (W2) apply.

The next result describes the construction of equilibrium states using transfer operators.

Proposition 7. *Let $T\ I \to I$ be an EMR transformation. Assume that ϕ satisfies (W1) and (W2). Then there exists a unique equilibrium state.*

Sketch of Proof. The associated transfer operator satisfies the following [Wa1, p. 128]

(1) There exists $\lambda > 0$ and $h \in C^0(I)$ with $L_{\psi_q} h = \lambda h$;

(2) There exists a T-invariant probability measure μ such that for $f \in C^0(I)$ we have that $\lambda^{-n} L^n_{\psi_q} f(x) \to h \int f d\mu$.

If we denote $g = \exp(\psi)h/(\lambda h \circ T)$ we have that $\lambda^{-n} L^n_{\log g} f(x) \to \int f d\mu$ [Wa1, p. 128]. Moreover, $dT^*\mu/d\mu = 1/g$ [Wa1, p. 124], where $T^*\mu$ denotes the pull-back of the measure μ by T. The proof that μ is the required equilibrium state comes from (2). A useful property of g is that there exists $C_0 > 0$ such that for all $x, y \in I$,

$$\exp(-C_0 d(x,y)) \leq \prod_{i=0}^{n-1} \frac{|g(T^i x)|}{|g(T^i y)|} \leq \exp(-C_0 d(x,y))$$

[Wa1, p. 130]. In particular, this implies that there exists $C > 0$ such that for all $x \in I$,

$$\frac{1}{C} \leq \frac{\mu(I_n(x))}{\prod_{i=0}^{n-1} g(T^i x)} \leq C.$$

Uniqueness comes by the ergodicity of μ and the fact that any two solutions are absolutely continuous. □

6. Proofs of Theorems 1 and 2

Consider the transformation $T = T_1$ or $\widehat{T}_2$. Statement (1) in our definition of multifractal analysis easily follows from the Birkhoff ergodic theorem. Given $x \in I \setminus \mathcal{O}$, it follows from Proposition 3 that $d_\mu(x) = -\overline{\log\psi}(x)/\lambda(x)$. Since the equilibrium state μ is an ergodic measure we have that $\lambda(x) = \int_I \log|T'|d\mu$ and $\overline{\log\psi}(x) = \int_I \log\psi\, d\mu$ for μ-almost every $x \in I$. Since $P(\log\psi) = 0$, it follows from the Variational Principle that $h_\mu(T) = -\int_I \log\psi\, d\mu$. It follows that $d_\mu(x) = h_\mu(T)/\int_I \log|T'|\, d\mu$ for μ-almost every $x \in I$.

Recall that $t(q)$ is the unique solution to $P(\phi_{(q,t(q))}) = P(-t(q)\log|T'|+q\log\psi) = 0$ and μ_q is the equilibrium state for $\phi_q = \phi_{(q,t(q))}$. For each map T and each potential ϕ, the function $t(q)$ is defined for a certain range $(q_{\min}, q_{\max})$ of q which we have discussed in Sect. 6. By Proposition 6 the function $t(q)$ is smooth and strictly convex on this interval. Using the derivative formula in Proposition 6, we define for each q the function

$$\alpha(q) \equiv -t'(q) = \frac{\int_I \log\psi\, d\mu_q}{-\int_I \log|T'|d\mu_q},$$

and consider the level sets

$$K^M_{\alpha(q)} = \left\{x : \delta_\mu(x) = \alpha(q)\right\}.$$

We remind the reader that

$$K^M_{\alpha(q)} = \left\{x \in I \setminus \mathcal{O} : \lim_{n\to\infty} \frac{\log\mu(I_n(x))}{\log\ell(I_n(x))} = \lim_{n\to\infty} \frac{\sum_{i=0}^{n-1}\log\psi(T^i x)}{\sum_{i=0}^{n-1}\log|T'(T^i(x))|} = \frac{\int_I \log\psi\, d\mu_q}{\int_I \log|T'|d\mu_q}\right\}.$$

An immediate consequence of the Birkhoff ergodic theorem is that the Markov pointwise dimension satisfies $\mu_q(K^M_{\alpha(q)}) = 1$.

Proposition 3 immediately implies that for all $x \in K^M_{\alpha(q)}$,

$$\delta_{\mu_q}(x) = -\frac{\overline{\phi}_q(x)}{\lambda(x)} = \frac{t(q)\overline{\log|T'|}(x) - q\overline{\log\psi}(x)}{\lambda(x)} = t(q) + q\alpha(q).$$

It follows from Proposition 3(2) that the upper pointwise dimension $\overline{d}_{\mu_q}(x) \leq \delta_{\mu_q}(x) = t(q) + q\alpha(q)$ for *all* $x \in K^M_{\alpha(q)}$.

Since the Birkhoff average $\overline{\log\psi}(x)$ exists for μ_q-almost every $x \in K^M_{\alpha(q)}$ (and equals $\int_I \log\psi\, d\mu_q$), it follows from Proposition 3(3) that $\underline{d}_{\mu_q}(x) \geq \delta_{\mu_q}(x) = t(q) + q\alpha(q)$

for μ_q-almost every $x \in K^M_{\alpha(q)}$. By standard arguments in dimension theory [PW1, pp. 253–254], this aėinequality implies that $\dim_H(K^M_{\alpha(q)}) \geq t(q) + q\alpha(q)$, and these two estimates imply that $\dim_H(K^M_{\alpha(q)}) = t(q) + q\alpha(q)$.

The smoothness and convexity properties of $t(q)$ follow from Proposition 6. □

Remark. As mentioned before the statement of Theorem 3, the above proof of Theorems 1 and 2 provides a sequence of ergodic invariant measures for T_1 or $\widehat{T}_2$ such that

$$\lim_{k\to\infty} \dim_{\mathrm{H}}(\mu_k) = \dim_{\mathrm{H}}(I) = 1.$$

Since $\delta_{\mu_q}(x) = t(q) + q\alpha(q)$ for μ_q almost-all $x \in K^M_{\alpha(q)}$ and $\mu_q(K^M_{\alpha(q)}) = 1$, a weaker conclusion than we obtained in the theorem above is that for the (ergodic) equilibrium state μ_q we have that $\dim_{\mathrm{H}}(\mu_q) = t(q) + q\alpha(q)$ [Pe, p. 42]. Since $t(0) = 1$ we have that $\lim_{q\to 0} \dim_{\mathrm{H}}(\mu_q) = 1$. Thus any allowable potential $\phi \in \mathcal{W}$ gives rise to such a one parameter family (and hence such a sequence) of measures.

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