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Continued fractions, modular symbols, and noncommutative geometry

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We dedicate this paper to Friedrich Hirzebruch, in occasion of his 75th birthday

Abstract. Using techniques introduced by D. Mayer, we prove an extension of the classical Gauss–Kuzmin theorem about the distribution of continued fractions, which in particular allows one to take into account some congruence properties of successive convergents. This result has an application to the Mixmaster Universe model in general relativity. We then study some averages involving modular symbols and show that Dirichlet series related to modular forms of weight 2 can be obtained by integrating certain functions on real axis defined in terms of continued fractions. We argue that the quotient $PGL(2,\mathbf{Z})\backslash\mathbf{P}^1(\mathbf{R})$ should be considered as noncommutative modular curve, and show that the modular complex can be seen as a sequence of K_0-groups of the related crossed-product C^*-algebras.

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0. Introduction and summary

In this paper we study the interrelation between several topics: a generalization of the classical Gauss problem on the distribution of continued fractions, certain averages of modular symbols, the properties of geodesics on modular curves, the Mixmaster Universe model in general relativity, and the noncommutative geometry of the quotient $PGL(2,\mathbf{Z}) \backslash \mathbf{P}^1(\mathbf{R})$.

Our main motivation is a picture of a tower of "noncommutative modular curves", parameterizing two-dimensional noncommutative tori. This ties in with the project of studying Stark's conjectures for real quadratic extensions of $\mathbf{Q}$ via a theory of real multiplication on noncommutative elliptic curves [Man6].

The traditional algebro-geometric compactification of a modular curve $X_{G_0} = G_0\backslash H$, for $G_0 \subset PSL(2,\mathbf{Z})$ a finite index subgroup and H the upper half plane, is given by the set of cusps $G_0\backslash\mathbf{P}^1(\mathbf{Q})$. Our main philosophy is that this should be replaced by the quotient $G_0\backslash\mathbf{P}^1(\mathbf{R})$ considered as a noncommutative space. This point of view fits into the context of recent work [CoDS], [Man4], [Soi].

We support this philosophy by results of two types. First, we show that certain exact sequences of [Mer], related to the modular complex introduced in [Man1], which gives a combinatorial definition of the homology of modular curves, can be identified with the Pimsner exact sequence for K-theory of our noncommutative modular curves. Second, we demonstrate that cusps forms of weight two for congruence subgroups (or rather their Mellin transforms) can be obtained by integrating along the real axis certain "automorphic series" defined in terms of continued fractions and modular symbols (cf. identity (0.16)).

In a different but related perspective, we also show that the classical definition of modular symbols can be generalized to "limiting modular symbols" which take into account geodesics on the upper half plane which end at irrational points. We show that quadratic irrationalities give rise to limiting cycles while, for generic irrational points, there is a vanishing result in suitable averaged sense. This result depends on the properties of a generalization of the Ruelle transfer operator (or Gauss–Kuzmin operator) for the shift on the continued fraction expansion, and its spectral properties. In the case of the group $\Gamma_0(2)$ these properties have applications to the Mixmaster Universe model in general relativity.

0.1. Continued fractions

We start by fixing the notation which will be used throughout the paper. Considering first $k_1, \dots, k_n$ as independent variables, put for $n \geq 1$

$$[k_1, \dots, k_n] := \frac{1}{k_1 + \frac{1}{k_2 + \dots \frac{1}{k_n}}} = \frac{P_n(k_1, \dots, k_n)}{Q_n(k_1, \dots, k_n)} \tag{0.1}$$

where P_n, Q_n are polynomials with integral coefficients which can be calculated inductively from the relations

$$Q_{n+1}(k_1, \dots, k_n, k_{n+1}) = k_{n+1} Q_n(k_1, \dots, k_n) + Q_{n-1}(k_1, \dots, k_{n-1}),$$

$$P_n(k_1, \dots, k_n) = Q_{n-1}(k_2, \dots, k_n). \tag{0.2}$$

It is convenient also to put formally $Q_{-1} = 0, Q_0 = 1$ which is compatible with (0.2), (0.1).

From (0.2) one readily sees that

$$[k_1, \dots, k_{n-1}, k_n + x_n]$$

$$= \frac{P_{n-1}(k_1, \dots, k_{n-1})\, x_n + P_n(k_1, \dots, k_n)}{Q_{n-1}(k_1, \dots, k_{n-1})\, x_n + Q_n(k_1, \dots, k_n)} = \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} (x_n) \tag{0.3}$$

where we use the standard matrix notation for fractional linear transformations defining the action of $GL(2)$ on $\mathbf{P}^1$:

$$z \mapsto \frac{az+b}{cz+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} (z).$$

If $\alpha \in (0,1)$ is an irrational number, there is a unique sequence of integers $k_n(\alpha) \geq 1$ such that α is the limit of $[\,k_1(\alpha), \dots, k_n(\alpha)\,]$ as $n \to \infty$. Moreover, there is a unique sequence $x_n(\alpha) \in (0,1)$ such that

$$\alpha = [\,k_1(\alpha), \dots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)\,]$$

for each $n \geq 1$. Rational numbers in $(0,1]$ can be accommodated by allowing finite sequences of $k_i \geq 1$ complemented by zeroes, and similarly for $x_n(\alpha)$.

We can specialize (0.3) at the point α and get by induction

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & k_1(\alpha) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n(\alpha) \end{pmatrix} (x_n(\alpha)). \tag{0.4}$$

Put

$$p_n(\alpha) := P_n(k_1(\alpha), \dots, k_n(\alpha)), \; q_n(\alpha) := Q_n(k_1(\alpha), \dots, k_n(\alpha))$$

so that $p_n(\alpha)/q_n(\alpha)$ is the sequence of convergents to α. Denote also

$$g_n(\alpha) := \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix}.$$

For further use, we reproduce a description of the total set of matrices $g_n(\alpha)$. Put

$$\mathrm{Red}_n = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n \end{pmatrix} \mid k_1, \dots, k_n \geq 1;\, k_i \in \mathbf{Z} \right\}, \tag{0.5}$$

and $\mathrm{Red} := \cup_{n \geq 1} \mathrm{Red}_n \subset GL(2, \mathbf{Z})$.

In the terminology of [LewZa1], Red_n consists of reduced matrices of length n. The following properties of Red are proved in that paper:

(i) *A matrix in $GL(2,\mathbf{Z})$ is reduced iff it has non-negative entries which are non-decreasing downwards and to the right.*

(ii) *The length $l(g)$ of a reduced matrix g and its representation in the form* (0.5) *are uniquely defined.*

(iii) *Reduced matrices are hyperbolic and have two distinct fixed points on the real axis. Every conjugacy class g of hyperbolic matrices in $GL(2,\mathbf{Z})$ contains reduced representatives. They all have the same length $l(g)$, and there are exactly $l(g)/k(g)$ of them where $k(g)$ is the maximal integer such that $g = h^{k(g)}$ for some h.*

0.1.1. Generalized Gauss problem. Consider a subgroup of finite index $G \subset GL(2,\mathbf{Z})$ and the coset space $\mathbf{P} = GL(2,\mathbf{Z})/G$ with the left transitive action of $GL(2,\mathbf{Z})$ on it:

$$t \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix} (t).$$

For any $x \in [0,1]$, $t \in \mathbf{P}, n \geq 0$ put

$$m_n(x,t) := \text{measure of the set } \{\alpha \in (0,1) \,|\, x_n(\alpha) \leq x,\, g_n(\alpha)^{-1}(t_0) = t\} \qquad (0.6)$$

where t_0 is the base point of $\mathbf{P}$, the coset of G. Notice that $x_n(\alpha) = g_n(\alpha)^{-1}(\alpha)$ so that m_n is essentially the pullback of the Lebesgue measure on $(0,1) \times \mathbf{P}$ with respect to the operator $g_n(\alpha)$ acting upon α and t simultaneously. Notice also that $g_n(\alpha)^{-1}$ is another notation for the n-th power of the shift operator

$$T:\, (\alpha,t) \mapsto \left(\frac{1}{\alpha} - \left[\frac{1}{\alpha}\right], \begin{pmatrix} -[1/\alpha] & 1 \\ 1 & 0 \end{pmatrix} (t).\right)$$

The first result of this paper is the following generalization of the Gauss–Kuzmin–Lévy formula:

0.1.2. Theorem. *Assume that* $\mathrm{Red}\,(t) = \mathbf{P}$ *for each* $t \in \mathbf{P}$ *(transitivity condition), with* $\mathrm{Red}\,(t) := \{gt | g \in \mathrm{Red} \subset GL(2,\mathbf{Z})\}$. *Then the limit* $m(x,t) = \lim_{n\to\infty} m_n(x,t)$ *exists and equals*

$$m(x,t) = \frac{1}{|\mathbf{P}|\log 2} \log(1+x). \qquad (0.7)$$

A proof is given in Section 1 below. It starts with a straightforward generalization of the Gauss–Kuzmin inductive expression of m_{n+1} through m_n. To write it down, consider first the following sets: for $y \in (0,1), s \in \mathbf{P}$, put

$$M_n(y,s) := \{\beta \in (0,1) \,|\, x_n(\beta) \leq y,\, g_n(\beta)^{-1} t_0 = s\}.$$

Then we have, using (0.4) and neglecting the rationals which have measure zero:

$$M_{n+1}(x,t) = \coprod_{k=1}^{\infty} \left\{ M_n\left(\frac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right) - M_n\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right)\right\}.$$

Therefore,

$$m_{n+1}(x,t) = \sum_{k=1}^{\infty} \left\{ m_n\left(\frac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right) - m_n\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right)\right\}. \qquad (0.8)$$

Derivating (0.8) in x, we get the following equation for the densities, which introduces an important operator L, the generalized Gauss–Kuzmin operator:

$$m'_{n+1}(x,t) = (Lm'_n)(x,t) := \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} m'_n\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)\right). \qquad (0.9)$$

In the classical case, when $|\mathbf{P}| = 1$, at least four ways to deduce (0.7) from (0.9) are known: two "elementary" deductions going back to R. Kuzmin and P. Lévy respectively (see e.g. [Sch]), and two functional analytic methods, based upon the spectral analysis of the operator L formally defined by (0.9); see [Ba], [BaYu] and [May1].

Clearly, the limiting measure must be an eigenfunction of L corresponding to the eigenvalue 1, and both analytic proofs show the convergence of $L^n m'_0$ to this limiting measure by establishing that 1 is the multiplicity one eigenvalue with maximal modulus for the extension of L to an appropriate function space. K. Babenko realizes L as a selfadjoint operator in a Hilbert space, whereas D. Mayer works in the context of nuclear and trace class operators in Banach spaces.

Of these four proofs, we were able to generalize to our context only Mayer's method. Babenko's representation seems to be inadequate for proving convergence. However, it might still be useful for numerical calculations, and we present it in Section 1.3.

The operator L and its deformation L_s ((1.1) below) were introduced and studied also in a recent paper [ChMay], of which we became aware only after the first draft of this paper was written.

0.2. Modular curves and geodesics

Our study of the generalized Gauss measure (0.7) was motivated by the relationship between continued fractions and one-dimensional homology of modular curves. We will start by recalling the basic features of this relationship in the form given in [Man1]; see also [Mer] for additional information.

Let G_0 be a subgroup of finite index in the fractional linear group $PSL(2,\mathbf{Z}) = SL(2,\mathbf{Z})/(\pm 1)$. It determines the noncompact modular curve $G_0 \setminus H$ (where H is the upper complex half-plane). This curve admits a smooth compactification by a finite number of cusps which are in a natural bijection with the set $G_0 \setminus \mathbf{P}^1(\mathbf{Q})$. Let X_{G_0} (or more precisely, $X_{G_0}(\mathbf{C})$) denote this compactification and φ the respective covering map.

For any two points $\alpha, \beta \in \overline{H} := H \cup \mathbf{P}^1(\mathbf{Q})$ we can define a real homology class ("modular symbol") $\{\alpha,\beta\}_{G_0} \in H^1(X_{G_0}, \mathbf{R})$ by integrating the lifts of differentials ω of the first kind on X_{G_0} along the geodesic path connecting α to β:

$$\int_{\{\alpha,\beta\}} \omega := \int_{\alpha}^{\beta} \varphi^*(\omega). \qquad (0.10)$$

Modular symbols satisfy the basic additivity and invariance properties:

$$\{\alpha,\beta\}_{G_0}+\{\beta,\gamma\}_{G_0}=\{\alpha,\gamma\}_{G_0},$$
$$\forall g\in G_0,\ \{g\alpha,g\beta\}_{G_0}=\{\alpha,\beta\}_{G_0}. \tag{0.11}$$

The integrals (0.10) can be related to finite, stably periodic, or general infinite continued fractions, depending on the arithmetical nature of the ends α,β. We will briefly treat these three cases separately.

(i) *Finite continued fractions.* Assume first that α,β are cusps. It is known that in this case the modular symbol represents a rational homology class (Manin–Drinfeld's theorem).

By additivity, it suffices to look at the integrals of the form $\int_0^\alpha$, $\alpha\in\mathbf{Q}$. Let

$$g_k:=\begin{pmatrix} p_{k-1}(\alpha) & p_k(\alpha)\\ q_{k-1}(\alpha) & q_k(\alpha)\end{pmatrix},\qquad k=1,\dots,n,\ \alpha=\frac{p_n(\alpha)}{q_n(\alpha)}.$$

Then, again by additivity, we have

$$\int_0^\alpha \varphi^*(\omega)=\sum_{k=1}^n\int_{p_{k-1}(\alpha)/q_{k-1}(\alpha)}^{p_k(\alpha)/q_k(\alpha)}\varphi^*(\omega)=-\sum_{k=1}^n\int_{\{g_k(0),g_k(i\infty)\}}\omega. \tag{0.12}$$

Finally, in view of (0.11), the k-th integral in (0.12) depends only on the class of g_k in $G_0\backslash PSL(2,\mathbf{Z})$. Thus, (0.12) establishes a connection between the distribution of modular symbols $\{\alpha,\beta\}_{G_0}$ and the distribution of pairs of consecutive convergents to α and β in $\mathbf{P}_0:=G_0\setminus PSL(2,\mathbf{Z})$.

Notice that there is a slight discrepancy with 0.1.1 where we dealt instead with $G\subset GL(2,\mathbf{Z})$ and $GL(2,\mathbf{Z})/G$. In order to reduce the current problem to the former one, we can first replace G_0 by its lift to $SL(2,\mathbf{Z})$ and then denote by G the subgroup generated by this lift and $\begin{pmatrix}-1&0\\0&1\end{pmatrix}$. We will have a natural identification $\mathbf{P}_0=G\setminus GL(2,\mathbf{Z})$, and this set in turn identifies with $\mathbf{P}$ in 0.1.1 under the map $g\mapsto g^{-1}$ which is implicit in (0.6).

If α and/or β in (0.10) is real irrational, the integral diverges at this end. In this case, it is natural to study its asymptotic behavior. We will define the "limiting modular symbol" by the following expression whenever it makes sense:

$$\{\{*,\beta\}\}_{G_0}:=\lim\frac{1}{T(x,y)}\{x,y\}_{G_0}\in H^1(X_{G_0},\mathbf{R}). \tag{0.13}$$

Here $x,y\in H$ are two points on the geodesic joining α to β, x is arbitrary but fixed, $T(x,y)$ is the geodesic distance between them, and the limit is taken as y tends to β. In Section 2 we will prove that if the limit does exist, it depends neither on x nor on α, hence the first argument is replaced by $*$, whereas the double curly brackets remind about the limit. We will discuss the relation of this symbol to continued fractions in two situations.

