Thursday May 6 ①

The GL_2-system $(\mathcal{A}_{2,\mathbb{Q}}, \sigma)$

2-dim $\mathbb{Q}$-lattices up to commens.

(Λ, ϕ) Λ = lattice ϕ = "degenerate" level structure

What if also degenerate Λ lattice

$\mathbb{C}/\Lambda = E_\tau(\mathbb{C})$ elliptic curve $= E_q(\mathbb{C}) = \mathbb{C}^*/q^{\mathbb{Z}}$

$\Lambda = \mathbb{Z} + \mathbb{Z}\tau$ $q = e^{2\pi i \tau}$

$|q| \neq 1$

if $\tau \to 0$ i.e. $|q| \to 1$

$\mathbb{C}^*/q^{\mathbb{Z}} \rightsquigarrow \mathbb{C}^* = S^1 \times \mathbb{R}^*_+$ $S^1/q^{\mathbb{Z}}$ $q = e^{2\pi i \theta}$ $\theta \in \mathbb{R}$

NC tori $\mathcal{A}_\theta = C(S^1) \rtimes_\theta \mathbb{Z}$ irrational rotation $\theta \in \mathbb{R} \setminus \mathbb{Q}$

degeneration of a lattice $\Lambda = \mathbb{Z} + \mathbb{Z}\tau \subset \mathbb{C}$ to a pseudolattice $L = \mathbb{Z} + \mathbb{Z}\theta \subset \mathbb{R}$

$(\theta \in \mathbb{Q}$ Morita equiv. to commutative $C(T^2))$

Morita equivalent $\mathcal{A}_{\theta_1} \sim \mathcal{A}_{\theta_2}$ $\theta_1 = g(\theta_2)$ $g \in SL_2(\mathbb{Z})$ (frac. lin. transf.)

$\mathbb{P}^1(\mathbb{R})/SL_2(\mathbb{Z})$ Morita equiv. classes (also NC space) moduli space

$C(\mathbb{P}^1(\mathbb{R})) \rtimes \Gamma$

NC geometry of "boundary" of modular curves

$\mathbb{H}/\Gamma$ $\Gamma \subset PSL_2(\mathbb{Z})$ or $\Gamma \subset PGL_2(\mathbb{Z})$ of finite index

boundary classically $\mathbb{P}^1(\mathbb{Q})/\Gamma$

NC boundary $\mathbb{P}^1(\mathbb{R})/\Gamma$

$\mathbb{P}_\Gamma = PGL_2(\mathbb{Z})/\Gamma$ coset space $\simeq$ level structure

ell. curves with

Preliminary: a different way to describe action of $PGL_2(\mathbb{Z})$ on $\mathbb{P}^1(\mathbb{R})$

Continued fraction expansion:

$$[k_1, \ldots, k_n] := \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cdots}}} = \frac{P_n(k_1, \ldots, k_n)}{Q_n(k_1, \ldots, k_n)}$$

$k_i \in \mathbb{N}$

P_n, Q_n polynomials w/ integer coefficients satisfying recursion relation

$$Q_{n+1}(k_1, \ldots, k_n, k_{n+1}) = k_{n+1} Q_n(k_1, \ldots, k_n) + Q_{n-1}(k_1, \ldots, k_{n-1})$$

$$P_n(k_1, \ldots, k_n) = Q_{n-1}(k_2, \ldots, k_n)$$

starting with $Q_{-1} = 0$, $Q_0 = 1$

then one has

$$[k_1, \ldots, k_{n-1}, k_n + x_n] = \frac{P_{n-1}(k_1, \ldots, k_{n-1})\, x_n + P_n(k_1, \ldots, k_n)}{Q_{n-1}(k_1, \ldots, k_{n-1})\, x_n + Q_n(k_1, \ldots, k_n)} =$$

$$= \begin{pmatrix} P_{n-1} & P_n \\ Q_{n-1} & Q_n \end{pmatrix} (x_n)$$

fract lin. transf.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(x) = \frac{ax+b}{cx+d}$$

$\alpha \in (0,1)$ irrational number:

$\exists !$ sequence $k_i(\alpha) \in \mathbb{N}$ s.t.

$$\alpha = \lim_{n \to \infty} [k_1(\alpha), \ldots, k_n(\alpha)] \qquad \text{i.e. } \alpha = \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cfrac{1}{k_3 + \cdots}}}$$

equiv. $\exists !$ seq. $x_n(\alpha)$: $\alpha = [k_1(\alpha), \ldots, k_n(\alpha) + x_n(\alpha)]$

- rational numbers $=$ finite sequences $[k_1, \dots, k_n]$
- for irrational:

$$\alpha = \begin{pmatrix} 0 & 1 \\ 1 & k_1(\alpha) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & k_2(\alpha) \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n(\alpha) \end{pmatrix} (x_n(\alpha))$$

