H scaling hamiltonian in quantum system is infinitesimal generator of scaling action of $\mathbb{R}_+^*$ on $\mathcal{H}$

N_E = spectral projection on $[-E, E]$ in $\mathbb{R}$ (dual to $\mathbb{R}_+^*$ under λ^{it} pairing)

$$N_E = \vartheta_a(h_E) \qquad \text{for} \qquad h_E(u) = |u|^{-1/2} \frac{1}{2\pi} \int_{-E}^{E} |u|^{is} ds$$

using $|\lambda|^{-1/2} \vartheta_a(\lambda)$ unitary action

from $\chi_{[-E,E]}(t) = \int \lambda^{it} k(\lambda)\, d^*\lambda$ where the function $k(\lambda)$ is

characteristic function

$$k(\lambda) = \frac{1}{2\pi} \int_{-E}^{E} \lambda^{is} ds \qquad \lambda \in \mathbb{R}_+^*$$

Then counting # quantum states of $H \leq E$ is done by computing dimension of inters. of Q_Λ & N_E

where Q_Λ = projection onto $B_\Lambda \subset \mathcal{H}$ span of prolate spheroidal wave functions $2n \leq 4\Lambda^2$

<u>i.e.</u> compute $\mathrm{Tr}(Q_\Lambda N_E)$

$$\sim \mathrm{Tr}(R_\Lambda \vartheta_a(h_E))$$

$R_\Lambda = \hat{P}_\Lambda P_\Lambda$ error of order $\Lambda^{-\alpha} \log \Lambda$ some $\alpha > 0$

with respect to Q_Λ

Connes: if $h \in \mathcal{S}(\mathbb{R}^*)$ $h(-u) = h(u)$ even, compact support

Then $\mathrm{Tr}(R_\Lambda \vartheta_a(h)) = 2h(1) \log \Lambda + \int' \frac{h(u^{-1})}{|1-u|} d^*u + o(1)$

$\int'$ is a principal value (2)

$$\lim_{\varepsilon \to 0} \left(\int_{|x| \geq \varepsilon} f(x)\, d^*x + f(0) \log \varepsilon \right)$$ when f loc. const. near zero

<u>More precisely</u>: K local field ($\mathbb{R}$, $\mathbb{C}$, $\mathbb{Q}_p$, fin. ext.)

α character $\mathcal{F}_\alpha$ = Fourier transf. w/ char. α $\quad \int_K f(x)\, \alpha(xy)\, dx$

$\exists!$ distribution ρ_α extending d^*u

(normalized Haar measure of K^*) to $u = 0$

with $\mathcal{F}_\alpha(\rho_\alpha)(1) = 0$

$$\int'_{(K,\alpha)} \frac{f(u^{-1})}{|1-u|}\, d^*u = \langle \rho_\alpha, g \rangle \qquad g(\lambda) = \frac{f((\lambda+1)^{-1})}{|\lambda+1|}$$

But ... h_E is not of compact support so

cannot apply directly $\mathrm{Tr}(R_\Lambda \mathcal{U}(h)) = 2h(1)\log\Lambda + \int' \frac{h(u^{-1})}{|1-u|} d^*u + o(1)$

Can still get same result unitary equiv. of

$N_E \hat{P}_\Lambda P_\Lambda$ with other operator

Assume then can apply

$$\mathrm{Tr}(R_\Lambda \vartheta_a(h)) = 2h(1)\log\Lambda + \int' \frac{h(u^{-1})}{|1-u|}\, d^*u + o(1)$$

also to $h = h_E$ even though it does not have compact support (use Connes' quantized calculus argument)

then obtain

$$2\, h_E(1) \log\Lambda = \frac{1}{2\pi} 2E \cdot 2\log\Lambda$$

while other term gives

$$\int' \frac{h_E(u^{-1})}{|1-u|}\, d^*u \qquad \left(d^*u = \frac{1}{2}\frac{du}{|u|} \right)$$

$$= -2(\langle N(E)\rangle - 1)$$

to see this: have $\langle N(E)\rangle = N(E) - N_{osc}(E)$

with $N_{osc}(E) = \frac{1}{\pi} \mathrm{Im}(\log \zeta(\frac{1}{2} + iE))$

use an explicit formula for principal value, for even $f(-u) = f(u)$ with $f(u) = f(u^{-1})$

$$\int' f(u) \frac{|u|^{1/2}}{|1-u|}\, d^*u = (\log\pi + \gamma) f(1) + \lim_{\varepsilon\to 0}\left(\int_0^1 f(u) \frac{2u^{1/2}}{(1-u^2)^{1-\varepsilon}} \frac{du}{u} - \frac{1}{\varepsilon} f(1) \right)$$