(ii) *Stably periodic continued fractions.* Among geodesics with two irrational ends there is an important subclass consisting of geodesics that connect two fixed points of a hyperbolic element in G_0. More precisely, any hyperbolic $g \in G_0$ has two fixed points $\alpha^{\pm}$, repelling and attracting, on the real line. Let $\Lambda_g^{\pm}$ be the respective eigenvalues, $0 < \Lambda_g^+ < 1$. The oriented geodesic in H connecting α_g^- to α_g^+ is g-invariant, and the action of g induces on it the shift by the geodesic distance $\lambda(g) := \log \Lambda_g^-$.

For any point x on this geodesic, the image of its segment $[x, gx]$ is a parameterized closed loop on X_{G_0} missing the cusps. (The supporting set-theoretic loop is run over exactly $k(g)$ times where $k(g)$ is the maximal k such that g is a k-th power in G_0). The homology class of this loop is $\{0, g(0)\}$. When we integrate from a fixed x to $g^n x$, we run over the parameterized loop n times, the geodesic length of the path is $n\,\lambda(g)$, and its homology class is $n\,\{0, g(0)\}$. Therefore the limiting modular symbol (0.13) exists and equals

$$\{\{*, \alpha_g^+\}\} = \frac{\{0, g(0)\}}{\lambda(g)}. \tag{0.14}$$

The most important generating function for closed geodesics is the Selberg zeta function. However, it encodes only the lengths of closed geodesics in the hyperbolic metric. The usual modular symbol in the numerator of (0.14) depends only on the class of g modulo $[G_0, G_0]$ and is additive in g (see [Man1], Prop. 1.4). Perhaps, one can construct a generating function for (0.14) as a combination of Selberg's zetas with abelian characters.

The usual Selberg's zetas were studied in [May1], and then in [LewZa1] for GL and SL separately. It turned out that they could be represented as Fredholm determinants $\det(1 - L_s)$ and $\det(1 - L_s^2)$ respectively. Literally the same is true in our generalized setting, when subgroups G or G_0 are introduced. This is proved in [ChMay], and we supply a brief discussion of this in Section 3.

As we have briefly explained, the distribution of continued fractions and modular symbols at cusps is encoded in the eigenvalues of L_1, 1 being the dominant value producing the distribution (0.7).

From the identity $\det(1 - L_s) = Z(s)$ it follows that the zeroes of $Z(s)$ are exactly those values for which the deformed operator L_s has eigenvalue 1.

J. Lewis and D. Zagier produced also an in-depth study of the respective eigenfunctions for the full modular groups $GL(2, \mathbf{Z})$ and $SL(2, \mathbf{Z})$.

(iii) *General infinite continued fractions.* For this case, we will prove in Section 2 two results. Namely, we will establish that with an additional assumption the limiting modular symbol (0.13) exists in a weak sense and is zero.

0.2.1. Theorem. *Assume that* $\mathrm{Red}(t) = \mathbf{P}_0$ *for each* $t \in \mathbf{P}_0$. *Then (0.13) weakly converges to zero.*

For the precise description of the sense in which this convergence holds, see Section 2.3 and, in particular, (2.21). Our prof is based upon Theorem 0.1.2.

This vanishing can be compared with a well-known interpretation of the Selberg trace formula for compact surfaces: quantum mechanical averages for the geodesic flow can be calculated as if this flow were classically concentrated on closed geodesics.

The transitivity assumption for Red will be checked in Section 2 in the case $G_0 = \Gamma_0(N)$.

We show that the case $N = 2$ has a nice little application to the study of the $t = 0$ singularity of the Bianchi IX model in general relativity.

The next result of Section 2 concerns a series of averaging formulas of a different kind. Drawing on a lemma of P. Lévy, we will explain in Section 2 how to calculate averages (over $[0,1]$) of some functions of α defined by the sums over all pairs of consecutive convergents of α. Here we will state an interesting particular case, providing averages of weighted modular symbols.

Fix a prime number $N > 0$ and put $G_0 = \Gamma_0(N)$. We will assume that the genus of $X_{G_0} = X_0(N)$ is ≥ 1, otherwise our identities become trivial. Consider a $\Gamma_0(N)$-invariant differential $\varphi^*(\omega)$ on H which is a cusp eigenform for all Hecke operators and denote by $L_\omega^{(N)}(s)$ (resp. $\zeta^{(N)}(s)$) its Mellin transform (resp. Riemann's zeta) with omitted Euler N-factor. More precisely, the coefficients of $L_\omega^{(N)}(s)$ are Hecke eigenvalues of $\varphi^*(\omega)/dz$.

0.2.2. Theorem. *We have for* $\mathrm{Re}\, t > 0$*:*

$$\int_0^1 d\alpha \sum_{n=0}^{\infty} \frac{q_{n+1}(\alpha) + q_n(\alpha)}{q_{n+1}(\alpha)^{1+t}} \int_{\{0, q_n(\alpha)/q_{n+1}(\alpha)\}} \omega$$

$$= \left[\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{L_\omega^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2} \right] \int_0^{i\infty} \varphi^*(\omega). \tag{0.16}$$

Our calculation of averages of modular symbols like (0.16) and various generalizations in Section 2.1–2.2 point a way towards understanding what function theory on noncommutative modular curves may be used in order to recover the theory of modular forms on the upper half plane. In fact, (0.16) represents Mellin transforms of weight two cusp forms in terms of the quantities that can be defined entirely in terms of the *boundary* of the moduli space, and not the traditional integrals along geodesics: in fact, the integral in the left hand side of (0.16) is taken along the real axis.

For us this boundary is $\mathbf{P}^1(\mathbf{R})$, and not $\mathbf{P}^1(\mathbf{Q})$ as in the traditional algebro-geometric compactification, and it is exactly the consideration of this boundary that leads us into the land of noncommutative geometry: since modular symbols, Hecke operators and continuous fractions all can be expressed in terms of the (noncommutative) geometry of the boundary, so can our L-series as well. (The appearance of ω can be avoided since we could consider only eigenvalues of Hecke operators acting on the modular complex.)

0.3. Relations with noncommutative geometry

As is well-known, the quotient $PGL(2,\mathbf{Z})\setminus\mathbf{P}^1(\mathbf{R})$ can be identified with the space of classes of continued fractions modulo the equivalence relation "$k_{n+n_0}(\alpha) = k_{n+n_1}(\beta)$ *for some* n_0, n_1 *and all* n". Classical results on various averages like

$$\lim_{n\to\infty} n^{-1}\sum_{i=1}^{n} f(k_i(\alpha))$$

state that these averages are almost everywhere constant functions on this space.

On the other hand, this space and its finite coverings corresponding to subgroups $G\subset PGL(2,\mathbf{Z})$ constitute the boundary of the analytic moduli stack of elliptic curves whose irrational points are invisible in algebraic geometry (only cusps admit classical algebraic interpretation). According to the emerging general philosophy, this boundary is a bridge to the world of noncommutative geometry. In particular, the geometric objects parameterized by this boundary, which are two-dimensional noncommutative tori modulo Morita equivalence, can be treated as limiting elliptic curves. For some explanations, see [CoDS], [Soi], [Man4].

Accordingly, the boundary itself should be considered as (a tower of) "noncommutative modular curves" in Connes' spirit. The modular complex introduced in [Man1] and further studied in [Mer], [Gon] and other papers, provides a combinatorial definition of the homology of the modular tower. In Section 4 we show that essentially the same complex calculates K-theory of the crossed-products describing the noncommutative boundary modular tower.

This viewpoint presents in new light also the identity (0.16) and its generalizations considered in Section 2.1–2.2. Namely, it demonstrates that at least a part of the theory of modular forms in the upper half plane can be recast as the study of averages of certain functions defined on the boundary $\mathbf{R}$ as sums of the type (2.1). Their behavior with respect to fractional linear transformations is not modular in the traditional sense, but their expression via pairs of successive denominators can be seen as remnants of modularity. For another family of similar phenomena, see [Za] and [LawZa].

It is interesting to remark in this context that the Gauss–Kuzmin operator L_s vaguely looks like a "Hecke operator at the arithmetical infinity", and has some properties that might be expected of such an operator.

As a final remark, in [Man2] it was shown that after a choice of Schottky uniformization, the Arakelov geometry of a complex curve X at arithmetical infinity can be described in terms of the hyperbolic geometry of geodesics not on X itself, but rather in the hyperbolic handlebody having X as the boundary at infinity. It would be interesting to clarify the statistical aspects of the closed and infinite geodesics in the same vein as above and to relate them to Arakelov geometry. We hope to return to this question later.

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1. Gauss–Kuzmin operator

1.1. Operator L_s

Consider the operator L_s acting on functions of two variables (x, t) and formally given by

$$(L_s f)(x,t) := \sum_{k=1}^{\infty} \frac{1}{(x+k)^{2s}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} (t)\right). \tag{1.1}$$

The variable x here varies in a subset of $\mathbf{C}$ stable with respect to all maps $x \mapsto (x+k)^{-1}$, $k = 1, 2, \dots$. In our context this subset will always contain $[0,1]$. The variable t belongs to a finite $GL(2, \mathbf{Z})$-set $\mathbf{P}$ endowed with a base point t_0. The parameter s here is real and $> 1/2$. We are mostly interested in the case $s = 1$. In Section 3 we will allow s to take complex values. Finally, f will vary in a linear space of functions stable with respect to L_s and containing the function $m_0'(x,t) = \delta_{t,t_0}$ (cf. 0.1.1).

For the proof of Theorem 0.1.2, we want to create a functional analytic context in which the machinery of Krasnoselskii's theorem as stated in [May1], 7.25, becomes applicable. To this end we make the following choices, slightly generalizing Mayer. (They were also made in [ChMay]).

(i) *Definition domain.* It will be $\mathbf{D} \times \mathbf{P}$ where $\mathbf{D} := \{z \in \mathbf{C} \,|\, |z-1| < 3/2\}$. We will call the subsets $\mathbf{D} \times \{t\}$ *sheets.* Notice that each map $z \mapsto (z+k)^{-1}$ transforms $\mathbf{D}$ strictly into itself.

(ii) *Functional spaces.* We shall consider the complex Banach space $B_{\mathbf{C}} := V_{\mathbf{C}}(\mathbf{D} \times \mathbf{P})$ consisting of functions holomorphic on each sheet and continuous on its boundary. We shall also consider the real Banach space $B := V(\mathbf{D} \times \mathbf{P})$ of functions holomorphic on each sheet and continuous on its boundary, real at the real points of each sheet. Both spaces are endowed with the supremum norm. These spaces obviously contain δ_{t,t_0} and are stable with respect to L_s for real $s > 1/2$. The space $B_{\mathbf{C}}$ is also stable with respect to L_s with $\mathrm{Re}\, s > 1/2$.

(iii) *Positive cone.* Denote by $K \subset B$ the cone consisting of functions taking non-negative values at real points of each sheet. We have $K \cap -K = 0$ (K is proper) because a nonzero analytic function cannot vanish on an interval. We also have $B = K - K$ (K is reproducing) because $f = (f + r) - r$, and if r is large and positive, $f + r, r \in K$. Finally, functions positive at all real points of all sheets form the interior of K.

We write $f \leq g$ if $f - g \in K$.

(iv) L_s *is* K*-positive.* This means that $L_s(K) \subset K$ which is obvious.

1.1.1. Lemma. *Assume that* $\mathbf{P}$ *contains no proper invariant subsets with respect to the operators* Red *(see* (0.5)*).*

Then for each nonzero $f \in K$ *there exist two real positive constants* a, b *and an integer* $p \geq 1$ *such that* $a \leq L_s^p f \leq b$.

Proof. The upper bound is trivial. Assume that for some f and all p the lower bound is zero. This means that for each $p \geq 1$, $L_s^p f$ vanishes at some point (x_p, t_p) with x_p real in the closure of $\mathbf{D}$. Since in (1.1) all summands are non-negative at real points, when $f \in K$, we see that f must vanish at all points contained in the set $\cup_{p\geq 1}\mathrm{Red}_p(x_p, t_p)$.

From our assumption it follows that for some q and any $t \in \mathbf{P}$, $\mathrm{Red}_q(t) = \mathbf{P}$. By downward induction we deduce first that for any p, $L_s^p f$ has real zeroes on all sheets. Then, again by downward induction, one sees that for each t there exists a sequence of integers $q_n \to \infty$ and real points y_n in $\mathbf{D}$ such that $f(x, t)$ vanishes at all $x \in \cup_n \mathrm{Red}_{q_n}(y_n)$. However, the intersection of the latter set with $[0, 1]$ is dense in $[0, 1]$. A nonvanishing holomorphic function cannot have as many zeroes. This contradiction proves our assertion. □

1.1.2. Lemma. $L_s : B_{\mathbf{C}} \to B_{\mathbf{C}}$ *is a nuclear operator of order zero, in particular compact and trace class for* $\mathrm{Re}\, s > 1/2$.

Proof. The reasoning is the same as in [May1], and we only sketch it. Denote the k-th summand in (1.1) by $(\pi_{s,k}f)(x, t)$. Each $\pi_{s,k}$ is nuclear, and $\sum_k \|\pi_{s,k}\|$ converges for $\mathrm{Re}\, s > 1/2$. In fact, the spectrum of $\pi_{s,k}$ can be easily calculated. Let z_k be the unique fixed point of $\gamma_k := \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$ in $\mathbf{D}$, and $\mu_i^{(k)}$ the spectrum of the permutation induced by this matrix on $\mathbf{P}$. Then the spectrum of $\pi_{s,k}$ is $\{(-1)^n (z_k + k)^{-2(s+n)} \mu_i^{(k)}\}$, $n \geq 0$. □

1.2. Proof of the generalized Gauss–Kuzmin Theorem

We have now checked all the conditions for the applicability of the Theorem 7.25 of [May1]; see also [KraLS] for more details.

Using this theorem, we conclude that there exists exactly one eigenfunction of norm one f_s of L_s in the interior of K and its eigenvalue $\lambda_{0,s}$ is positive and simple. All other eigenvalues have strictly lesser modulus. For any $f \in B$ and $\epsilon > 0$, we have

$$L_s^n f = \text{const}\, \lambda_{0,s}^n\, f_s + O(c(\epsilon)(q+\epsilon)^n \lambda_{0,s}^n) \tag{1.2}$$

as $n \to \infty$, where $q = q(L_s) < 1$ is the spectral margin of L_s.

In the case $s = 1$ we know a positive eigenfunction: this is Gauss' density $\dfrac{1}{x+1}$, independent of t. The normalization constant is straightforward.

This argument completes the proof of the generalized Gauss–Kuzmin theorem. We will, however, provide some more details about the deduction of (1.2), because this technique can be useful also in the treatment of lower eigenvalues.

The basic result that ensures the existence of eigenfunctions in certain invariant cones is the following ([KraLS], Theorem 9.2):

If K is a cone in a real Banach space B satisfying $\overline{K-K} = B$ and L is a compact operator with $LK \subseteq K$, and with positive spectral radius $r(L)$, then $r(L)$ is an eigenvalue of L with a corresponding eigenfunction in K.

We also recall some results which enable us to establish that the top eigenvalue of an operator with an invariant cone is simple, and when the rest of the spectrum is of strictly smaller absolute value.

Following [May1], we say that an operator L is u-bounded, with respect to a function $u \in K$, if for any $f \in K$ there exists some $n > 0$ and $a, b > 0$ such that

$$au \le L^n f \le bu.$$

(Since we allow a power of L, this is a weaker definition of u-boundedness than the one on p. 110 of [KraLS], cf. their remark on p. 111.)