Also set

$$p_n(\alpha) = P_n(k_1(\alpha), \dots, k_n(\alpha))$$
$$q_n(\alpha) = Q_n(k_1(\alpha), \dots, k_n(\alpha))$$

p_n / q_n = convergents of α rationals approximating α

$$g_n(\alpha) = \begin{pmatrix} p_{n-1}(\alpha) & p_n(\alpha) \\ q_{n-1}(\alpha) & q_n(\alpha) \end{pmatrix} \in GL_2(\mathbb{Z})$$

these matrices given by semigroup

$$Red_n = \left\{ \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_n \end{pmatrix} : k_1, \dots, k_n \geq 1, \ k_i \in \mathbb{Z} \right\}$$

$$Red = \bigcup_{n \geq 1} Red_n$$ reduced matrices of length n

Shift of the continued fraction expansion

$$\alpha = [k_1, k_2, k_3, \dots, k_n, \dots] \longmapsto T\alpha = [k_2, k_3, \dots, k_n, \dots]$$

$$T\alpha = \frac{1}{\alpha} - \left[\frac{1}{\alpha}\right]$$

$g_n(\alpha)^{-1}$ = n-th power of T

$$x_n(\alpha) = g_n(\alpha)^{-1}(\alpha)$$

Let $G \subset \Gamma$ with $\Gamma = GL_2(\mathbb{Z})$ and G a finite index subgroup

$\Gamma/G = \mathbb{P}$ a finite set

extend shift map $T \cdot [0,1) \to [0,1)$ to

$$T : (0,1) \times \mathbb{P} \longrightarrow (0,1) \times \mathbb{P} \quad \text{by}$$

$$T(\alpha, s) = \left(\frac{1}{\alpha} - \left[\frac{1}{\alpha}\right], \begin{pmatrix} -\left[\frac{1}{\alpha}\right] & 1 \\ 1 & 0 \end{pmatrix}(s) \right)$$

action of Γ on Γ/G on the left

Note: the set $[0,1] \times \mathbb{P}$ meets every orbit of the action of Γ on $\mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$

(equiv to action of G on $\mathbb{P}^1(\mathbb{R})$)

and two points (x_1, s_1) (x_2, s_2) are ~~eq~~ in the same Γ-orbit iff

$\exists\ n, m$ s.t. $T^n(x_1, s_1) = T^m(x_2, s_2)$

(without $\mathbb{P}$-part this is saying: two real numbers are in same $GL_2(\mathbb{Z})$-action iff they have the same tail of the continued fraction expansion)

So action of T on $[0,1) \times \mathbb{P}$ is a way to describe the NC quotient $\mathbb{P}^1(\mathbb{R})/G$

There is an invariant measure on $[0,1] \times \mathbb{P}$ for action of T:

let $m_n(x,s) =$ (Lebesgue) measure of set $\{ \alpha \in (0,1) \mid x_n(\alpha) \leq x;\ g_n(\alpha)^{-1}(s_0) = s \}$
where $s_0 =$ base pt of $\mathbb{P}$ = coset of G

Then the limit of these exists

$$\lim_{n \to \infty} m_n(x,s) = m(x,s) = \frac{1}{\#\mathbb{P} \cdot \log(2)} \log(1+x)$$

provided that the semigroup Red acts transitively on $\mathbb{P}$

Procedure: recursive relation for the $m_n(x,s)$

$$m_{n+1}(x,s) = \sum_{k=1}^{\infty} \left(m_n\left(\tfrac{1}{k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right) - m_n\left(\tfrac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right) \right)$$

deriving in x:

$$m_{n+1}'(x,s) = (\mathcal{L}\, m_n')(x,s) \quad \text{where}$$

$$(\mathcal{L} f)(x,s) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^2} f\left(\tfrac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} s\right)$$

$\mathcal{L}$ = transfer operator associated to shift operator T

So invariant measure becomes a fixed point problem
$f = \mathcal{L} f \implies$ density of inv. measure

Operators (1-parameter family) $s \in \mathbb{R}$, $s > \frac{1}{2}$
(or complex values w/ $Re(s) >$

$$(\mathcal{L}_s f)(x, t) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^{2s}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} t\right)$$

x here in a region of $\mathbb{C}$ stable under
$x \longmapsto (x+k)^{-1}$ (and containing $[0,1]$)

t in $\mathbb{P} = \Gamma/G$, base pt t_0 (in particular will want $s=1$ case)

$$\mathbb{D} = \{ z \in \mathbb{C} : |z-1| < \tfrac{3}{2} \}$$

$z \longmapsto (z+k)^{-1}$ maps $\mathbb{D}$ to itself

$\mathcal{B}_{\mathbb{C}}$ = Banach space of holomorphic functions on each sheet $\mathbb{D} \times \{t\}$ $t \in \mathbb{P}$
continuous to boundary $\partial \mathbb{D}$

$\mathcal{B}_{\mathbb{R}} \subset \mathcal{B}_{\mathbb{C}}$ functions w/ real values at real pts.

$K \subset \mathcal{B}$ positive cone : non-negative values at real points

- $\mathcal{L}_s(K) \subset K$
- if $\mathbb{P}$ no proper invariant subsets under action of Red
  then $\forall f \in K$, $f \not\equiv 0$, $\exists\ c_1, c_2 > 0$ constants
  and $m \geq 1$ s.t.
  $$c_1 \leq \mathcal{L}_s^m f \leq c_2$$
- $\mathcal{L}_s : \mathcal{B}_{\mathbb{C}} \to \mathcal{B}_{\mathbb{C}}$ compact operator of trace class

unique fixed pt $\begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix}$ $\mathbb{D}$

sp. of mat acting on

k-th summand $\pi_{s,k}(f)$: spectrum of $\pi_{s,k}$ $= \{ (-1)^n (z_k^* + k)^{-2(s+n)} \mu_i^{(k)} \}_{n \geq 0}$