Models based on Finite Spectral Triple

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Ma148b Spring 2016
Topics in Mathematical Physics

References

- M.Marcolli, W.van Suijlekom, *Gauge Networks in Noncommutative Geometry*, J. Geom. Phys., Vol.75 (2014) 71–91
- J.W. Barrett, *Matrix geometries and fuzzy spaces as finite spectral triples*, arXiv:1502.05383
- J.W. Barrett, L. Glaser, *Monte Carlo simulations of random non-commutative geometries*, arXiv:1510.01377

Gaugenetworks

- using finite spectral triple for a model combining gauge theory on a lattice (or graph) and spin networks approach to gravity
- an action functional (in terms of Dirac operator) that recovers the Wilson action (which in continuum limit gives Yang–Mills) will additional terms for a Higgs field in adjoint representation
- build a category of finite spectral triples with morphisms built from algebra morphisms and unitary operators
- representations of quivers (oriented graphs) in this category of finite spectral triples
- configuration space (of such representation) modulo gauge action
- morphisms between gauge networks by correspondences (bimodules); Hamiltonian and time evolution
- discretized Dirac operator and continuum limit

$\mathcal{C}_0$ Category of finite spectral triples with trivial Dirac $D = 0$

• objects $(\mathcal{A}, \pi, \mathcal{H})$, fin. dim. algebra $\mathcal{A}$ and fin. Hilbert space rep. $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$

• morphisms $\Phi : (\mathcal{A}_1, \pi_1, \mathcal{H}_1) \to (\mathcal{A}_2, \pi_2, \mathcal{H}_2)$ pair $\Phi = (\phi, L)$ $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ morphism of unital $\star$-algebras, $L : \mathcal{H}_1 \to \mathcal{H}_2$ unitary

$$L\pi_1(a)L^* = \pi_2(\phi(a))$$

$\mathcal{C}$ Category of finite spectral triples

• objects $(\mathcal{A}, \pi, \mathcal{H}, D)$ fin spectral triples

• morphisms $\Phi : (\mathcal{A}_1, \pi_1, \mathcal{H}_1, D_1) \to (\mathcal{A}_2, \pi_2, \mathcal{H}_2, D_2)$ as above with also $LD_1L^* = D_2$

Bratteli diagrams

- Wedderburn theorem:

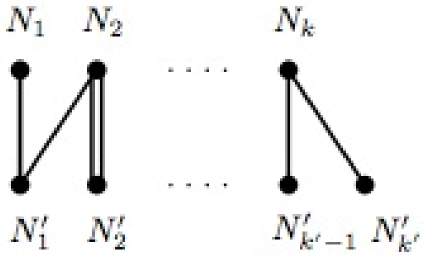
$$\mathcal{A}_1 = \bigoplus_{i=1}^{k} M_{N_i}(\mathbb{C}), \quad \mathcal{A}_2 = \bigoplus_{j=1}^{k'} M_{N'_j}(\mathbb{C})$$

- unital $*$-algebra morphism $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ direct sum

$$\phi_j : \bigoplus_{i=1}^{k} M_{N_i}(\mathbb{C}) \to M_{N'_j}(\mathbb{C})$$

ϕ_j splits as a direct sum of representation $\phi_{ij} : M_{N_i}(\mathbb{C}) \to M_{N'_j}(\mathbb{C})$ with multiplicity $d_{ij} \geq 0$, with $N'_j = \sum_i d_{ij} N_i$

- Bratteli diagrams: two rows of vertices: top k vertices labeled $N_1, \ldots, N_k$, bottom k' vertices labeled by $N'_1, \ldots, N'_{k'}$; d_{ij} edges between vertex i (top row) and j (bottom row)



$\phi : \mathcal{A}_1 \to \mathcal{A}_2$ unital, so all vertices in bottom row reached by an edge, but top row can have vacant vertices