Euler constant

apply to $f(u) = |u|^{is} + |u|^{-is}$

and compute $\int_0^1 u^{is} \frac{2u^{1/2}}{(1-u^2)^{1-\varepsilon}} \frac{du}{u}$ using

$$\int_0^1 x^{(\frac{1}{4} + i\frac{s}{2})} (1-x)^{-1+\varepsilon} \frac{dx}{x} = \frac{\Gamma(\frac{1}{4} + i\frac{s}{2})\Gamma(\varepsilon)}{\Gamma(\frac{1}{4} + i\frac{s}{2} + \varepsilon)}$$

At $\varepsilon = 0$ residue $= 1$
finite part $= -\frac{\Gamma'}{\Gamma}(\frac{1}{4} + i\frac{s}{2}) - \gamma$

$$\Rightarrow \int' f(u) \frac{|u|^{1/2}}{|1-u|}\, d^*u = 2(\log\pi + \gamma) - 2\gamma - \frac{\Gamma'}{\Gamma}(\tfrac{1}{4} + i\tfrac{s}{2}) - \frac{\Gamma'}{\Gamma}(\tfrac{1}{4} - i\tfrac{s}{2})$$

$$\Rightarrow \int' \frac{h_E(u^{-1})}{|1-u|}\, d^*u = \frac{1}{2\pi} \int_0^E \left(2\log\pi - \frac{\Gamma'}{\Gamma}(\tfrac{1}{4} + i\tfrac{s}{2}) - \frac{\Gamma'}{\Gamma}(\tfrac{1}{4} - i\tfrac{s}{2})\right) ds$$

but in fact

$$\langle N(E)\rangle = 1 + \frac{1}{2\pi}\int_0^E \left(-\log\pi + \mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac14 + i\frac{s}{2}\right)\right)\right)ds$$

from

$$d\,\mathrm{Im}\log\Gamma\left(\frac14 + i\frac{s}{2}\right) = \frac{1}{2i}\,d\left(\log\Gamma\left(\frac14 + i\frac{s}{2}\right) - \log\Gamma\left(\frac14 - i\frac{s}{2}\right)\right)$$
$$= \frac12\,\mathrm{Re}\left(\frac{\Gamma'}{\Gamma}\left(\frac14 + i\frac{s}{2}\right)\right)ds$$

and from

$$\langle N(E)\rangle = 1 - \frac{E}{2\pi}\log\pi + \frac{1}{\pi}\,\mathrm{Im}\log\Gamma\left(\frac14 + i\frac{E}{2}\right)$$

which follows from $\langle N(E)\rangle = 1 + \frac{\theta(E)}{\pi}$

with $\theta(E) = -\frac{E}{2}\log\pi + \mathrm{Im}\log\Gamma\left(\frac14 + i\frac{E}{2}\right)$

So obtain

$$\mathrm{Tr}(Q_\Lambda N_E) \sim \frac{4E}{2\pi}\log\Lambda - 2(\langle N(E)\rangle - 1) + o(1)$$

meaning of this +2 ?