Lemma 1.1.1 shows that L_s is u-bounded with respect to the constant function $u(x,t) = 1$. The lower bound guarantees the positivity of the spectral radius $r(L_s)$ and hence the applicability of the Theorem 9.2 of [KraLS]. This fact follows from Lemma 9.2 of [KraLS].

We then have the following result ([KraLS], Theorem 11.1). Assume that the cone K is reproducing and the K-positive operator L is u-bounded. Assume moreover that L has an eigenvalue $\lambda_0 > 0$ with an eigenvector $f \in K$. Then the eigenvalue λ_0 is simple. A proof of this fact can be obtained using Lemma 11.1 of [KraLS]. This is actually a simple result of linear algebra which uses only the fact that f is an interior point of the cone (nowhere vanishing), and that iterates of the operator L map boundary points of the cone different from $\{0\}$ to interior points of the cone.

Furthermore, Theorem 11.4 of [KraLS] shows that with the same hypothesis as in the previous result, every eigenvalue λ of L different from λ_0 satisfies $|\lambda| < \lambda_0$.

This result follows from the observation that if the operator L is u-bounded, then it is also f-bounded, where again f is the eigenfunction in K with eigenvalue λ_0. Then, if h is an eigenfunction with eigenvalue λ the estimate

$$-\alpha(\lambda_0 - \epsilon)f \leq \lambda h \leq \alpha(\lambda_0 - \epsilon)f$$

for some $\epsilon > 0$ follows easily, where $\alpha > 0$ is the smallest positive number such that $-\alpha f \leq h \leq \alpha f$ is satisfied. This gives $|\lambda| \leq \lambda_0 - \epsilon$.

Since in our case we know that the operator L_s is compact, the previous result implies that all the other eigenvalues λ (hence all the points in the spectrum of L_s) satisfy the estimate $|\lambda| < q\lambda_0$, for some $q < 1$.

Finally, we have a result on the convergence of iterates; cf. Theorem 15.4 of [KraLS].

The cone K in our case contains some ball of positive radius. In this case, Theorem 9.11 on p. 97 of [KraLS] ensures that the adjoint operator L^* acting on the dual Banach space B' has an eigenfunctional f^* in the adjoint cone K^* of linear K-positive functionals, with eigenvalue $\lambda \leq r(L)$ where $r(L)$ is the spectral radius of L. In our case L_s has an eigenvalue $\lambda_0 = r(L_s)$ and a corresponding eigenfunction f which is an interior point of the cone K. Thus, if $L^* f^* = \lambda f^*$, for a non-trivial $f^* \in K^*$, and if moreover $f^*(f) > 0$, then $\lambda = \lambda_0$, because

$$\lambda_0 f^*(f) = f^*(\lambda_0 f) = f^*(Lf) = (L^* f^*)(f) = \lambda f^*(f).$$

Assume that the operator L has a simple eigenvalue equal to the spectral radius, $\lambda_0 = r(L)$, and the remaining part of the spectrum lies in the disk $|\lambda| < q\, r(L)$ for some $q < 1$. Let $f \in K$ be the eigenfunction of the eigenvalue λ_0. Let f^* be an eigenfunctional for L^* in K^*, with eigenvalue λ_0 as above, satisfying $f^*(f) = 1$. Then the sequence of iterates

$$f_{n+1} = L f_n$$

converges to the eigenfunction f in the following sense.

Define the operators $Uh := f^*(h)f$, and $U^\perp h := h - f^*(h)f$. We have

$$\lim_n \frac{\|U^\perp f_n\|}{\|U f_n\|} = 0$$

and the rate of convergence is estimated by

$$\frac{\|U^\perp f_n\|}{\|U f_n\|} \leq c(\epsilon)(q+\epsilon)^n \frac{\|U^\perp f_1\|}{\|U f_1\|},$$

for arbitrarily small $\epsilon > 0$. In other words, the iterates converge as fast as a geometric progression with ratio arbitrarily close to the spectral margin $q = q(L)$, cf. Section 15.2 of [KraLS], and in particular Theorem 15.3, where a more refined estimate of the coefficient $c(\epsilon)$ is also given.

Sometimes similar techniques may be applied to the study of the second eigenvalue, by supplying a suitable real Banach space with an invariant cone for the operator $h \mapsto L_s h - \lambda_{0,s} f^*(h) f$, cf. [May1].

As an example, we will now show that the condition of Lemma 1.1.1 holds for the congruence subgroup $\Gamma_0(N)$. For $N = 2$, the generalized Gauss–Kuzmin Theorem has a nice application to the dynamical system arising in the general relativity, the so called "Mixmaster Universe".

1.2.1. Proposition. *Let G be the subgroup generated by the lift of $\Gamma_0(N)$ and the sign change. Then $\mathrm{Red}_3(t) = \mathbf{P}$ for any t.*

Proof. In fact, elements of $\mathbf{P}$ can be thought of as points of the projective line over $\mathbf{Z}/N$, that is, formal quotients of residues mod N that can be represented by pairwise prime integers ([Man1]). Moreover, this encoding can be chosen compatible with the usual action of $GL(2)$ upon $\mathbf{P}^1$.

Let us break these points into three groups:

(I) $\{u/1 \,|\, (u,N) = 1\}$.

(II) $\{du/1 \,|\, d/N, d > 1, (u,N) = 1\}$.

(III) $\{1/du\} \,|\, d/N, d > 1, (u,N) = 1$.

Let us say that s can be obtained from t in one step, if $s \in \mathrm{Red}_1(t)$, that is, $s = (t+k)^{-1}$ for some k. The following statements are straightforward, and taken together, prove our claim.

From any single element of $\mathrm{I} \cup \mathrm{II}$ we can obtain in one step all elements of the set $\mathrm{I} \cup \mathrm{III}$.

From any single element of III one can obtain in one step an element of II, by adding zero and inverting. Hence from the total III one can obtain the total II in one step. □

1.2.2. Application to the Mixmaster Universe. "Mixmaster Universe" is defined as the space of solutions of the vacuum Einstein equations admitting $SO(3)$ symmetry of the space-like hypersurfaces (Bianchi IX model, see [Bo]) whose metric acquires a singularity as near $t \to +0$. The metric in appropriate coordinates takes the following form:

$$ds^2 = dt^2 - a(t)^2 dx^2 - b(t)^2 dy^2 - c(t)^2 dz^2.$$

The coefficients $a(t)$, $b(t)$, $c(t)$ are called scale factors.

A family of such metrics satisfying Einstein equations is given by *Kasner solutions*:

$$a(t) = t^{p_1},\ b(t) = t^{p_2},\ c(t) = t^{p_3};\ \sum p_i = \sum p_i^2 = 1.$$

Around 1970, V. Belinskii, I. M. Khalatnikov, E. M. Lifshitz and I. M. Lifshitz discovered that most of the trajectories in Mixmaster Universe exhibit a chaotic behavior as $t \to +0$: see [BeKhLi] and subsequent amplifications in [BoN], [KhLiKSS], [Bar], [May2]. Roughly speaking, the behaviour of a typical trajectory followed backwards in time (to the "Big Bang") is described in these papers in the following way.

Introduce the local logarithmic time Ω along this trajectory: $d\Omega := -\frac{dt}{abc}$. Then $\Omega \to +\infty$ approximately as $-\log t$ as $t \to +0$, and we have:

(i) The time evolution can be divided into "Kasner eras" $[\Omega_n, \Omega_{n+1}]$, $n \geq 1$.

(ii) Within each era, the evolution of a, b, c is approximately described by Kasner's formula, with variable p_i's which depend on an additional parameter u. If we arrange p_i in the increasing order, $p_1 < p_2 < p_3$, we have

$$p_1 = -\frac{u}{1+u+u^2}, \; p_2 = \frac{1+u}{1+u+u^2}, \; p_3 = \frac{u(1+u)}{1+u+u^2}$$

(see [KhLiKSS], formula (2.1).) The evolution starts with a certain value $u_n > 1$, and proceeds as u diminishes with growing Ω until u becomes less than 1. After a brief transitional period a new Kasner era starts, with the remarkable transition formula for the parameter u_{n+1} suggesting that continued fractions can be used to model the situation:

$$u_{n+1} = \frac{1}{u_n - [u_n]}.$$

(iii) The arrangement of exponents $p_i(u)$ of the scaling coefficients a, b, c in the increasing order induces generally a non-identical permutation of these coefficients. Moreover, during each era several such permutations (Kasner cycles) occur: as u diminishes by 1, the old permutation is multiplied by (12)(3) (see [KhLiKSS], formula (2.3).) When the era finishes, the permutation (1)(23) occurs (this is [KhLiKSS], formula (2.2).)

This means that during one era, the largest exponent decreases monotonically, and governs the same scale factor, a, b, or c which we will call *the leading one.* Two other exponents oscillate between the remaining pair of scaling coefficients. The number of oscillations is about $k_n := [u_n]$. Denote $x_n = u_n - k_n$.

Summarizing, we see that in this degree of approximation, the individual evolution of a typical trajectory is determined by a number $\alpha \in (0, 1)$ whose continued fraction $[k_1, k_2, k_3, \dots]$ determines the number of oscillations in each successive Kasner era. Of course, α is defined only up to a shift, because the initial point of the backward evolution can be chosen arbitrarily. Hence the relevant measure is the Gauss one.

If we want to keep track of the sequence of the leading scale factors as well, we should introduce a set **P**. We claim that in this case it corresponds to the lift of the group $\Gamma_0(2)$. In fact, consider the action of $GL(2, \mathbf{Z})$ upon $\mathbf{P}^1(\mathbf{F}_2) = \{1, 0, \infty\}$. Then the fractional linear transformation $u \mapsto 1/u$ corresponding to the transition to the new era, introduces the permutation (1)(23) of $\{1, 0, \infty\}$, whereas the passage to a new cycle within one era is described by the transformation $u \mapsto u - 1$ which produces the permutation (12)(3).

Hence the generalized Gauss–Kuzmin theorem leads in this case to the conclusion that during evolution along a typical trajectory (i.e., on the set of α of measure 1) each scale factor becomes the leading one in about one third of Kasner eras.

(iv) We have not yet connected the proper time Ω with the variable u. This is not directly relevant to our discussion, but we will do it for completeness, and because this connection can be beautifully rephrased in terms of "double-sided continued fractions", see [May2], formula (8), [Bar], formulas (43)–(45), and formula (4.32) below.

We start with a formula relating the endpoints of Ω_n and Ω_{n+1} of the n-th era with the initial value u_n inside this era. Namely, introduce one more parameter $\delta_n > 0$ characterizing the relative length of the era:

$$\Omega_{n+1} = (1 + \delta_n k_n (u_n + 1/x_n))\, \Omega_n.$$

If we put $\eta_n = (1 - \delta_n)/\delta_n$, we have the following recursion relation:

$$\eta_{n+1} x_n = \frac{1}{k_n + \eta_n x_{n-1}}.$$

This means that in terms of the variables $(x_n, y_n := \eta_{n+1} x_n)$ the transition to the next era is described by invertible double-sided shift operator

$$\tilde{T} : (x, y) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y + [1/x]} \right),$$

which is studied in [May2] and [KhLiKSS].

Having thus completed our discretized description of the evolution along an individual trajectory, we have to warn the reader that it refers, strictly speaking, to *another dynamical system* which is defined on the boundary of a certain compactification of the phase space of the Mixmaster Universe. This boundary whose construction involves a nontrivial real blow up at the $t = 0$ subspace was first constructed in [BoN]; see details in [Bo]. The boundary is an attractor, it supports an array of fixed points and separatrices, and the jumps between separatrices which result from subtle instabilities account for jumps between Kasner's regimes. In what sense this picture approximates the actual trajectories, is not quite trivial question: cf. the last three paragraphs of the Section 2 of [KhLiKSS], where it is explained that among these trajectories there can exist "anomalous" cases when the description in terms of Kasner eras does not make sense, but that they are, in a sense, infinitely rare.

1.3. The integral kernel operator

In this subsection, we consider the formal operator (1.1) in another functional space, and show that it admits there a representation with integral kernel, generalizing that of [Ba], [BaYu].

We will have to assume additionally that the $GL(2,\mathbf{Z})$-set $\mathbf{P}$ (see 1.1) is such that the action

$$t \mapsto \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}(t)$$

depends only on $k \bmod N$ for an appropriate integer N. This assumption is satisfied for instance when G is a congruence subgroup. In the following, we fix such N.

With this assumption, (1.1) can be written as

$$(L_s f)(z,t) = \sum_{p=-N+1}^{0} \sum_{k:\, k\geq 1,\, k\equiv p\,(N)} \frac{1}{(k+z)^{2s}} f\left(\frac{1}{k+z}, \begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}(t)\right). \tag{1.3}$$

Our new functional space $\mathbf{H}$ will consist of functions $f(z,t)$ holomorphic on the sheets $\{\operatorname{Re} z > -1/2\} \times t$.

We first recall a useful identity which is the essential ingredient in the arguments of [Ba], [BaYu], [May1], namely

$$\begin{gathered}\sum_{k\geq 1} \frac{1}{(k+z)^{2s}} \exp\left(\frac{-\xi}{k+z}\right) = \\ \xi^{-s+1/2} \int_0^\infty \eta^{s-1/2} e^{-\eta(z+1/2)} \frac{J_{2s-1}(2\sqrt{\xi\eta})}{2\sinh(\eta/2)} d\eta.\end{gathered} \tag{1.4}$$

(This is the formula (111) of [May1] rewritten in a way that looks more similar to the corresponding formula in [Ba], [BaYu].)

For our purpose, it is useful to consider also the following corollary of (1.4).

1.3.1. Lemma. *Let* $p \in \{-N+1, \dots, 0\}$. *We have*

$$\sum_{k:\, k\geq 1,\, k\equiv p\,(N)} \frac{1}{(k+z)^{2s}} \exp\left(-\frac{\xi}{k+z}\right)$$

$$= N^{2s-2} \xi^{-s+1/2} \int_0^\infty \eta^{s-1/2} \exp\left(-\eta\left(z+p+N/2\right)\right) \frac{J_{2s-1}(2\sqrt{\xi\eta})}{2\sinh\left(N\eta/2\right)} d\eta. \tag{1.5}$$

Proof. We have

$$\sum_{l\geq 1} \frac{1}{(p+lN+z)^2} \exp\left(-\frac{\xi}{p+lN+z}\right)$$

$$= \frac{1}{N^2} \sum_{l \geq 1} \frac{1}{(l+(z+p)/N)^2} \exp\left(-\frac{\xi/N}{l+(z+p)/N}.\right).$$

Plugging (1.4) in and redenoting $N\eta$ as the new η, we get (1.5).

It is convenient to write the functions $f(z,t)$ in the form

$$f(z,t) = \sum_{j=0}^{|\mathbf{P}|-1} f_j(z) \delta_j(t),$$

with respect to a basis of delta functions. We have

$$\delta_j \left(\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix} (t) \right) = \sum_l A_{jl}(p)\, \delta_l(t),$$

where $\mathbf{A}(p)$ is the matrix representing the permutation of the sheets.