Example

- $\mathcal{A}_1 = \mathbb{C} \oplus M_2(\mathbb{C})$, $\mathcal{H}_1 = \mathbb{C} \oplus \mathbb{C}^2$, $\mathcal{A}_2 = M_3(\mathbb{C})$, $\mathcal{H}_2 = \mathbb{C}^3$
- unital $*$-algebra map $\phi : \mathcal{A}_1 \to \mathcal{A}_2$ two possibilities

$$(z, a) \in \mathbb{C} \oplus M_2(\mathbb{C}) \quad \mapsto u \begin{pmatrix} z & \\ & a \end{pmatrix} u^* \in M_3(\mathbb{C})$$

with $u \in U(3)$ or

$$(z, a) \in \mathbb{C} \oplus M_2(\mathbb{C}) \quad \mapsto z1_3 \in M_3(\mathbb{C})$$

with kernel $M_2(\mathbb{C})$

- unitary map of $\mathcal{H}_1$ to $\mathcal{H}_2$

$$(x, y) \in \mathbb{C} \oplus \mathbb{C}^2 \mapsto U \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{C}^3$$

with $U \in U(3)$

- compatibility of ϕ and L: first case OK with $u = U$

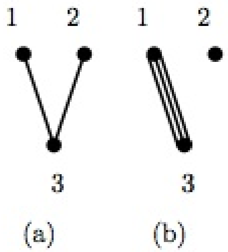
$$u \begin{pmatrix} z & \\ & a \end{pmatrix} u^* = U \begin{pmatrix} z & \\ & a \end{pmatrix} U^*.$$

but in second case

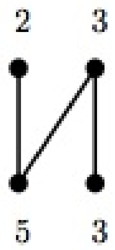
$$z 1_3 = U \begin{pmatrix} z & \\ & a \end{pmatrix} U^*.$$

cannot be satisfied for arbitrary $(z, a) \in \mathcal{A}_1$

- so get $\mathrm{Hom}((A_1, H_1), (A_2, H_2)) \simeq U(3)$ and Bratteli diagram

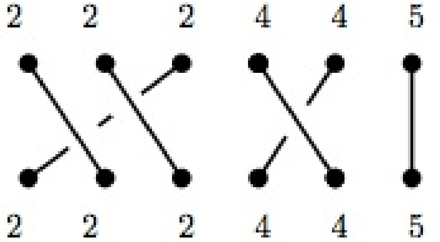


Example



Bratteli diagram for the only unital $*$-algebra map $M_2(\mathbb{C}) \oplus M_3(\mathbb{C}) \to M_5(\mathbb{C}) \oplus M_3(\mathbb{C})$ given $(a, b) \mapsto (a \oplus b, b)$

• to better take care also of permutations of matrix blocks of the same dimension: braid Bratteli diagrams



braid Bratteli diagram with permutations of matrix blocks of same dim in $M_2(\mathbb{C})^{\oplus 3} \oplus M_4(\mathbb{C})^{\oplus 2} \oplus M_5(\mathbb{C})$

Quiver representations in categories

• Quiver Γ directed graph

• representation π of a quiver Γ in a category $\mathcal{C}$:
- object π_v for each vertex v
- morphism π_e in $\mathrm{Hom}(\pi_{s(e)}, \pi_{t(e)})$ for each directed edge e.

• two representations π, π' of Γ in same category equivalent if $\pi_v = \pi'_v$, for all $v \in V(\Gamma)$ and $\exists$ family of invertible morphisms $\phi_v \in \mathrm{Hom}(\pi(v), \pi(v))$ for $v \in V(\Gamma)$ such that

$$\pi_e = \phi_{t(e)} \circ \pi'_e \circ \phi_{s(e)}^{-1}$$

• For categories $\mathcal{C}$ (or $\mathcal{C}_0$) of finite spectral triples, representation π of a quiver Γ assigns
- spectral triples $(\mathcal{A}_v, \mathcal{H}_v, D_v)$ ($D_v = 0$ for $\mathcal{C}_0$) to vertices $v \in V(\Gamma)$
- pairs $(\phi, L) \in \mathrm{Hom}((\mathcal{A}_{s(e)}, \mathcal{H}_{s(e)}, D_{s(e)}), (\mathcal{A}_{t(e)}, \mathcal{H}_{t(e)}, D_{t(e)}))$ to edges $e \in E(\Gamma)$