This is the symplectic volume of $\mathbf{W}(E,\Lambda)$ as in the classical Hamiltonian system

$$W(E,\Lambda) = \{(\lambda,s)\in \mathbb{R}_+^*\times\mathbb{R} : |\log\lambda| \le \log\Lambda \text{ and } |s|\le E\}$$

So want to match the cutoffs of original classical system to those implemented in quantum system

$$|q|\le\Lambda \rightsquigarrow |\lambda|\le\Lambda$$
$$|p|\le\Lambda \rightsquigarrow |\lambda|^{-1}\le\Lambda$$
$$|h(p,q)|\le E \rightsquigarrow |s|\le E$$

Construct a map from even functions of $u \in \mathbb{R}$ $f(u)$
to functions of $\lambda \in \mathbb{R}_+^*$ so that the cutoffs match

$$\mathcal{E}(f)(\lambda) \quad ?$$

* if map $f(u) \longmapsto \lambda^{1/2} f(\lambda)$ first cutoff ok
* if map $\mathcal{E}$ also satisfies $\mathcal{E}(f)(\lambda) = \mathcal{E}(\hat{f})(\lambda^{-1})$
  with $\hat{f}$ = Fourier transform $\int e^{ixy} f(x)\,dx$
  then second cutoff also OK

but restriction $f(u) \mapsto \lambda^{1/2} f(\lambda)$ does not satisfy this

correct to give

$$\mathcal{E}(f)(\lambda) = \lambda^{1/2} \sum_{n \in \mathbb{Z}} f(n\lambda) \qquad \lambda \in \mathbb{R}_+^*$$

if impose conditions $f(0) = \hat{f}(0) = 0$ ← these two condition $\Rightarrow$ extra 2 in $\mathrm{Tr}(Q_\Lambda N_E)$
then Poisson summation formula gives second property

Note functions f even so just

$$\lambda^{1/2} \sum_{n=1}^{\infty} f(n\lambda) \qquad \text{(up to factor 2)}$$

then third cutoff also matches:

$$(\vartheta_a(\lambda) f)(x) = f(\lambda^{-1} x) \quad \text{gives}$$

$$\mathcal{E} \circ \vartheta_a(\lambda) = |\lambda|^{1/2}\, \vartheta_m(\lambda) \circ \mathcal{E}$$

action of $\mathbb{R}_+^*$ on $L^2(\mathbb{R})_{(u)}$

regular rep. of $\mathbb{R}_+^*$ on $L^2(\mathbb{R}^*)_{(\lambda)}$

warning: $\mathcal{E}$ not an isometry

Refine this quantum system:

instead of just the scaling Hamiltonian $h(q,p) = 2\pi\, p\cdot q$

view this as a component at the archimedean place of a system that also has a contribution for each non-archimedean place in the adeles & ideles

$$\mathbb{A}_{\mathbb{Q}} = \mathbb{A}_{\mathbb{Q},f} \times \mathbb{R} \qquad \mathbb{A}^*_{\mathbb{Q}} = \mathbb{A}^*_{\mathbb{Q},f} \times \mathbb{R}^*$$

finitely many degrees of freedom approximation

$$\mathbb{A}_{\mathbb{Q},S} = \prod_{p \in S} \mathbb{Q}_p \times \mathbb{R}$$

S = finite sets of primes $\cup \{\infty\}$

$$C_{\mathbb{Q},S} = GL_1(\mathbb{A}_{\mathbb{Q},S}) / \mathbb{Q}_S^*$$

$$\mathbb{Q}_S^* = \{ \gamma \in GL_1(\mathbb{Q}) : |\gamma|_v = 1 \quad \forall v \notin S \}$$

$$\mathbb{Q}_S = \{ q \in \mathbb{Q} : |q|_v \leq 1 \quad \forall v \notin S \}$$

(rational numbers whose denominator only involves the primes $p \in S$)

$$\mathbb{Q}_S^* = \{ \pm p_1^{n_1} \cdots p_k^{n_k} : p_j \in S \setminus \{\infty\} ;\ n_j \in \mathbb{Z} \}$$

invertible elements of $\mathbb{Q}_S$

inverse also has denom's in S only

Then have again scaling action

$$(\vartheta_a(\lambda)\xi)(x) = \xi(\lambda^{-1}x) \qquad \forall x \in \mathbb{A}_{\mathbb{Q},S}$$

$$\forall \xi \in \mathcal{S}(\mathbb{A}_{\mathbb{Q},S})$$

$$\lambda \in GL_1(\mathbb{A}_{\mathbb{Q},S})$$

The map $\mathcal{E} : \mathcal{S}(\mathbb{R}) \to L^2(\mathbb{R}^*)$

extends to a map $\mathcal{E} : \mathcal{S}(\mathbb{A}_{\mathbb{Q},S}) \longrightarrow L^2(C_{\mathbb{Q},S})$
with dense range given by