With this notation, we can write L_s in the form

$$\begin{aligned}(L_s f)(z,t) &= \sum_{j=0}^{|\mathbf{P}|-1} \sum_{p=-N+1}^{0} \sum_{k:\, k \geq 1,\, k \equiv p\, (N)} f_j\left(\frac{1}{k+z}\right) \frac{1}{(k+z)^{2s}} \delta_j(\begin{pmatrix} 0 & 1 \\ 1 & p \end{pmatrix}(t)) \\ &= \sum_{j,l=0}^{|\mathbf{P}|-1} \sum_{p=-N+1}^{0} \sum_{k:\, k \geq 1,\, k \equiv p\, (N)} f_j\left(\frac{1}{k+z}\right) \frac{1}{(k+z)^{2s}} A_{jl}(p) \delta_l(t).\end{aligned}$$

We now introduce the following operators. Let $\mathcal{L}$ denote the Fourier–Laplace transform

$$(\mathcal{L}g)(z,t) := \int_0^\infty e^{-\xi z} g(\xi,t) d\xi.$$

We also define the multiplication operator

$$(Tg)(\xi,t) := e^{-\xi/2} g(\xi,t),$$

as in [Ba], [BaYu], and

$$(Sg)(\xi,t) := S(\xi)\, g(\xi,t),$$

with

$$S(\xi) = \frac{(1-e^{-N\xi})^{1/2}}{\xi^{-1/2+s}}.$$

The operator $\mathcal{L}T$ is an isometric isomorphism between the space $\mathcal{L}_2((0,\infty)\times\mathbf{P})$ and $\mathbf{H}$ endowed with the norm

$$\|f\|^2 := \frac{1}{2\pi} \sum_{j=0}^{|\mathbf{P}|-1} \int_{-\infty}^{\infty} |f_j(0+iy)|^2 dy.$$

Define the matrix function

$$\Theta_{jl}(\xi) := e^{-(N-1)\xi/2} \sum_{p=-N+1}^{0} e^{-\xi p} A_{jl}(p). \tag{1.6}$$

We can and will choose its square root with complex valued real analytic entries $\Theta_{jl}^{1/2}(\xi)$:

$$\Theta_{jl}(\xi) = \sum_{k=0}^{|\mathbf{P}|-1} \Theta_{jk}^{1/2}(\xi)\Theta_{kl}^{1/2}(\xi),$$

for all $\xi \in (0,\infty)$. We define

$$\hat{\Theta}_{jl}(\xi) := e^{-(N-1)\xi/4}\Theta_{jl}^{1/2}(\xi). \tag{1.7}$$

The results of the following Lemma and Proposition represent the analog in our setting of the main result of [Ba], see also 7.4.2 of [May1].

1.3.2. Lemma. *For a function $f(z,t) = \sum_j f_j(z)\,\delta_j(t)$ such that*

$$f_j = \mathcal{L}TS^{-1}h_j,$$

we can write the operator L_s in the form

$$(L_s f)(z,t) = \sum_{l=0}^{|\mathbf{P}|-1} \left(\sum_{j=0}^{|\mathbf{P}|-1} \int_0^\infty e^{-\eta z} S(\eta)^{-1} \Theta_{jl}(\eta) e^{-\eta/2} e^{-N\eta/4} \times \int_0^\infty \tilde{\kappa}(\xi,\eta) e^{N\xi/4} e^{-\xi/2} h_j(\xi) d\xi d\eta \right) \delta_l(t) \tag{1.8}$$

with a function $\tilde{\kappa}(\xi,\eta)$ satisfying $\tilde{\kappa}(\xi,\eta) = \tilde{\kappa}(\eta,\xi)$.

Proof. Assume that the functions $f_j(z)$, for $j = 0,\ldots,|\mathbf{P}|-1$, are in the range of the operator $\mathcal{L}T$, namely

$$f_j = \mathcal{L}Tg_j,$$

for some function g_j in $\mathcal{L}_2(0,\infty)$. This means that we can write

$$f_j(z) = \int_0^\infty e^{-\xi z} e^{-\xi/2} g_j(\xi) d\xi.$$

We now apply (1.5) and obtain

$$= N^{2s-2} \int_0^\infty \xi^{-s+1/2} \int_0^\infty \eta^{s-1/2} e^{-\eta(z+p+N/2)} \frac{J_{2s-1}(2\sqrt{\xi\eta})}{2\sinh(N\eta/2)} e^{-\xi/2} g_j(\xi) d\xi d\eta.$$

With our previous definition of $\Theta(\xi)$ as in (1.6) we can write

$$\sum_p \sum_{k:\, k\geq 1,\, k\equiv p\,(N)} f_j\left(\frac{1}{k+z}\right)\frac{1}{(k+z)^{2s}} A_{jl}(p)$$

$$= N^{2s-2}\int_0^\infty e^{-\eta z}\int_0^\infty (\frac{\eta}{\xi})^{1/2-s}\frac{J_{2s-1}(2\sqrt{\xi\eta})}{2\sinh(N\eta/2)} e^{-(\xi+\eta)/2}\Theta_{jl}(\eta) g_j(\xi) d\xi d\eta.$$

Now, for $g_j(\xi) = S(\xi)^{-1} h_j(\xi)$, this can be rewritten as

$$\int_0^\infty e^{-\eta z} S(\eta)^{-1}\Theta_{jl}(\eta)\int_0^\infty \frac{N^{2s-2} J_{2s-1}(2\sqrt{\xi\eta})}{\sqrt{2}\sinh(N\eta/2)^{1/2}} e^{-(\xi+\eta)/2} e^{-N\eta/4}\frac{g_j(\xi)}{\xi^{1/2-s}} d\xi\, d\eta$$

$$= \int_0^\infty e^{-\eta z} S(\eta)^{-1}\Theta_{jl}(\eta)\int_0^\infty \tilde{\kappa}(\xi,\eta) e^{-(\xi+\eta)/2} e^{-N\eta/4} e^{N\xi/4} h_j(\xi) d\xi\, d\eta,$$

where we have set

$$\tilde{\kappa}(\xi,\eta) := \frac{N^{2s-2} J_{2s-1}(2\sqrt{\xi\eta})}{2\sinh(N\eta/2)^{1/2}\sinh(N\xi/2)^{1/2}}.$$

Now the final step.

1.3.3. Proposition. *On the range of $\mathcal{R} := \mathcal{L} T S^{-1}\hat{\Theta}$, the operator L_s satisfies*

$$\mathcal{R}^{-1} L_s \mathcal{R} = \mathcal{M},$$

where $\mathcal{M}$ is the integral kernel operator

$$(\mathcal{M}\zeta)(\eta,t) = \sum_{i=0}^{|\mathbf{P}|-1}\int_0^\infty \sum_{j=0}^{|\mathbf{P}|-1} \mathcal{M}_{ij}(\eta,\xi)\zeta_j(\xi) d\xi\ \delta_i(t),$$

with $\zeta(\xi,t) = \sum_j \zeta_j(\xi)\delta_j(t)$. The integral kernel is of the form

$$\mathcal{M}_{ij}(\eta,\xi) = \kappa(\eta,\xi)\sum_r \hat{\Theta}_{jr}(\xi)\hat{\Theta}_{ri}(\eta),$$

where the function

$$\kappa(\xi,\eta) = e^{(N/4-1/2)(\xi+\eta)}\tilde{\kappa}(\xi,\eta)$$

still satisfies $\kappa(\xi,\eta) = \kappa(\eta,\xi)$, but the integral kernel is in general not symmetric.

Proof. For a function

$$f(z,t) = \sum_{j=0}^{|\mathbf{P}|-1} f_j(z)\delta_j(t) = \sum_{j=0}^{|\mathbf{P}|-1}\int_0^\infty e^{-\xi z} e^{-\xi/2} S(\xi)^{-1}\sum_{i=0}^{|\mathbf{P}|-1}\hat{\Theta}_{ij}(\xi)\zeta_i(\xi) d\xi\, \delta_j(t),$$

we have

$$(L_s f)(z,t) = \sum_{l=0}^{|\mathbf{P}|-1} (L_s f)_l(z)\delta_l(t),$$

with $(L_s f)_l(z)$ of the form

$$\sum_{j=0}^{|\mathbf{P}|-1} \int_0^\infty e^{-\eta z}\Theta_{jl}(\eta)e^{-\eta/2}S(\eta)^{-1}$$

$$\int_0^\infty e^{-N\eta/4}\tilde{\kappa}(\xi,\eta)e^{N\xi/4}e^{-\xi/2}\sum_{i=0}^{|\mathbf{P}|-1}\hat{\Theta}_{ij}(\xi)\zeta_i(\xi)d\xi d\eta.$$

We can write this equivalently as

$$\sum_{k,i=0}^{|\mathbf{P}|-1} \int_0^\infty e^{-\eta z}\hat{\Theta}_{kl}(\eta)e^{-\eta/2}S(\eta)^{-1}\int_0^\infty \mathcal{M}_{ik}(\eta,\xi)\zeta_i(\xi)d\xi d\eta,$$

where we define

$$\mathcal{M}_{ki}(\eta,\xi) := e^{(N/4-1/2)(\eta+\xi)}\tilde{\kappa}(\eta,\xi)\sum_{j=0}^{|\mathbf{P}|-1}\hat{\Theta}_{ij}(\xi)\hat{\Theta}_{jk}(\eta).$$

Thus, we have obtained

$$L_s\,\mathcal{L}TS^{-1}\hat{\Theta}\,\zeta = \mathcal{L}TS^{-1}\hat{\Theta}\,\mathcal{M}\,\zeta.$$

1.4. Remarks about the l-adic case

Let l be a prime number, $\mathbf{Z}_l$ (resp. $\mathbf{Q}_l$) the ring of l-adic integers (resp. the field of all l-adic numbers). Put also

$$\overline{\mathcal{Z}}_l := \left\{\frac{n}{l^r}\,|\,n,r\in\mathbf{Z}, r\geq 0\right\}\cap(0,l)\subset\mathbf{Q},\ \ \mathcal{Z}_l := \overline{\mathcal{Z}}_l\setminus\{1,2,\ldots,l-1\}. \qquad (1.9)$$

Every irrational $\alpha\in\mathbf{Q}_l$ has a unique representation in the form

$$\alpha = \frac{a_{-r}}{l^r}+\cdots+\frac{a_{-1}}{l}+a_0+a_1 l+\cdots+a_s l^s+\ldots,\ \ a_i\in\{0,1,\ldots,l-1\}.$$

The l-adic norm of this α is $|\alpha|_l = l^r$ if $a_{-r}\neq 0$. We then put

$$[\alpha]_l := \frac{a_{-r}}{l^r}+\cdots+\frac{a_{-1}}{l}+a_0\in\overline{\mathcal{Z}}_l. \qquad (1.10)$$

In more invariant terms, $[\alpha]_l$ is the unique element in $\overline{\mathcal{Z}}_l$ such that $|\alpha - [\alpha]_l| < 1$.

Repeating the usual reasoning, one sees that each irrational l-adic α with $|\alpha|_l < 1$ determines a unique sequence of $k_n(\alpha) \in \mathcal{Z}_l$ and $x_n(\alpha) \in l\mathbf{Z}_l \setminus \{0\}$ such that

$$\alpha = [\, k_1(\alpha), \dots, k_{n-1}(\alpha), k_n(\alpha) + x_n(\alpha)\,] \tag{1.11}$$

for each $n \geq 1$. We get thus the formalism of the theory of continued fractions in the l-adic setting, in which $\mathbf{R}, \mathbf{Z}, [0,1)$ are replaced respectively by $\mathbf{Q}_l, \overline{\mathcal{Z}_l}, l\mathbf{Z}_l$. Notice that the successive convergents are still rational numbers, but the incomplete quotients $k_n(\alpha)$ generally are not integral.

It will be convenient to restrict ourselves to irrational α in $\mathbf{Q}_l^* \setminus \mathbf{Z}_l^*$. In this case all $k_n(\alpha)$ will belong to $\mathcal{Z}_l$.

The shift operator is given by $T: \alpha \mapsto \alpha^{-1} - [\alpha^{-1}]_l$, and it transforms $\mathbf{Q}_l^* \setminus \mathbf{Z}_l^*$ into itself.

The definition of a (deformed) transfer operator, however, presents interesting new problems. We can consider two basic options.

(A) We can try to define the formal transfer operator by the classical formula

$$(L_s f)(x) := \sum_{k \in \mathcal{Z}_l} \frac{1}{(x+k)^{2s}} f\left(\frac{1}{x+k}\right) \tag{1.12}$$

in which $\mathbf{Z}$ is replaced by $\mathcal{Z}_l$.

We could have included a second argument t, but did not do it in order to focus on the peculiarity of (1.12) apparent already in this straightforward version of the classical setting. Namely, we should not imagine f as a function taking real or complex values. Otherwise, the term $\dfrac{1}{(x+k)^{2s}}$ would only be defined at rational points x and would not tend to zero as k runs over $\mathcal{Z}_l$. In fact, from the archimedean viewpoint, $\mathcal{Z}_l$ is a dense set inside $[0, l)$, so that (1.12) would tend to diverge, unless f is highly discontinuous and tends to zero when the denominator of the argument tends to infinity. However, this last property would be lost after an application of L_s.

However, $\mathcal{Z}_l$ is discrete and unbounded in the l-adic sense, so that (1.12) still makes sense as an operator in various l-adic function spaces. For example, one can consider the space of analytic functions on $l\mathbf{Z}_l$ represented by convergent series $\sum_{n \geq 0} a_n x^n$.

This remark, and parallels with the theories of l-adic uniformization and Drinfeld modules, suggest the following problems.

(i) Find a natural Banach space of l-adic functions in which (1.12) would define a compact operator.

Since compact operators are nuclear in the l-adic theory, this would allow us to define l-adic Selberg's zeta values at integral points $2s > 0$ as $\det(1 - L_{2s})$.

(ii) Define the set of l-adic reduced matrices by the same prescription as (0.5), but this time with k_i running over $\mathcal{Z}_l$.

Can one find a characterization of this set similar to that given in [LewZa1] and reproduced in 0.1? Assuming we know an l-adic zeta, can one find an Euler product for it similar to (3.1) below?

(iii) Again assuming a positive answer to the first question, is there an eigenfunction with eigenvalue 1 of L_1? Can one find its measure-theoretic interpretation in terms of Mazur's theory of l-adic integration? (See [Man5], Section 8 and 9).

At this point, we may notice that the passage from T to L_1 in the classical case implicitly involves integration with respect to the additively invariant measure, since these operators are adjoint via the obvious bilinear form determined by this integration, cf. formula (2.19) below.

It is well known that this invariant measure becomes inadequate for integrating l-adic valued functions, for the simple reason that the smaller is, say, an l-adic ball, the larger is its invariant measure l-adically.

Instead, the l-adic integration invented by B. Mazur for treating l-adic L-functions, utilizes finitely additive functions on open/closed subsets of, say, $\mathbf{Z}_l$ which take values in bounded subsets of finite-dimensional l-adic spaces. Such measures produce linear functionals on the spaces of functions satisfying the Lipschitz condition. See [Man5], Section 8 and Section 9, where a more general class of measures of moderate growth is introduced and studied as well.

Hence another option, perhaps more natural than the formal prescription (1.12) is:

(B) Define and study the transfer operators on various spaces of l-adic measures, as well as appropriate modifications of the questions (i)–(iii).

2. Calculation of certain averages

2.1. P. Lévy's lemma

Let f be a complex valued function defined on pairs of coprime integers (q, q') such that $q \geq q' \geq 1$ and $f(q, q') = O(q^{-\varepsilon})$ for some $\varepsilon > 0$. Put for $\alpha \in (0, 1]$

$$l(f, \alpha) = \sum_{n=1}^{\infty} f(q_n(\alpha), q_{n-1}(\alpha)). \tag{2.1}$$

2.1.1. Proposition (P. Lévy, 1929). *We have*

$$\int_0^1 l(f,\alpha)d\alpha = {\sum}' \, \frac{f(q,q')}{q(q+q')}. \tag{2.2}$$

Sums and integrals in (2.1), (2.2) *converge absolutely and uniformly.*

A notational convention: prime at the summation sign as in (2.2) refers to the domain $q \geq q' \geq 1$, $(q,q')=1$.