Example $U(N)$ spin networks (John Baez)

• If $(\mathcal{A}_v, \mathcal{H}_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ and $D = 0$, unitary $u_e \in U(N)$ along each edge and gauge action $g_v \in U(N)$ at each vertex with

$$u_e \mapsto g_{t(e)} u_e g^*_{s(e)}$$

• only possible Bratteli diagram in this case for $\phi : M_N(\mathbb{C}) \to M_N(\mathbb{C})$ is single edge between one upper row vertex and one lower row vertex

• J.C. Baez, *Spin network states in gauge theory*, Adv. Math. 117 (1996) 253–272

General case: gauge networks

$$\{\Gamma, (A_v, \lambda_v, H_v; \iota_v)_v, (\rho_e, \mathbb{B}_e)_e\}$$

- Γ directed graph
- (A_v, λ_v, H_v) is an object in the category $\mathcal{C}_0^s$ for each vertex $v \in V(\Gamma)$
- Edge $e \in E(\Gamma)$: representation ρ_e of unitary group $G_e = \mathrm{Aut}_{\widetilde{A}_{t(e)}}(H_{t(e)}) \times \mathcal{U}(\ker \lambda_{t(e)})$
- Edge $e \in E(\Gamma)$: Bratteli diagram $\mathbb{B}_e$ for $*$-algebra maps $A_{s(e)} \to A_{t(e)}$
- subdiagrams $\widetilde{\mathbb{B}}$ for $\widetilde{A}_{s(e)} \to \widetilde{A}_{t(e)}$ and $\mathbb{B}_0$ for $A_{s(e)} \to \ker \lambda_{t(e)}$
- Vertex v: the intertwiners ι_v for the group $\mathcal{G}_v = U(\mathcal{A}_v) \rtimes S(\mathcal{A}_v)$:

$$\iota_v : \rho_{e_1'} \otimes \cdots \otimes \rho_{e_k'} \to \rho_{e_1}^{K_{\mathbb{B}_{e_1}}} \circ \phi_{\mathbb{B}} \otimes \cdots \otimes \rho_{e_l}^{K_{\mathbb{B}_{e_l}}} \circ \phi_{\mathbb{B}}$$

$e_1', \ldots, e_k'$ incoming edges, $e_1, \ldots, e_l$ outgoing edges at v; isotropy group $K_{\mathbb{B}_e} = \mathcal{U}(\ker \lambda_{t(e)})_{\mathbb{B}_{e0}}$.

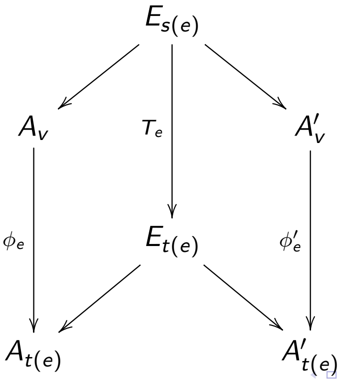
Correspondences between gauge networks

- two π, π' quiver reps of Γ
- $\mathcal{A}_v - \mathcal{A}'_v$ Bimodules $\mathcal{E}_v$

$$\mathcal{H}_v = \mathcal{E} \otimes_{\mathcal{A}'_v} \mathcal{H}'_v$$

- morphisms $T_e : \mathcal{E}_{s(e)} \to \mathcal{E}_{t(e)}$ compatible with alg maps ϕ_e, ϕ'_e

$$T_e(a\eta b) = \phi_e(a) T_e(\eta) \phi'_e(b), \quad a \in A_{s(e)}, \eta \in E_{s(e)}, b \in A'_{s(e)}$$