Proof. This proposition is an immediate consequence of the following statement.

For any $q \geq q' \geq 1$ with $(q,q')=1$ there exists a unique $n \geq 0$ such that one can find $\alpha \in (0,1]$ with $q_n(\alpha)=q$, $q_{n-1}(\alpha)=q'$. Moreover, all such α form a semi-interval of length $\dfrac{1}{q(q+q')}$.

In fact, assume that such an α and n exist, and let $p_n(\alpha)/q_n(\alpha)$, $p_{n-1}(\alpha)/$ $/q_{n-1}(\alpha)$ be the respective convergents to α; then we have

$$p_{n-1}(\alpha)\, q_n(\alpha) - p_n(\alpha)\, q_{n-1}(\alpha) = (-1)^n.$$

Together with the conditions $p_k(\alpha) \leq q_k(\alpha)$ this allows us to reconstruct n uniquely by induction and shows that all α with this property fill the semi-interval

$$\frac{p_{n-1}(\alpha)z + p_n(\alpha)}{q_{n-1}(\alpha)z + q_n(\alpha)}, \quad z \in (0,1].$$

(cf. (0.3)). Conversely, for any (q,q') we can start with complementing this line by $p \leq q, p' \leq q'$ to a reduced $(2,2)$-matrix with determinant ± 1, and then produce the continued fraction for p/q with neighboring convergents p/q, p'/q'. This proves the lemma. □

It is often convenient to have the summation domain in (2.2) extended to all $q \geq q' \geq 1$. One can do this by first extending f to this domain in the following way: choose a function $\kappa : \mathbf{N} \to \mathbf{Z}$ and a number t and put

$$F(q,q') := \kappa(d)\, d^{-t} f(q,q'),\ d := \text{g. c. d.}\,(q,q'). \tag{2.3}$$

Then

$$ {\sum}' \, \frac{f(q,q')}{q(q+q')} = \zeta(\kappa,t)^{-1} \sum_{q \geq q' \geq 1} \frac{F(q,q')}{q(q+q')}, \tag{2.4}$$

where $\zeta(\kappa,t) := \sum_{d\geq 1} \kappa(d)\, d^{-t}$. (This remark is also contained in [L].)

We will now combine this with the results of [Man1] in order to prove Theorem 0.2.2.

2.2. Averaging weighted modular symbols

In this subsection we keep the notation explained before the statement of the Theorem 0.2.2. In particular, modular symbols refer to the group $\Gamma_0(N)$. We start with the identity (20) in [Man1]:

$$\sum_{d/m}\sum_{b=1}^{d}\int_{\{0,b/d\}}\omega=(\sigma(m)-c_m)\int_0^{i\infty}\varphi^*(\omega). \tag{2.5}$$

Here $(m,N)=1$, $\phi^*(\omega)/dz$ is a cusp form for $\Gamma_0(N)$ with eigenvalue c_m with respect to the Hecke operator T_m, $\sigma(m)$ is the sum of the divisors of m.

Multiply this identity by m^{-2-t} and sum over all m prime to N:

$$\sum_{m\colon (m,N)=1}\frac{1}{m^{2+t}}\sum_{d/m}\sum_{b=1}^{d}\int_{\{0,b/d\}}\omega=$$

$$\left[\sum_{m\colon (m,N)=1}\frac{\sigma(m)}{m^{2+t}}-L_\omega^{(N)}(2+t)\right]\int_0^{i\infty}\varphi^*(\omega). \tag{2.6}$$

Any symbol $\{0,\frac{q'}{q}\}$, $(q,q')=1$, occurs in the guise $\{0,\frac{b}{d}\}$ for some d/m only when q divides m, and then exactly $\tau(mq^{-1})$ times where τ is the number of divisors. Hence the integration path in the left hand of (2.6) can be rewritten in the following way:

$$\sum_{m\colon (m,N)=1}\sum_{q/m}\frac{\tau(mq^{-1})}{m^{2+t}}\sum_{q'\le q,\,(q,q')=1}\{0,\frac{q'}{q}\}$$

$$=\sum_{n\colon (n,N)=1}\frac{\tau(n)}{n^{2+t}}\sum_{q\colon (q,N)=1}\frac{\sum_{q'\le q,\,(q,q')=1}\{0,\frac{q'}{q}\}}{q^{2+t}}$$

$$=\zeta^{(N)}(2+t)^2\left[\sum_{q\colon (q,N)=1}\frac{1}{q^{2+t}}\sum_{q'\le q,\,(q,q')=1}\{0,\frac{q'}{q}\}\right]. \tag{2.7}$$

Moreover, the first series inside the square brackets in (2.6) equals

$$\zeta^{(N)}(1+t)\,\zeta^{(N)}(2+t).$$

Hence (2.6) divided by $\zeta^{(N)}(2+t)^2$ can be rewritten as

$$\sum_{q\colon (q,N)=1}\frac{1}{q^{2+t}}\sum_{q'\le q,\,(q,q')=1}\int_{\{0,\frac{q'}{q}\}}\omega$$

$$= \left[\frac{\zeta^{(N)}(1+t)}{\zeta^{(N)}(2+t)} - \frac{L_\omega^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2} \right] \int_0^{i\infty} \varphi^*(\omega). \tag{2.8}$$

The left hand side of (2.8) can be represented as the right hand side of (2.2) with the function $f(q,q')$ which vanishes for q/N and otherwise equals

$$f(q,q') = \frac{q+q'}{q^{1+t}} \{0, \frac{q'}{q}\}. \tag{2.9}$$

Let us define a new function $\tilde{f}(q,q')$ by the same formula (2.9) *for all* relatively prime (q,q'). Since N is prime, we have $\{0, \frac{q'}{q}\} = \{0, i\infty\}$ for N/q. Therefore, writing (2.2) for this $\tilde{f}$, and integrating $\varphi^*(\omega)$, we get on the left hand side the same expression as on the left hand side of (0.13). The right hand side becomes the sum of the right hand side of (2.8) and

$$\sum_{d=1}^{\infty} \frac{\phi(Nd)}{(Nd)^{2+t}} \int_0^{i\infty} \varphi^*(\omega) = \left[\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{\zeta^{(N)}(1+t)}{\zeta^{(N)}(2+t)} \right] \int_0^{i\infty} \varphi^*(\omega) \tag{2.10}$$

where ϕ is the Euler function. This sum equals the right hand side of (0.16). This completes the proof. □

2.2.1. Comments and variations. The distribution of modular symbols was studied by D. Goldfeld ([Gol1], [Gol2]) who has found interesting connections between the conjectural asymptotic behavior of certain sums involving such symbols and other number-theoretical problems, e.g., the *abc*-conjecture. One of Goldfeld's conjectures reads:

$$\sum_{\substack{c^2M^2+d^2\le X \\ c\equiv 0\,(N)}} \{0, \frac{b}{d}\}_N \sim R(iM)\, X$$

as $X \to \infty$, where the sum is taken over matrices in $\Gamma_0(N)$, $R(iM) := \int_{iM}^{i\infty}$, and both sides are considered as functionals on the space of $\Gamma_0(N)$ cusp forms of weight two.

D. Goldfeld and C. O'Sullivan introduced a class of Eisenstein series twisted by modular symbols and established their analytic properties. The simplest series of this kind can be represented as the right hand side of (2.2) if one chooses for f the following function (depending on z, s as parameters):

$$\frac{f(q,q')}{q(q+q')} := \chi(q)\, \{0, \frac{q'}{q}\}_N \sum_{g \in A_{q,q'}} \mathrm{Im}\,(gz)^s$$

where $A_{q,q'}$ is the set of matrices in $\Gamma_0(N)$ with the second column $(q',q)^t$.

Here is another class of quite simple functions f that might produce interesting specializations of (2.2):

$$\frac{f(q,q')}{q(q+q')} := \frac{\chi_1(q)\,\chi_2(q')}{q^{s_1}q'^{s_2}}.$$

They lead to some identities involving double logarithms at roots of unity at the right hand side of (2.2). As Goncharov has shown in [Gon], relations between these numbers can be described in terms of the modular complex for $\Gamma_1(N)$. This stresses the relevance of the modular symbols in the study of the distribution of continued fractions.

Our last example is a function that was introduced in [AlZa]:

$$\frac{f(q,q')}{q(q+q')} := x^{\sum k_j(q/q')-1}\,q\log_2 q.$$

2.3. Proof of the Theorem 0.2.1

We now return to the notation and conventions explained in the paragraph around formula (0.13). In particular, β is real irrational. We start with proving that whenever the limit (0.13) exists, it does not depend on α (independence on $x \in H$ with fixed α is obvious).

We will compare the behavior of (0.13) for two geodesics Γ_1, Γ_2 ending at β. It suffices to consider the case when Γ_1 starts from $i\infty$, whereas Γ_2 starts from some real $\alpha < \beta$.

Denote by $p_n(\beta)/q_n(\beta) = p_n/q_n$ the convergents to β. If n is large enough and has the appropriate parity, the respective convergents will have the following positions on the real line:

$$\frac{\alpha+\beta}{2} < \frac{p_{n-1}}{q_{n-1}} < \beta < \frac{p_n}{q_n}\,. \tag{2.11}$$

Besides, we always have

$$\left|\frac{p_{n-1}}{q_{n-1}} - \beta\right| > \left|\frac{p_n}{q_n} - \beta\right|\,. \tag{2.12}$$

Let $[a,b]$ generally denote the geodesic joining a to b. Define two sequences of points in H by

$$z_n := \Gamma_1 \cap \left[\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}\right],\ \zeta_n := \Gamma_2 \cap \left[\frac{p_{n-1}}{q_{n-1}}, \frac{p_n}{q_n}\right]\,. \tag{2.13}$$

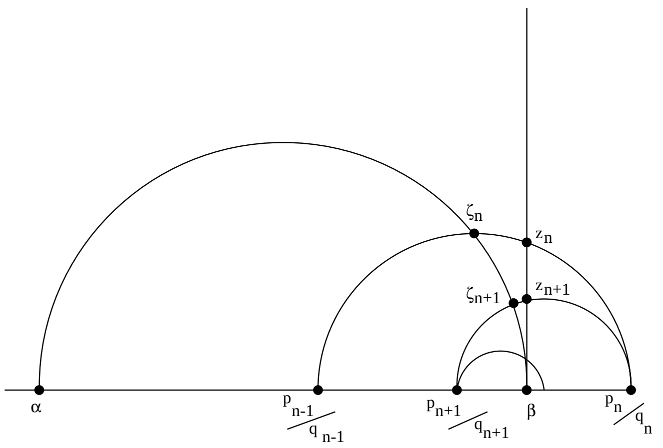
From (2.11)–(2.13) it follows that

$$\frac{1}{2q_nq_{n+1}} < \operatorname{Im} z_n < \frac{1}{2q_{n-1}q_n},$$

and if moreover n is large enough,

$$\frac{\theta}{2q_{n+2}q_{n+1}} < \theta \operatorname{Im} z_{n+1} < \operatorname{Im} \zeta_n \le \frac{1}{2q_{n-1}q_n}$$

where θ is some fixed constant between 0 and 1. The easiest way to convince oneself of this is to look at a picture containing all the relevant geodesics.



The geodesic distance from any fixed $x_1 \in \Gamma_1$ to $z \in \Gamma_1$ equals $-\log \operatorname{Im} z + O(1)$ as $z \to \beta$. The similar distance from a fixed $x_2 \in \Gamma_2$ to $\zeta \in \Gamma_2$ to β equals $-\log \operatorname{Im} \zeta + O(1)$ as $\zeta \to \beta$.

Taking into account our inequalities and the additivity of modular symbols, we obtain

$$\frac{1}{T(x_1, z_n)} \{x_1, z_n\} = \frac{1}{T(x_2, \zeta_n) + O(1)} [\{x_2, \zeta_n\} + O(1)]. \tag{2.15}$$

From this and (2.14) it follows that both limits exist or otherwise simultaneously, and have a common value whenever they both exist.

Moreover, according to the Khintchin–Lévy theorem we have for almost all β

$$\log q_n = Cn(1 + o(1)),\ C = \frac{\pi^2}{12 \log 2} \tag{2.16}$$

as $n \to \infty$. Hence for almost all β we can replace the limit (0.13) by

$$\lim_{n\to\infty} \frac{1}{2Cn} \{i\infty, z_n\} = \lim_{n\to\infty} \frac{1}{2Cn} \sum_{i=1}^{n} \left\{ \frac{p_{i-1}(\beta)}{q_{i-1}(\beta)}, \frac{p_i(\beta)}{q_i(\beta)} \right\}. \tag{2.17}$$

Temporarily fixing n, we will consider the sum of modular symbols on the right hand side of (2.17) as a function of β, and then prove that the resulting sequence of functions weakly converges to zero in the $\mathcal{L}_2$-sense. For this, we need the following lemma.

2.3.1. Lemma. *Let $\mathbf{P}_0$ be a finite left $GL(2,\mathbf{Z})$-set such that $\mathrm{Red}^{-1}(t) = \mathbf{P}_0$ for each $t \in \mathbf{P}_0$. Let $\varphi : \mathbf{P}_0 \to H$ be a function with values in an $\mathbf{R}$-vector space, $t_0 \in \mathbf{P}_0$. Then we have*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} \varphi(g_i(x)^{-1}\, t_0) = \frac{1}{|\mathbf{P}_0|} \sum_{s\in\mathbf{P}_0} \varphi(s), \tag{2.18}$$

where the limit is taken in the sense of weak convergence in $\mathcal{L}_2([0,1]\times\mathbf{P}_0)$, and

$$g_k(x) = \begin{pmatrix} p_{k-1}(x) & p_k(x) \\ q_{k-1}(x) & q_k(x) \end{pmatrix}.$$

Proof. Consider the shift

$$T(x,t) = \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} (t) \right).$$

We denote by λ the measure on $[0,1]\times\mathbf{P_0}$ given by the standard Lebesgue measure on $[0,1]$ and the counting measure on $\mathbf{P_0}$.

The Gauss–Kuzmin operator $L = L_1$ that we discussed in Section 1 is the adjoint of this shift T, in the sense that, for any function $h \in \mathcal{L}_1([0,1]\times\mathbf{P_0},\lambda)$ and any $f \in B_{\mathbf{C}}$ we have

$$\int_{[0,1]\times\mathbf{P_0}} f \cdot Lh\, d\lambda = \int_{[0,1]\times\mathbf{P_0}} (f\circ T)\, h\, d\lambda. \tag{2.19}$$

The eigenfunctional of of L^* denoted $f^* \in K^*$ in Section 1.2 can be taken as

$$h \mapsto \int_{[0,1]\times\mathbf{P_0}} h(x,t) d\lambda(x,t).$$

From 1.2 it follows that for any $h \in B_{\mathbf{C}}$ we have strong convergence

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} (L^k h)(x,t) = \frac{1}{|\mathbf{P}_0|\log 2}\, \frac{1}{1+x} \int_{[0,1]\times\mathbf{P_0}} h\, d\lambda. \tag{2.20}$$

According to (2.19), this is equivalent to the convergence

$$\int_{[0,1]\times\mathbf{P_0}} \frac{1}{n} \sum_{k=1}^{n} f(T^k(x,t)) h(x,t) d\lambda(x,t) \to$$

$$\frac{1}{|\mathbf{P}_0|\log 2}\left(\int_{[0,1]\times\mathbf{P_0}} \frac{f(x,t)}{1+x} d\lambda(x,t)\right)\int_{[0,1]\times\mathbf{P_0}} h\, d\lambda,$$

for any $f \in B_{\mathbf{C}}$ and any test function $h \in B_{\mathbf{C}}$. If we consider a function $f(x,t) = \varphi(t)$, independent of $x \in [0,1]$, we obtain that for any t

$$\frac{1}{n}\sum_{k=1}^{n}\varphi(g_k(x)^{-1}t) \to \frac{1}{|\mathbf{P_0}|}\sum_{s\in\mathbf{P_0}}\varphi(s)$$

weakly in $\mathcal{L}_2$, because among the test functions we have all polynomials.