Algebra of gauge networks and correspondences

- given gauge networks

$$\psi = (\Gamma, (A_v, H_v, \iota_v)_v, (\rho_e, \mathbb{B}_e)_e), \qquad \psi' = (\Gamma, (A'_v, H'_v, \iota_v)_v, (\rho'_e, \mathbb{B}'_e)_e)$$

and correspondences ${}_{\psi}\Psi_{\psi'}$

$$\Psi = \{\Gamma, ({}_{A_v}E_{A'_v}, \iota_v \otimes \iota'_v)_v, (\rho_e \otimes \rho'_e, \mathbb{B}_e \times \mathbb{B}'_e)_e\}$$

- composition of correspondences (tensor product of bimodules)

$$\Psi_1 = \{\Gamma, ({}_{A_v}E_{A'_v}, \iota_v \otimes \iota'_v)_v, (\rho_e \otimes \rho'_e, \mathbb{B}_e \times \mathbb{B}'_e)_e\}$$

$$\Psi_2 = \{\Gamma, ({}_{A'_v}F_{A''_v}, \iota'_v \otimes \iota''_v)_v, (\rho'_e \otimes \rho''_e, \mathbb{B}'_e \times \mathbb{B}''_e)_e\}$$

$$\Psi_1 \circ \Psi_2 = \{\Gamma, ({}_{A_v}E \otimes_{A'_v} F_{A''_v}, \iota_v \otimes \iota''_v)_v, (\rho_e \otimes \rho''_e, \mathbb{B}_e \times \mathbb{B}''_e)_e\}$$

• $\mathcal{S}$ = category of gauge networks with correspondences as morphisms

• algebra $\mathbb{C}[\mathcal{S}]$ elements $a = \sum_\Psi a_\Psi \Psi$ convolution product

$$(a * b)_\Psi = \sum_{\Psi = \Psi_1 \circ \Psi_2} a_{\Psi_1} b_{\Psi_2}.$$

• can be completed to a C^*-algebra represented on a Hilbert space

• dynamical: Hamiltonian and time evolution, built using quadratic Casimir (kind of Lie group Laplacian) on $\mathcal{U}(A_{t(e)})$

Spectral action and lattice field theory

• Γ embedded in a Riemannian spin manifold M: pullback spin geometry of M to Γ

• $\mathcal{S}$ fiber of spinor bundle on M; take $\mathcal{S}^{V(\Gamma)}$ space of spinors on Γ

• holonomy $\mathrm{Hol}(e, \nabla^S)$ of spin connection along edges e of Γ

$$\mathrm{Hol}(e, \nabla^S) = \mathcal{P} e^{\int_e \omega \cdot dx} \sim 1 + l_e \omega_e(s(e)) + \mathcal{O}(l_e^2)$$

$\omega_e(v)$ pairing of 1-form ω and vector $\dot{e}$ at vertex v

• Dirac operator on Γ:

$$(D_\Gamma \psi)_v = \sum_{t(e)=v} \frac{1}{2l_e} \gamma_e \, \mathrm{Hol}(e, \nabla^S) \psi_{s(e)} + \sum_{s(e)=v} \frac{1}{2l_{\bar{e}}} \gamma_{\bar{e}} \, \mathrm{Hol}(\bar{e}, \nabla^S) \psi_{t(e)};$$

l_e = geodesic length of embedded edge e; $\bar{e}$ = opposite orientation

• gamma matrices γ_e defined so that (discretization/continuum)

$$\sum_{e \in S(v)} \gamma_e \omega_e = \gamma^\mu \omega_\mu$$

Continuum limit of Dirac operator

• lattice spacing l_e goes to zero; assume $l_e = l$ for all edges and square lattice

$$(D_\Gamma \psi)_v = \sum_{v_1, v_2} \frac{1}{2l} \gamma_e (\psi_{v_1} - \psi_{v_2}) + \frac{1}{2} \gamma_e \omega_e(v) (\psi_{v_1} + \psi_{v_2}) + \mathcal{O}(l).$$

sum over all collinear $\quad v_1 \xrightarrow[e']{} v \xrightarrow[e]{} v_2$

• formally, when $l \to 0$

$$(D_\Gamma \psi)_v \longrightarrow \gamma^\mu (\partial_\mu + \omega_\mu) \psi(v)$$