This is equivalent to (2.18) for $\mathbf{C}$-valued functions and therefore also for vector-valued ones.

We can now conclude the proof of the Theorem 0.2.1. If our modular curve is $G_0 \setminus \overline{H}$, we put $\mathbf{P}_0 = PSL(2,\mathbf{Z})/G_0$ and consider $\mathbf{P}_0$ as a left $GL(2,\mathbf{Z})$-set as explained in the Introduction.

Since modular symbols are left G_0-invariant, we can find a function φ and $t_0 \in \mathbf{P}_0$ such that

$$\varphi(g_k(\beta)^{-1}\, t_0) = \{g_k(\beta)(0), g_k(\beta)(i\infty)\} = \left\{\frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)}\right\}.$$

It follows from (2.18) that the weak limit (2.17) is

$$\frac{1}{2C\,|\mathbf{P}_0|}\sum_k \{h_k(0), h_k(i\infty)\}$$

where h_k now run over a complete set of representatives of $\mathbf{P}_0$. But this last sum vanishes. In fact, let

$$\sigma = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then $\{h_k\sigma\}$ as well is a complete system of representatives, and

$$\{\sigma(0), \sigma(i\infty)\} = -\{0, i\infty\}.$$

Let us stress that the pointwise behavior of (2.17) might be wildly oscillating. We proved only that for any measurable set E we have

$$\lim_{n\to\infty}\frac{1}{n}\int_E \sum_{i=1}^{n}\left\{\frac{p_{i-1}(\beta)}{q_{i-1}(\beta)}, \frac{p_i(\beta)}{q_i(\beta)}\right\} d\beta = 0. \tag{2.21}$$

3. Selberg's zeta function

3.1. Notation

In this section we explain the definition of Selberg's zeta for subgroups of finite index $G \subset GL(2,\mathbf{Z})$ and $G_0 \subset SL(2,\mathbf{Z})$, their representation as Fredholm determinant, and relations to geodesics on modular curves. We closely follow [LewZa1], pp. 3–6, whose version requires only minor modifications. For a much more comprehensive treatment, see [ChMay].

As in [LewZa1], for $g \in GL(2,\mathbf{Z})$ put

$$D(g) = \mathrm{Tr}\,(g)^2 - 4\det(g),\ \ N(g) = \left(\frac{\mathrm{Tr}\,(g) + D(g)^{1/2}}{2}\right)^2,$$

and call g hyperbolic if $\mathrm{Tr}\,(g)$ and $D(g)$ are positive. A hyperbolic matrix is primitive if it is not a nontrivial power of an element of $GL(2,\mathbf{Z})$. For a hyperbolic g set

$$\chi_s(g) = \frac{N(g)^{-s}}{1 - \det(g)\,N(g)^{-1}}.$$

As above, put $\mathbf{P} := GL(2,\mathbf{Z})/G$ and denote by $\rho_{\mathbf{P}}$ the natural representation of $GL(2,\mathbf{Z})$ in the space of functions on $\mathbf{P}$. Finally, put

$$Z_G(s) := \prod_{g\in \mathrm{Prim}} \prod_{m=0}^{\infty} \det\,\left[1 - \det(g)^m\,N(g)^{-s-m}\,\rho_{\mathbf{P}}(g)\right] \qquad (3.1)$$

where Prim is a set of representatives of all $GL(2,\mathbf{Z})$-conjugacy classes of primitive hyperbolic elements of $GL(2,\mathbf{Z})$.

For $G_0 \subset SL(2,\mathbf{Z})$, we define $Z_{G_0}(s)$ in the same way, replacing Prim by Prim_0, a set of representatives of all $SL(2,\mathbf{Z})$-conjugacy classes of primitive hyperbolic elements of $SL(2,\mathbf{Z})$, and $\mathbf{P}$ by $\mathbf{P}_0$.

3.2. Theorem. *We have*

$$Z_G(s) = \det(1 - L_s),\ \ Z_{G_0}(s) = \det(1 - L_s^2) \qquad (3.2)$$

where L_s is given by (1.1) *and considered as a nuclear operator in the space $B_{\mathbf{C}}$.*

We give only a sketch of formal calculations for $Z_G(s)$. Using notation as in the proof of Lemma 1.1.2, we have

$$-\log\det(1 - L_s) = \sum_{l=1}^{\infty} \frac{\mathrm{Tr}\,L_s^l}{l} = \mathrm{Tr}\,\left(\sum_{l=1}^{\infty} \frac{1}{l}\left(\sum_{n=1}^{\infty} \pi_{s,n}\right)^l\right)$$

$$= \mathrm{Tr}\left(\sum_{g\in\mathrm{Red}} \frac{1}{l(g)}\pi_s(g)\right) = \sum_{g\in\mathrm{Hyp}} \frac{1}{k(g)}\,\chi_s(g)\,\tau_g. \tag{3.3}$$

Here we define the operator $\pi_s(g)$ for a reduced matrix g as in (0.5) as the product of the respective π_{s,k_i}, and $l(g)$ means its length. Hyp denotes a set of representatives of all conjugacy classes of hyperbolic matrices, and $k(g)$ the maximal integer such that $g = h^{k(g)}$. The last piece of notation is

$$\tau_g := \mathrm{Tr}\,(\rho_{\mathbf{P}}(g)) = \mathrm{card}\,\{t\in\mathbf{P}\,|\,g(t)=t\}. \tag{3.4}$$

The appearance of τ_g is explained by the fact that our $\pi_s(g)$ acts as the tensor product of the similar operator $\pi_s(g)$ for the case $G = GL(2,\mathbf{Z})$ and of $\rho_{\mathbf{P}}$, and our trace is the product of the respective traces.

Using the properties (i)–(iii) of Red summarized at the end of 0.1, we can keep rewriting (3.3):

$$\sum_{g\in\mathrm{Hyp}} \frac{1}{k(g)}\,\chi_s(g)\,\tau_g = \sum_{g\in\mathrm{Prim}}\sum_{k=1}^{\infty}\frac{1}{k}\,\frac{N(g)^{-ks}\,\tau_{g^k}}{1-\det(g)^k\,N(g)^{-k}}. \tag{3.5}$$

On the other hand, from (3.1) we find

$$-\log Z_G(s) = \sum_{g\in\mathrm{Prim}}\sum_{m=0}^{\infty}\mathrm{Tr}\sum_{k=1}^{\infty}\frac{1}{k}\det(g)^{mk}\,N(g)^{-(s+m)k}\,\rho_{\mathbf{P}}(g^k)$$

$$= \sum_{g\in\mathrm{Prim}}\sum_{k=1}^{\infty}\frac{1}{k}\frac{N(g)^{-ks}\,\tau_{g^k}}{1-\det(g)^k\,N(g)^{-k}} \tag{3.6}$$

This finishes the formal argument which differs from that of [LewZa1], Section 1, only by the presence of τ_g. The subsequent check of convergence in [LewZa1] and the argument of Section 3 concerning SL generalize in the same straightforward way.

Finally, in order to interpret (3.6) in the language of closed geodesics, it remains only to notice that if $G\subset GL(2,\mathbf{Z})$ is the lift of $G_0\subset PSL(2,\mathbf{Z})$ as in the Introduction, then $G_0\setminus H$ can be naturally identified with $GL(2,\mathbf{Z})\setminus(H\times\mathbf{P})$, and any closed geodesic on the respective modular curve is covered by the geodesics $[\alpha_g^-,\alpha_g^+]$ lying on those sheets which are left invariant by the respective hyperbolic matrix $g\in G$, in agreement with (3.4).

4. Noncommutative geometry and the modular complex

4.1. Noncommutative modular curves

As discussed in the Introduction, we want to regard the boundary $\mathbf{P}^1(\mathbf{R})$ with the action of $PSL(2,\mathbf{Z})$ as a moduli space of "noncommutative elliptic curves", where the quotient $PSL(2,\mathbf{Z})\backslash\mathbf{P}^1(\mathbf{R})$ is itself a noncommutative space. According to the general philosophy underlying noncommutative geometry, this is done by replacing the quotient with the crossed product of an algebra of functions on $\mathbf{P}^1(\mathbf{R})$ by the action of the group $PSL(2,\mathbf{Z})$. More generally, we can consider the quotients $G\backslash\mathbf{P}^1(\mathbf{R})$ as noncommutative spaces, where G is a finite index subgroup of $PSL(2,\mathbf{Z})$. The classical quotient

$$G\backslash(H \cup \mathbf{P}^1(\mathbf{Q})),$$

with H the upper half plane, is the modular curve $G\backslash H$ together with its algebro-geometric compactification by the set of cusps $G\backslash\mathbf{P}^1(\mathbf{Q})$. The quotient of the full $\mathbf{P}^1(\mathbf{R})$ can be regarded as that part of the analytic boundary which is invisible to the algebro-geometric compactification, and can be considered as a "noncommutative modular curve" when replaced by the crossed product. We can either consider the crossed product

$$C(\mathbf{P}^1(\mathbf{R})) \rtimes G \tag{4.1}$$

or, if $\mathbf{P}$ denotes the coset space $\mathbf{P} = PSL(2,\mathbf{Z})/G$, we can consider the (reduced) crossed product C^*-algebra

$$C(\mathbf{P}^1(\mathbf{R}) \times \mathbf{P}) \rtimes PSL(2,\mathbf{Z}). \tag{4.2}$$

The C^*-algebras (4.1) and (4.2) are strongly Morita equivalent.

For a discussion of some properties of crossed product C^*-algebras arising from the action of Fuchsian groups on their limit set; see, e.g., [An–De], [LaSp], [Spi].

We argued in the Introduction that the modular complex introduced in [Man1] and further studied in [Mer] provides a definition of cohomology of our boundary space compatible with passage to the limit. In this section we show that, in fact, the modular complex can be related to some standard homological constructions of noncommutative geometry for the noncommutative spaces (4.1) or (4.2).

4.2. Notation

In the following we denote by $\hat{X} = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}$, with $\mathbf{P}$ the coset space $\mathbf{P} = PSL(2,\mathbf{Z})/G$. Moreover, we have $PSL(2,\mathbf{Z}) = \mathbf{Z}/2 * \mathbf{Z}/3$, where we denote by

$$\sigma : x \mapsto -1/x \tag{4.3}$$

the generator of $\mathbf{Z}/2$ acting on $\mathbf{P}^1(\mathbf{R})$ and by

$$\tau : x \mapsto -1/(x-1) \tag{4.4}$$

the generator of $\mathbf{Z}/3$. This action is conjugate to the action on the unit circle by rotation by π or $2\pi/3$, respectively. Let $X_G = X_G(\mathbf{C})$ denote the modular curve

$$X_G = G\backslash\overline{H},$$

with

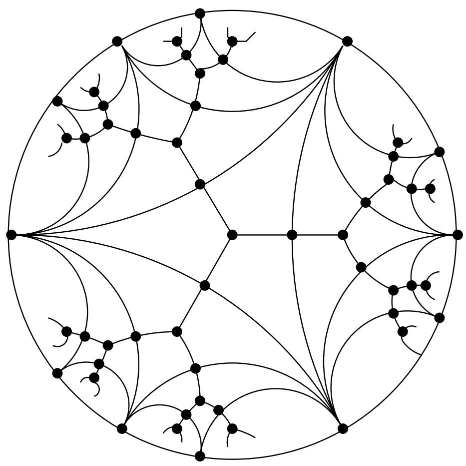
$$\overline{H} = H \cup \mathbf{P}^1(\mathbf{Q}).$$

We denote by $\tilde{I}$ and $\tilde{R}$ the elliptic points, namely the orbits $\tilde{I} = PSL(2,\mathbf{Z}) \cdot i$ and $\tilde{R} = PSL(2,\mathbf{Z}) \cdot \rho$. We denote by I and R the image in X_G of the elliptic points

$$I = G\backslash\tilde{I} \tag{4.5}$$

$$R = G\backslash\tilde{R}, \tag{4.6}$$

with $\rho = e^{\pi i/3}$. Finally, for x and y in $\overline{H}$ we denote by $\langle x, y\rangle$ the oriented geodesic arc connecting them.



4.3. Modular complex

We consider the following complex:

0–cells: the cusps $G\backslash\mathbf{P}^1(\mathbf{Q})$, and the elliptic points I and R.

1–cells: the oriented half-edges oriented from the parabolic to the elliptic point:

$$G\backslash\{\langle g(i\infty), gi\rangle, g\in PSL(2,\mathbf{Z})\}$$

and the edges

$$G\backslash\{\langle g(i), g(\rho)\rangle, g\in PSL(2,\mathbf{Z})\}.$$

2–cells: The images of $E=\{i,\rho,1+i,i\infty\}$,

$$G\backslash\{PSL(2,\mathbf{Z})\cdot E\}.$$

These cells correspond to the image under the projection $\pi:\overline{H}\to X_G$ of all the cells that appear in the figure, including the vertices on the boundary at infinity of the hyperbolic disk.

The boundary operators for this complex are given by

$$\partial: C_2\to C_1$$

$$gE\mapsto g\langle i,\rho\rangle+g\langle\rho,1+i\rangle+g\langle 1+i,i\infty\rangle+g\langle i\infty,i\rangle \tag{4.7}$$

and

$$\partial: C_1\to C_0$$

$$g\langle i\infty,i\rangle\mapsto g(i)-g(i\infty) \tag{4.8}$$

$$g\langle i,\rho\rangle\mapsto g(\rho)-g(i). \tag{4.9}$$

The arguments of [Man1] show that this complex computes the homology of X_G,

$$H_1(X_G)\cong\frac{\mathrm{Ker}(\partial: C_1\to C_0)}{\mathrm{Im}(\partial: C_2\to C_1)}. \tag{4.10}$$

4.3.1. Modular complex for relative homology. There are versions of the modular complex considered in [Mer], computing relative homology of X_G with respect to cusps and elliptic points. Here we consider two cases, which differ slightly from those considered in [Mer]. In the modular complex described before, we have $\mathbf{Z}[\text{cusps}] = C_0/\mathbf{Z}[R \cup I]$. The quotient complex

$$0 \to C_2 \xrightarrow{\partial} C_1 \xrightarrow{\tilde{\partial}} \mathbf{Z}[\text{cusps}] \to 0, \tag{4.11}$$

where $\tilde{\partial}$ is the quotient of the boundary operator of the original modular complex, computes the relative homology $H_1(X_G, R \cup I)$. The cycles are given by $\mathbf{Z}[\mathbf{P}]$, that is, combinations of elements $g\langle i, \rho\rangle$, g ranging over representatives of $\mathbf{P}$, and by the elements $\oplus a_g \langle g(i\infty), g(i)\rangle$ satisfying $\sum a_g g(i\infty) = 0$. In fact, these can be represented as relative cycles in $(X_G, R \cup I)$.

The subcomplex

$$0 \to \mathbf{Z}[\mathbf{P}] \xrightarrow{\partial} \mathbf{Z}[R \cup I] \to 0, \tag{4.12}$$

with $\mathbf{Z}[\mathbf{P}]$ generated by the elements $g\langle i, \rho\rangle$, computes the homology $H_1(X_G -$ cusps$)$, and the homology

$$H_1(X_G - \text{cusps}, R \cup I) \cong \mathbf{Z}[\mathbf{P}] \tag{4.13}$$

is generated by the relative cycles $g\langle i, \rho\rangle$.

For convenience of notation, we introduce the same notation used in [Mer] for the relative homology groups

$$H_A^B := H_1(X_G - A, B; \mathbf{Z}).$$

These groups are related by the pairing

$$H_A^B \times H_B^A \to \mathbf{Z}. \tag{4.14}$$

In particular, we consider the groups $H_{R\cup I}^{\text{cusps}}$, $H_{\text{cusps}}^{R\cup I}$, H^{cusps}, and H_{cusps}.

We consider an analog of Merel's exact sequences in this setting, given by the long exact sequence of relative homology

$$0 \to H_{\text{cusps}} \to H_{\text{cusps}}^{R\cup I} \xrightarrow{(\tilde{\beta}_R, \tilde{\beta}_I)} H_0(R) \oplus H_0(I) \to \mathbf{Z} \to 0, \tag{4.15}$$

with H_{cusps} and $H_{\text{cusps}}^{R\cup I}$ as above, and with

$$H_0(I) \cong \mathbf{Z}[\mathbf{P}_I], \qquad \mathbf{P}_I = \langle\sigma\rangle\backslash\mathbf{P} = G\backslash\tilde{I} \tag{4.16}$$

$$H_0(R) \cong \mathbf{Z}[\mathbf{P}_R], \qquad \mathbf{P}_R = \langle\tau\rangle\backslash\mathbf{P} = G\backslash\tilde{R}, \tag{4.17}$$

that is,

$$0 \to H_{\mathrm{cusps}} \to \mathbf{Z}[\mathbf{P}] \to \mathbf{Z}[\mathbf{P}_R] \oplus \mathbf{Z}[\mathbf{P}_I] \to \mathbf{Z} \to 0.$$

We want to compare modular symbols and the noncommutative topology of the boundary $G\backslash\mathbf{P}^1(\mathbf{R})$, in a way that is compatible with group restrictions $G' \subset G$.

In the case of $H^{R\cup I}_{\mathrm{cusps}}$ and $H^{\mathrm{cusps}}_{R\cup I}$ the pairing (4.14) gives the identification of $\mathbf{Z}[\mathbf{P}]$ and $\mathbf{Z}^{|\mathbf{P}|}$, obtained by identifying the elements of $\mathbf{P}$ with the corresponding delta functions. Thus, we can rewrite the sequence (4.15) as

$$0 \to H^{\mathrm{cusps}} \to \mathbf{Z}^{|\mathbf{P}|} \xrightarrow{(\beta_R,\beta_I)} \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} \to \mathbf{Z} \to 0, \tag{4.18}$$

with

$$H^{\mathrm{cusps}}_{R\cup I} \cong \mathbf{Z}^{|\mathbf{P}|}. \tag{4.19}$$

The map $(\tilde\beta_R, \tilde\beta_I)$ of the relative homology sequence (4.15) maps $s \mapsto ([s]_R, [s]_I)$, where $s \in \mathbf{P}$ corresponds to the generator $g\langle i, \rho\rangle$, for $g \in PSL(2,\mathbf{Z})$ the chosen representative of $s \in \mathbf{P}$, and $[s]_R \in \mathbf{P}_R = G\backslash\tilde R$ and $[s]_I \in \mathbf{P}_I = G\backslash\tilde I$ are the G-orbits of $g(\rho)$ and $g(i)$, respectively. The map (β_R, β_I) is given by $\delta_s \mapsto \delta_{[s]_R} \oplus \delta_{[s]_I}$.

4.3.2. Algebraic version. We recall the algebraic formulation of the modular complex computing $H^{\mathrm{cusps}} = H_1(X_G, \mathrm{cusps})$, following [Man1] Section 1.8 (a).

We consider the set of generators δ_s with $s \in \mathbf{P}$, given by the modular symbols $\{g(0), g(i\infty)\}_G$, with g in the chosen set of representatives of the cosets $\mathbf{P}$. The relations given by the 2-cells can be described as follows. Consider the subgroup C of $\mathbf{Z}^{|\mathbf{P}|}$ with generators δ_s and relations $\delta_s \oplus \delta_{\sigma s}$ or δ_s if $s = \sigma s$. Then the homology group H^{cusps} can be identified with the quotient of C by the subgroup generated by $\delta_s \oplus \delta_{\tau s} \oplus \delta_{\tau^2 s}$, or δ_s if $s = \tau s$. This follows from the arguments of [Man1] Section 1.8 (a).

In order to relate this description to the sequence (4.18), consider first the homology group (4.13), $H^{R\cup I}_{\mathrm{cusps}} = \mathbf{Z}[\mathbf{P}]$. This is generated by the images in X_G of the geodesic segments $g\gamma_0 := g\langle i, \rho\rangle$, with g ranging over the chosen representatives of the coset space $\mathbf{P}$.

Following [Mer], we can identify the dual basis δ_s of $H^{\mathrm{cusps}}_{R\cup I} = \mathbf{Z}^{|\mathbf{P}|}$ with the images in X_G of the paths $g\eta_0$, where for a chosen point z_0 with $0 < Re(z_0) < 1/2$ and $|z_0| > 1$ the path η_0 is given by the geodesic arcs connecting ∞ to z_0, z_0 to τz_0, and τz_0 to 0. These satisfy

$$[g\gamma_0] \bullet [g\eta_0] = 1$$

$$[g\gamma_0] \bullet [h\eta_0] = 0,$$

for $gG \neq hG$, under the intersection pairing (4.14).

The identification of H^{cusps}, given in terms of generators and relations as above, with $\operatorname{Ker}(\beta_R, \beta_I)$ in the sequence (4.18) is obtained by the identification

$$\{g(0), g(i\infty)\}_G \mapsto g\eta_0,$$

so that the relations imposed on the generators δ_s by the vanishing under β_I correpond precisely to the relations $\delta_s \oplus \delta_{\sigma s}$ (or δ_s if $s = \sigma s$) and the vanishing under β_R gives the other set of relations $\delta_s \oplus \delta_{\tau s} \oplus \delta_{\tau^2 s}$ (or δ_s if $s = \tau s$).

We shall use this algebraic formulation in the following, when we relate the group H^{cusps} to the noncommutative topology of $G\backslash \mathbf{P}^1(\mathbf{R})$.

4.4. Pimsner exact sequence

We consider the reduced crossed product C^*-algebra (4.2). We recall the setting of Pimsner [Pim] (cf. [LaSp] for the case of $C(\mathbf{P}^1(\mathbf{R})) \rtimes PSL(2, \mathbf{Z})$). With the notation introduced above, we consider

$$\Gamma = PSL(2, \mathbf{Z}) = \mathbf{Z}/2 * \mathbf{Z}/3$$

acting on a tree T with set of edges $T^1 = \Gamma$ and set of vertices T^0 given by the cosets Γ/Γ_0 and Γ/Γ_1, where $\Gamma_0 = \mathbf{Z}/2$ and $\Gamma_1 = \mathbf{Z}/3$. This tree T can be realized as a graph in the 2-dimensional hyperbolic space H, where the vertices are the elliptic points $\Gamma \cdot i$ and $\Gamma \cdot \rho$ and the edges are the geodesic segments $\Gamma \cdot \langle i, \rho \rangle$, represented by bold lines in the figure, giving rise to the subcomplex (4.12) of the modular complex.

Associated to this action on a tree, there is a six term exact sequence [Pim]:

$$\begin{array}{ccccc} K_0(C(\hat{X})) & \xrightarrow{\alpha} & K_0(C(\hat{X}) \rtimes \Gamma_0) \oplus K_0(C(\hat{X}) \rtimes \Gamma_1) & \xrightarrow{\tilde{\alpha}} & K_0(C(\hat{X}) \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(C(\hat{X}) \rtimes \Gamma) & \xleftarrow{\tilde{\beta}} & K_1(C(\hat{X}) \rtimes \Gamma_0) \oplus K_1(C(\hat{X}) \rtimes \Gamma_1) & \xleftarrow{\beta} & K_1(C(\hat{X})) \end{array} \tag{4.20}$$

We prove the following result that relates the six term exact sequence (4.20) to the modular complex.

4.4.1. Theorem. *There exists a natural isomorphism of the four terms exact sequence*

$$0 \to \operatorname{Ker}(\beta) \hookrightarrow K_1(C(\hat{X})) \xrightarrow{\beta} K_1(C(\hat{X}) \rtimes \Gamma_0) \oplus K_1(C(\hat{X}) \rtimes \Gamma_1) \to \operatorname{Im}(\tilde{\beta}) \to 0$$

and the exact sequence (4.18),

$$0 \to H^{\text{cusps}} \to H^{\text{cusps}}_{R\cup I} \to \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} \to \mathbf{Z} \to 0.$$

These isomorphisms are compatible with the restriction of the group $G' \subset G$. Moreover, the identification $\mathrm{Ker}(\beta) \cong H^{\mathrm{cusps}}$ *is given via the algebraic formulation of Section* 4.3.2*.*

Proof. First recall that we have natural identifications

$$K_0(C(\mathbf{P}^1(\mathbf{R}))) \cong \mathbf{Z} \qquad K_1(C(\mathbf{P}^1(\mathbf{R}))) \cong \mathbf{Z},$$

given, respectively, by the rank of projections and by the winding number of the determinant of elements in $GL_n(C(S^1))$.

Moreover, for the finite groups Γ_j, there are canonical isomorphisms,

$$K^0_{\Gamma_j}(\hat{X}) \cong K_0(C(\hat{X}) \rtimes \Gamma_j$$

given by

$$[E] \mapsto [\Gamma(E)],$$

with E a G-vector bundles and $\Gamma(E)$ its space of continuous sections. This gives natural identifications

$$K_i(C(\hat{X})) \cong \mathbf{Z}^{|\mathbf{P}|} \quad K_i(C(\hat{X}) \rtimes \mathbf{Z}/2) \cong \mathbf{Z}^{|\mathbf{P}_I|} \quad K_i(C(\hat{X}) \rtimes \mathbf{Z}/3) \cong \mathbf{Z}^{|\mathbf{P}_R|}. \tag{4.21}$$

Thus, we obtain from (4.20) and (4.21)

$$\begin{array}{ccccc} \mathbf{Z}^{|\mathbf{P}|} & \xrightarrow{\alpha} & \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} & \xrightarrow{\tilde{\alpha}} & K_0(C(\hat{X}) \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ K_1(C(\hat{X}) \rtimes \Gamma) & \xleftarrow[\tilde{\beta}]{} & \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} & \xleftarrow[\beta]{} & \mathbf{Z}^{|\mathbf{P}|} \end{array} \tag{4.22}$$

The maps in this sequence are defined as in [Pim] Section 1, and they depend on a choice of fundamental domain for the action of Γ on the tree T, which, in our case, is given by the edge $\langle i, \rho \rangle$ in T and the vertices $\{i, \rho\}$.

We can split the six term exact sequence (4.22) as

$$0 \to \mathrm{Ker}(\alpha) \hookrightarrow \mathbf{Z}^{|\mathbf{P}|} \xrightarrow{\alpha} \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} \to \mathrm{Im}(\tilde{\alpha}) \to 0 \tag{4.23}$$

and

$$0 \to \mathrm{Ker}(\beta) \hookrightarrow \mathbf{Z}^{|\mathbf{P}|} \xrightarrow{\beta} \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} \to \mathrm{Im}(\tilde{\beta}) \to 0. \tag{4.24}$$

With the notation of [Pim] Section 4, the morphism β (or α) in the Pimsner exact sequence is induced by the maps

$$\beta_y : C(\hat{X}) \to C(\hat{X}) \rtimes \Gamma_{t(y)} \qquad \beta_y(a) = \gamma_{y^t}^{-1}(a)$$

and

$$\beta_{\bar{y}} : C(\hat{X}) \to C(\hat{X}) \rtimes \Gamma_{o(y)} \quad \beta_{\bar{y}}(a) = \gamma_{y^o}^{-1}(a).$$

Here we denote by y the edge $\langle i, \rho \rangle$ in the chosen fundamental domain for the action of Γ on the tree T, and $o(y) = i$, and $t(y) = \rho$ its source and terminus. The groups $\Gamma_{o(y)} = \mathbf{Z}/2$ and $\Gamma_{t(y)} = \mathbf{Z}/3$ are the stabilizers of these points. Also, here y^o and y^t denote the edges of T with $t(y^t), o(y^o) \in \{i, \rho\}$, and γ_{y^t} and γ_{y^o} are the elements of Γ that satisfy $\gamma_{y^t} y^t = y$ and $\gamma_{y^o} y^o = y$, as in [Pim] Section 1. The element γ_{y^t} acts on $C(\hat{X})$ as the element τ and γ_{y^o} acts as the element σ.

Thus, the morphism

$$\beta : K_1(C(\hat{X})) \to K_1(C(\hat{X}) \rtimes \mathbf{Z}/2) \oplus K_1(C(\hat{X}) \rtimes \mathbf{Z}/3$$

is precisely the map that sends the generator δ_s, identified with the homotopy class of determinant functions of elements in $GL_n(C(\hat{X}))$ with winding number one around the circle $S^1 \times \{s\}$ and zero around the circles $S^1 \times \{t\}$ for $t \neq s$, to the element $\delta_{[s]_I} \oplus \delta_{[s]_R}$.

We can identify of the maps β in (4.24) and (β_I, β_R) of (4.18).

This means that there is a natural identification

$$\mathrm{Ker}(\beta) \cong H^{\mathrm{cusps}} \tag{4.25}$$

obtained via the algebraic formulation of Section 4.3.2. In fact, the kernel of β in (4.24) can be identified with the subgroup of $\mathbf{Z}^{|\mathbf{P}|}$ of elements $\sum a_s \delta_s$ satisfying the relations $a_s + a_{\sigma s} = 0$ (or $a_s = 0$ if $s = \sigma s$) and $a_s + a_{\tau s} + a_{\tau^2 s} = 0$ (or $a_s = 0$ if $s = \tau s$).