Dirac twisted with finite spectral triples

• if also quiver representation of Γ in the category of finite spectral triples

$$(D_{\Gamma,L}\psi)_v = \sum_{t(e)=v} \frac{1}{2l_e}\gamma_e \left(\mathrm{Hol}(e,\nabla^S)\otimes L_e\right)\psi_{s(e)}$$

$$+\sum_{s(e)=v} \frac{1}{2l_{\bar{e}}}\gamma_{\bar{e}} \left(\mathrm{Hol}(\bar{e},\nabla^S)\otimes L_{\bar{e}}\right)\psi_{t(e)} + \gamma D_v \psi_v$$

where $L_{\bar{e}} = L_e^*$ and γ grading on spinor bundle of M if even dimensional

• if $(A_v, H_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ at all vertices v, then morphism (ϕ, L) unitary in $U(N)$ holonomy of some gauge connection 1-form A_μ, then Dirac on Γ reduces to Dirac on M twisted by gauge field

Spectral action: finite spectral triples

$$S[\{L_e\},\{D_v\}] = \mathrm{Tr} f(D_{\Gamma,L})$$

some function f on the real line

- lattice gauge fields on $M=\mathbb{R}^4$, cutoff $\Lambda \propto l^{-1}$

$$S_\Lambda[\{L_e\},\{D_v\}] := \mathrm{Tr} f(D_{\Gamma,L}/\Lambda) \equiv l^4 \mathrm{Tr}((D_{\Gamma,L})^4)$$

- on square lattice $\mathbb{Z}^4$ find

$$S_\Lambda[\{L_e\},\{D_v\}] = -\frac{1}{4} \sum_{\partial p = e_4 \cdots e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1}\right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4}\right)\right) + \text{const}$$

$$+ \sum_v l^4 \mathrm{Tr} D_v^4 + 4 l^2 \sum_e \left(\mathrm{Tr} D_{s(e)}^2 + \mathrm{Tr} D_{t(e)}^2 - \mathrm{Tr} L_e^* D_{t(e)} L_e D_v \right)$$

from counting contributions of different cycles in the lattice

- flat case: holonomy of spin connection trivial: $S_\Lambda[\{L_e\}]$ is

$$= 4l^4 \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \frac{1}{(2l)^4} \mathrm{Tr}(\gamma_\nu \gamma_\mu)^2 \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right)$$

plus constant terms

$$= -\frac{1}{4} \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right) + \text{const}$$

Similar argument for the other terms

Continuum limit and Wilson action

- μ direction of e and A_μ continuous gauge field at $s(e)$

$$L_e = \mathcal{P} e^{i \int_e A \cdot dx} \sim e^{i A_\mu l} \quad \text{for } l \to 0$$

- with $(A_v, H_v) = (M_N(\mathbb{C}), \mathbb{C}^N)$ at all vertices v, limit $l \to 0$ and $\Lambda \propto l^{-1}$ spectral action S_Λ becomes

$$\frac{1}{4} \int_M \mathrm{Tr} F_{\mu\nu} F^{\mu\nu} + 2 \int_M \mathrm{Tr}(\partial_\mu \Phi - [iA_\mu, \Phi])(\partial^\mu \Phi - [iA^\mu, \Phi])$$