The identification (4.25) is compatible with restrictions. In fact, suppose given another finite index subgroup G' of $PSL(2, \mathbf{Z})$, with $G' \subset G$. This gives a branched cover $X_{G'} \xrightarrow{\pi} X_G$, and a surjection $\mathbf{P}' \xrightarrow{\pi} \mathbf{P}$, with $\mathbf{P}' = PSL(2, \mathbf{Z})/G'$, and a corresponding map $C(\hat{X}) \to C(\hat{X}')$ given by composition with π. Since the action of τ or σ on $\mathbf{P}'$ and $\mathbf{P}$ commutes with π, the maps β_y and $\beta_{\bar{y}}$ are also compatible with restrictions to $G' \subset G$, and the induced morphism $\mathbf{Z}^{|\mathbf{P}|} \xrightarrow{\pi^*} \mathbf{Z}^{|\mathbf{P}'|}$ with

$$\begin{array}{ccc}
\mathbf{Z}^{|\mathbf{P}|} & \longrightarrow & \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} \\
\downarrow \pi^* & & \downarrow \pi^* \\
\mathbf{Z}^{|\mathbf{P}'|} & \longrightarrow & \mathbf{Z}^{|\mathbf{P}'_I|} \oplus \mathbf{Z}^{|\mathbf{P}'_R|}
\end{array}$$

is given by $\delta_s \mapsto \oplus_{t \in \pi^{-1}(s)} \delta_t$, where δ_s is the homotopy class of determinant functions of elements in $GL_n(C(\hat{X}))$ with winding number one around the circle

$S^1 \times \{s\}$, in

$$\begin{array}{ccc} K_1(C(\hat{X})) & \xrightarrow{\beta} & K_1(C(\hat{X}) \rtimes \mathbf{Z}/2) \oplus K_1(C(\hat{X}) \rtimes \mathbf{Z}/3) \\ \downarrow & & \downarrow \\ K_1(C(\hat{X}')) & \xrightarrow{\beta} & K_1(C(\hat{X}') \rtimes \mathbf{Z}/2) \oplus K_1(C(\hat{X}') \rtimes \mathbf{Z}/3) \end{array}$$

Thus, the identification (4.25) is compatible with restrictions $G' \subset G$, and the induced map $\pi^* : H^{\mathrm{cusps}} \to H^{\mathrm{cusps}'}$ in

$$\begin{array}{ccccccccccc} 0 & \to & H^{\mathrm{cusps}} & \to & \mathbf{Z}^{|\mathbf{P}|} & \xrightarrow{\beta} & \mathbf{Z}^{|\mathbf{P}_I|} \oplus \mathbf{Z}^{|\mathbf{P}_R|} & \to & \mathbf{Z} & \to & 0 \\ & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow \pi^* & & \downarrow & & \\ 0 & \to & H^{\mathrm{cusps}'} & \to & \mathbf{Z}^{|\mathbf{P}'|} & \xrightarrow{\beta'} & \mathbf{Z}^{|\mathbf{P}'_I|} \oplus \mathbf{Z}^{|\mathbf{P}'_R|} & \to & \mathbf{Z} & \to & 0 \end{array}$$

has the following description in terms of modular symbols:

$$\{g(0), g(i\infty)\}_G \mapsto \oplus_{t\in\pi^{-1}(s)} \{g'(0), g'(i\infty)\}_{G'},$$

with $gG = s \in \mathbf{P}$ and $g'G' = t \in \mathbf{P}'$. This map is the dual, under the intersection pairing (4.14), to the map $\pi_* : H_{\mathrm{cusps}'} \to H_{\mathrm{cusps}}$ defined by $g\langle i, \rho\rangle \mapsto g'\langle i, \rho\rangle$ with $g'G' = t \in \mathbf{P}'$ and $gG = \pi(t) \in \mathbf{P}$.

The sequence (4.23) differs from (4.24) by the presence of a torsion term $\mathcal{T}$ in $\mathrm{Im}(\tilde{\alpha})$. The map α, which is the map induced on K_0 by the action of σ and τ on $\hat{X}$, still satisfies $\mathrm{Ker}(\alpha) \cong H^{\mathrm{cusps}}$, but with different multiplicities, since the morphism for the case $\hat{X} = \mathbf{P}^1(\mathbf{R})$ is given by

$$K_0(C(\mathbf{P}^1(\mathbf{R}))) \xrightarrow{(2,3)} K_0(C(\mathbf{P}^1(\mathbf{R})) \rtimes \mathbf{Z}/2) \oplus K_0(C(\mathbf{P}^1(\mathbf{R})) \rtimes \mathbf{Z}/3).$$

The torsion term $\mathcal{T}$ in $\mathrm{Im}(\tilde{\alpha})$ depends on the elliptic elements of G. Namely, we have

$$\mathrm{Im}(\tilde{\alpha}) = \mathbf{Z} \oplus \mathcal{T} \cong \mathbf{Z}^2/\mathbf{Z}(\ell, 1) \oplus T(n_1, \ldots n_k),$$

where the group G has signature $(g; n_1, \ldots, n_k; q)$, with g the genus and q the number of cusps of X_G. Here $T(n_1, \ldots n_k)$ is the term computed in [An–De], and $\ell = l.c.m.(n_1, \ldots n_k)$.

Thus, from (4.23) and (4.24) we obtain identifications

$$K_1(C(\hat{X}) \rtimes \Gamma) \cong H^{\mathrm{cusps}} \oplus \mathbf{Z}, \tag{4.26}$$

$$K_0(C(\hat{X}) \rtimes \Gamma) \cong H^{\mathrm{cusps}} \oplus \mathbf{Z} \oplus \mathcal{T}, \tag{4.27}$$

following the identifications of $\mathrm{Ker}(\alpha)$, $\mathrm{Im}(\tilde{\beta})$, $\mathrm{Ker}(\beta)$, and $\mathrm{Im}(\tilde{\alpha})$ in (4.23) and (4.24).

4.5. Cyclic homology

Another way of relating the modular complex and homological constructions of noncommutative geometry is via cyclic homology. We consider the *algebraic* crossed product of the algebra of smooth functions on $\hat{X} = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}$ by the group $\Gamma = PSL(2, \mathbf{Z})$,

$$\mathcal{B} := C^\infty(\hat{X}) \rtimes \Gamma.$$

The $\mathbf{Z}/2$-graded periodic cyclic cohomology $PHC^*(\mathcal{B})$ is defined as the direct limit over $S : HC^{n-1}(\mathcal{B}) \to HC^{n+1}(\mathcal{B})$, where S is the morphism in Connes' exact sequence relating Hochschild and cyclic cohomology, [Co] III.1.γ. It is proved in [Nis] that there is a six terms exact sequence, analogous to the Pimsner sequence in K-theory, for the the periodic cyclic cohomology (or dually for the periodic cyclic homology) of the algebraic crossed product by a group acting on a tree. With the notation $A := C^\infty(\hat{X})$ we have

$$\begin{array}{ccccc} PHC_0(A) & \xrightarrow{\alpha} & PHC_0(A \rtimes \Gamma_0) \oplus PHC_0(A \rtimes \Gamma_1) & \xrightarrow{\tilde{\alpha}} & PHC_0(A \rtimes \Gamma) \\ \uparrow & & & & \downarrow \\ PHC_1(A \rtimes \Gamma) & \xleftarrow[\tilde{\beta}]{} & PHC_1(A \rtimes \Gamma_0) \oplus PHC_1(A \rtimes \Gamma_1) & \xleftarrow[\beta]{} & PHC_1(A) \end{array} \quad (4.28)$$

Again we can split this six-term exact sequence as

$$0 \to \mathrm{Ker}(\beta) \to PHC_1(A) \xrightarrow{\beta} PHC_1(A \rtimes \mathbf{Z}/2) \oplus PHC_1(A \rtimes \mathbf{Z}/3) \to \mathrm{Im}(\tilde{\beta}) \to 0$$

and

$$0 \to \mathrm{Ker}(\alpha) \to PHC_0(A) \xrightarrow{\alpha} PHC_0(A \rtimes \mathbf{Z}/2) \oplus PHC_0(A \rtimes \mathbf{Z}/3) \to \mathrm{Im}(\tilde{\alpha}) \to 0.$$

Argument analogous to the case of K-theory show that we have an identification of these sequences with

$$0 \to H^{\mathrm{cusps}} \to k^{|\mathbf{P}|} \to k^{|\mathbf{P}_I|} \oplus k^{|\mathbf{P}_R|} \to k \to 0,$$

where we consider homology with coefficients in a field $k = \mathbf{R}$ or $\mathbf{C}$, and a corresponding identification

$$PHC_0(\mathcal{B}) \cong PHC_1(\mathcal{B}) \cong H^{\mathrm{cusps}} \oplus k \quad (4.29)$$

The relation between the modular complex and the periodic cyclic homology of $\mathcal{B}$ can be derived also via the approach of [BN].

4.5.1. Groupoids and cyclic homology. We introduce the groupoid $\mathcal{G}_\Gamma$ for the action of Γ on $\hat{X}$. This is a Hausdorff locally compact étale groupoid where the morphisms are

$$\mathcal{G}_\Gamma = \hat{X} \times \Gamma,$$

the objects are $\mathcal{G}^0_\Gamma = \hat{X}$, and the source and target maps $o, t : \mathcal{G}_\Gamma \to \mathcal{G}^0_\Gamma$ are given by

$$o((x,s),\gamma) = (x,s) \qquad t((x,s),\gamma) = \gamma(x,s),$$

with composition

$$((x,s),\gamma)(\gamma(x,s),\gamma') = ((x,s),\gamma'\gamma).$$

The C^*-algebra $C^\infty_c(\mathcal{G}_\Gamma)$ of this groupoid is just the crossed product.

It is proved in [BN] that the periodic cyclic homology $PHC_*(C^\infty_c(\mathcal{G}_\Gamma))$ is obtained as a sum of components associated to the torsion conjugacy classes in Γ. With the notation of [BN] Corollary 5.9, these components are given by

$$e_{O_\gamma} PHC_n(C^\infty_c(\mathcal{G}_\Gamma)) = H_{n+N+2\mathbf{Z}}(\hat{X}^\gamma \times_\Gamma E\Gamma), \tag{4.30}$$

where $\hat{X}^\gamma$ is the fixed point set of γ acting on $\hat{X}$, $N = \dim \hat{X}^\gamma$, and $\hat{X}^\gamma \times_\Gamma E\Gamma$ is the homotopy quotient.

In our case, the only component that contributes in (4.30) is the one corresponding to the identity, and, since we consider homology with coefficients in the field k, we can replace the homotopy quotient by $\Gamma\backslash(\hat{X} \times H)$, with H the hyperbolic plane. Using the cell decomposition for $\mathbf{P} \times H$ given by the modular complex, compatible with the action of Γ, it is possible to define a cell complex computing the homology of the quotient $\Gamma\backslash(\hat{X} \times H)$, which recovers the identification (4.29) via (4.30).

4.6. Noncommutative geometry and the shift operator

There is another possible way of constructing a "noncommutative space" representing the action of a finite index subgroup $G \subset PGL(2,\mathbf{Z})$ on the projective line at infinity $\mathbf{P}^1(\mathbf{R})$, using the shift operator T acting on $[0,1] \times \mathbf{P}$,

$$T : (x,t) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x}\right], \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} (t) \right), \tag{4.31}$$

as introduced in Section 0.1.1. In fact, the set $X = [0,1] \times \mathbf{P}$ meets every orbit of the action of Γ on $\hat{X} = \mathbf{P}^1(\mathbf{R}) \times \mathbf{P}$, and two points (x,s) and (y,u) in X are equivalent under the action of Γ iff there exist positive integers m,n such that $T^m(x,s) = T^n(y,u)$. Thus, it makes sense to consider the action of T on X as a way of defining a noncommutative analog of the quotient $\hat{X}/\Gamma$. The shift T is locally invertible, and it determines a singly generated pseudogroup in the sense of [Ren], and it defines on $X = [0,1] \times \mathbf{P}$ an essentially free singly generated dynamical system, in the sense of Definitions 2.3 and 2.5 of [Ren]. There is an associated semidirect product groupoid with arrows

$$\mathcal{G}(X,T) = \{((x,t), m-n, (y,s)) | \, T^m(x,t) = T^n(y,s)\}$$

and objects

$$\mathcal{G}(X,T)^0 = X \cong \{((x,t), 0, (x,t)\} \subset \mathcal{G}(X,T).$$

The source and range maps and the multiplication are given by

$$p((x,t), m-n, (y,s)) = (x,t) \qquad q((x,t), m-n, (y,s)) = (y,s)$$

$$((x,t), m-n, (y,s)) \cdot ((x',t'), m'-n', (y',s')) = ((x,t), m+m'-(n+n'), (y',s')).$$

This is a Hausdorff locally compact étale groupoid, with the topology generated by the basis of open sets

$$\mathcal{G}(T)_{U,V} = \{((x,t), m-n, (y,s)) | \, (x,t) \in U, (y,t) \in V, \, T^m(x,t) = T^n(y,s)\},$$

with U and V open sets where T^m and T^n respectively are invertible. It is possible to construct a corresponding C^*-algebra $C^*(\mathcal{G}(X,T))$, which we may also regard as a noncommutative version of the "boundary" $\hat{X}/\Gamma$.

It is also interesting to consider the double-sided shift operator

$$\tilde{T} : [0,1] \times [0,1] \times \mathbf{P} \to [0,1] \times [0,1] \times \mathbf{P}$$

$$\tilde{T} : (x,y,t) \mapsto \left(\frac{1}{x} - \left[\frac{1}{x} \right], \frac{1}{y + \left[\frac{1}{x} \right]}, \begin{pmatrix} -[1/x] & 1 \\ 1 & 0 \end{pmatrix} (t) \right), \tag{4.32}$$

The shift (4.32) is related to the Poincaré return map of the geodesic flow on the modular curve X_G, and the one-sided shift (4.31) is the restriction to the expanding directions (cf. [ChMay] Section 3.3). The double-sided shift (4.32) is invertible and it defines by composition an automorphism of the algebra $C(\hat{Y})$, with $\hat{Y} = [0,1] \times [0,1] \times \mathbf{P}$. The crossed product $C(\hat{Y}) \rtimes_{\tilde{T}} \mathbf{Z}$ gives a natural way of replacing the set of equivalence classes under the action of T by a noncommutative space.

The invariants of the noncommutative geometry of $C(\hat{Y}) \rtimes_{\tilde{T}} \mathbf{Z}$ should therefore contain some information on the geodesic flow on the compactified modular curve X_G. For instance the Pimsner–Voiculescu exact sequence

$$0 \to K_1(C(\hat{Y}) \rtimes_{\tilde{T}} \mathbf{Z}) \to K_0(C(\hat{Y})) \xrightarrow{I - \tilde{T}_*} K_0(C(\hat{Y})) \to K_0(C(\hat{Y}) \rtimes_{\tilde{T}} \mathbf{Z}) \to 0$$

should be related to the properties of the action of Red on the coset space $\mathbf{P}$, hence to the dynamical properties of the geodesic flow.

4.6.1. Further remarks. There are other possible ways of introducing noncommutative geometry at the boundary of the modular curves. For instance, if we consider the disconnection of $\mathbf{P}^1(\mathbf{R})$ at all the fixed points of the parabolic elements of G, as defined in Section 2 of [Spi], we obtain a totally disconnected compact Hausdorff space Σ_G. By the results of [Spi], the crossed product C^*-algebra $C(\Sigma_G) \rtimes G$ is isomorphic to a Cuntz–Krieger algebra O_A, where the matrix A of zeroes and ones corresponds to a subshift of finite type associated to a choice of the fundamental domain for the group G as in [BS]. The K_0 and K_1 of this C^*-algebra can be computed respectively as the cokernel and the kernel of $(I - A^t)$. These invariants should also contain some information on the boundary of the modular curves. Using this same technique we can construct a Cuntz–Krieger algebra O_A with the Markov partition determined by the action of $PSL(2, \mathbf{Z})$ on $\overline{H} \times \mathbf{P}$ with fundamental domain $E \times \mathbf{P}$, with $E = \{i, \rho, 1 + i, i\infty\}$. By [Spi], this determines a disconnection Σ of $\mathbf{P}^1(\mathbf{R})$ along $\mathbf{P}^1(\mathbf{Q})$, such that the algebra $C(\Sigma \times \mathbf{P}) \rtimes PSL(2, \mathbf{Z})$ contain an image of O_A. Similarly, if we consider the disconnection Σ of $[0, 1]$ at all the rational points and the compact totally disconnected space $X' = \Sigma \times \mathbf{P}$ with the action of the shift operator T, we obtain a Markov shift as in Section 4 of [Ren], such that the C^*-algebra $C^*(\mathcal{G}(X', T))$ is a generalized Cuntz–Krieger algebra for infinite matrices, in the sense of [EL], with partial isometries $S = \{(x, 1, Tx), x \in U\}$, and with U the sets of the Markov partition. Again, it should be possible to relate in interesting ways the calculation of the K-theory for this algebra, according to the techniques of [EL], to the dynamical properties of the shift operator T and to the boundary of the modular curves.

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