$$+ 8\Lambda^2 \int_M \mathrm{Tr}\Phi^2 + \int_M \mathrm{Tr}\Phi^4 .$$

Yang–Mills coupled to a Higgs field with quartic potential

- For a plaquette

$$\mathrm{Tr}\,(L_{\bar{e}_4}L_{\bar{e}_3}L_{e_2}L_{e_1}) = \mathrm{Tr} e^{-ilA_\nu(x)}e^{-ilA_\mu(x+l\hat{\nu})}e^{ilA_\nu(x+l\hat{\mu})}e^{ilA_\mu(x)}$$

$$\sim \mathrm{Tr} e^{il^2 F_{\mu\nu}} \quad \text{for } l \to 0$$

and similarly for $\mathrm{Tr}\,(L_{\bar{e}_1}L_{\bar{e}_2}L_{e_3}L_{e_4})$

- so for $l \to 0$ (and $\Lambda \to \infty$)

$$S_\Lambda \sim \frac{1}{4}\int_M \mathrm{tr} F_{\mu\nu}F^{\mu\nu}$$

- Higgs terms: vertex v at position x

$$\mathrm{Tr} e^{-iA_\mu l}\Phi(x+l\hat{\mu})e^{iA_\mu l}\Phi(x) \sim$$

$$\mathrm{Tr}\Big(\Phi(x)\Phi(x+l\hat{\mu}) + l\Phi(x+l\hat{\mu})[iA_\mu,\Phi(x)]$$

$$-\frac{1}{2}l^2[iA_\mu,\Phi(x+l\hat{\mu})][iA_\mu,\Phi(x)]\Big) + \mathcal{O}(l^3)$$

$\Phi(x)$ continuous (hermitian) Higgs field corresponding to D_x and L_e is expanded in A_μ

- modulo $\mathcal{O}(l^3)$ find in S_Λ

$$S_\Lambda = -\frac{1}{4} \sum_{\partial p = \bar{e}_4 \bar{e}_3 e_2 e_1} \left(\mathrm{Tr}\left(L_{\bar{e}_4} L_{\bar{e}_3} L_{e_2} L_{e_1} \right) + \mathrm{Tr}\left(L_{\bar{e}_1} L_{\bar{e}_2} L_{e_3} L_{e_4} \right) \right)$$

$$+ \sum_v l^4 \mathrm{Tr} D_v^4 + 4 l^2 \sum_e \left(\mathrm{Tr} D_{s(e)}^2 + \mathrm{Tr} D_{t(e)}^2 - \mathrm{Tr} L_e^* D_{t(e)} L_e D_{s(e)} \right)$$

$$\sim \frac{1}{2} \mathrm{Tr} e^{i l^2 F_{\mu\nu}} + l^4 \mathrm{Tr} \Phi^4(x) + 2 l^2 \sum_\mu \mathrm{tr} \Phi^2(x) + \mathrm{tr} \Phi^2(x + l\hat{\mu})$$

$$+ 2 l^4 \sum_\mu \frac{1}{l^2} \mathrm{Tr} (\Phi(x + l\hat{\mu}) - \Phi(x))^2$$

$$- \frac{2}{l} \mathrm{Tr} \Phi(x + l\hat{\mu}) [i A_\mu(x), \Phi(x)] + \mathrm{Tr}([i A_\mu(x), \Phi(x)])^2$$

John Barret's Random noncommutative geometries

• a geometry: $(\mathcal{A}, \mathcal{H}, D, J, \gamma)$ finite spectral triple with real structure

• random geometry: fixed fermion space $(\mathcal{A}, \mathcal{H}, J, \gamma)$ and varying Dirac operator D up to unitary equivalences

• a random geometry is a "random" (in a suitable probability distribution) point in the moduli space of Dirac operators

• want measure to reflect some action functional, as in path integral:

$$e^{-S(D)}\, dD$$

- view this as a random matrix model where the matrices D are constrained by the properties of Dirac operators of finite spectral triples
- take action functional as a spectral action

$$S(D) = \mathrm{Tr}(f(D)) = \sum_{\lambda \in Spec(D)} f(\lambda)$$

- here want some function $f(x)$ with $f(x) \to \infty$ for $|x| \to \infty$ for convergence of

$$Z = \int_{\mathcal{M}} e^{-S(D)}\, dD$$

- simplest choice quartic polynomial: $g_4 > 0$ (or $g_4 = 0$, $g_2 > 0$)

$$f(D) = g_2 D^2 + g_4 D^4$$

- observables $\mathcal{O}(D)$ functions of D

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int_{\mathcal{M}} \mathcal{O}(D) e^{-S(D)}\, dD$$

behavior in limit $N \to \infty$ of large matrices

• use only Dirac operators that resemble those on manifolds

• different possibilities for Dirac operators: action on $\mathcal{H} = V \otimes M_n(\mathbb{C})$ with $V = \mathbb{C}^k$ a Clifford module signature (p, q) (with $k = 2^{d/2}$ or $k = 2^{(d-1)/2}$)

• express all the possibilities for (p, q) writing Dirac operators in terms of gamma matrices and commutators $[L, \cdot]$ or anticommutators $\{L, \cdot\}$ with given hermitian matrices H and anti-hermitian L

• Example: $(1, 0)$ has $D = \{H, \cdot\}$ and $(0, 1)$ has $D = -i[L, \cdot]$

• Example: $(1, 1)$ has $(\gamma^1)^2 = 1$ and $(\gamma^2)^2 = -1$ and

$$D = \gamma^1 \otimes \{H, \cdot\} + \gamma^2 \otimes [L, \cdot]$$

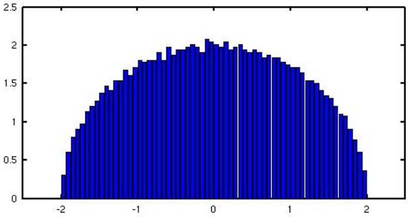
etc.

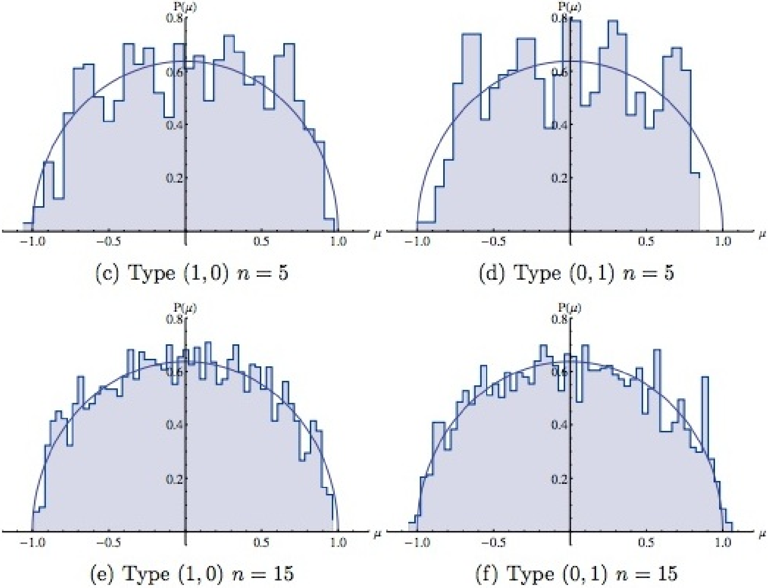
Monte Carlo simulation

• start with random D and construct $D + \delta D$ by δH_i and δL_i

• accept if $\Delta S(D) = S(D_{new}) - S(D_{old}) < 0$ or (to escape local minima) if $\exp(S(D_{old}) - S(D_{new})) > p$ uniformly distributed random number on $[0, 1]$ otherwise keep D_{old}

• compare results with Wigner's semicircle law for random matrix model with real symmetric matrices large order N





Density of states for *H* and *L* from Barrett and Glaser arXiv:1510.01377