$$\mathcal{E}(f)(x) = |x|^{1/2}\, \Sigma(f)(x) \quad \text{where}$$

$$\Sigma(f)(x) = \sum_{q \in \mathbb{Q}_S^*} f(qx)$$

Then one obtains

$$\mathrm{Tr}(\vartheta_a(h) R_\Lambda) = 2h(1)\log\Lambda + \sum_{v \in S} \int'_{\mathbb{Q}_v^*} \frac{h(u^{-1})}{|1-u|} d^*u + o(1)$$

for $\vartheta_a(h) = \int_{C_{\mathbb{Q},S}} h(g)\, \vartheta_a(g)\, dg$

trace formula computed on Hilbert space $L^2(X_{\mathbb{Q},S})$
where $X_{\mathbb{Q},S}$ = ~~finite~~ semi-local adeles class space $\mathbb{A}_{\mathbb{Q},S}/\mathbb{Q}_S^*$

and $L^2(X_{\mathbb{Q},S})$ = completion of $\mathcal{S}(\mathbb{A}_{\mathbb{Q},S})$ in inner product

$$\langle f_1, f_2 \rangle_S := \langle \mathcal{E}(f_1), \mathcal{E}(f_2) \rangle$$

also obtained as convergent series

$$\langle f_1, f_2 \rangle_S = \sum_{\gamma \in \mathbb{Q}_S^*} \langle f_1, \vartheta_a(\gamma) f_2 \rangle$$

inner prod in $L^2(\mathbb{A}_{\mathbb{Q},S})$

Relation of the trace formula to the Weil explicit formula

Weil's distributional form of the explicit formulae

$\mathbb{K}$ = number field

$$\hat{f}(\chi, \rho) = \int_{C_{\mathbb{K}}} f(u)\,\chi(u)\,|u|^{\rho}\,d^{*}u$$

χ character of $C_{\mathbb{K}}$ equal 1 on id. comp.

$h \in \mathcal{S}(C_{\mathbb{K}})$ w. compact support

$$\hat{h}(0) + \hat{h}(1) - \sum_{\chi} \sum_{\rho \in Z_{\tilde{\chi}}} \hat{h}(\chi, \rho) = \sum_{v} \int'_{\mathbb{K}_v^{*}} \frac{h(u^{-1})}{|1-u|}\,d^{*}u$$

$Z_{\tilde{\chi}}$ = set of zeros of $L(\tilde{\chi}, s)$ in strip $0 < \mathrm{Re}(\rho) < 1$

(analogous for function fields)

So if $\mathcal{H}$ = Coker of map E

from $L^2_\delta(X_{\mathbb{A}_{\mathbb{K}}}) \longrightarrow L^2_\delta(C_{\mathbb{K}})$

weighted norms

$\cup$ Subspace with conditions $f(0) = \hat{f}(0) = 0$

$$\Rightarrow \quad \sum \hat{h}(\tilde{\chi}, \rho) = \mathrm{Tr}(\vartheta_m(h)) \qquad \text{(Connes '97)}$$

zeros on the line only

scaling action induced on $\mathcal{H}$

does <u>not</u> prove RH

Return to the endomotive formulation

find $h \in \mathcal{S}(C_{\mathbb{K}})$ (Schwartz space w/ additional cond. "strong Schwartz space")

$$\vartheta(h) = \int_{C_{\mathbb{K}}} h(g)\,\vartheta(g)\,dg \qquad \text{acting on } H(\text{Co}(D(A,\varphi)))$$

$$\mathrm{Tr}(\vartheta(h)) = \hat{h}(0) + \hat{h}(1) - \Delta\cdot\Delta\, h(1) - \sum_v \int'_{\mathbb{K}_v^*} \frac{h(u^{-1})}{|1-u|}\,d^*u$$

$\Delta\cdot\Delta = -\log|D|$

↑ discriminant of the number field ; "size of the ring of integers" = Volume of fundam domain (measures how many primes are ramified)

(analog of $\chi(C)$ for function field: self inters of diag.)

Now: get counting of all zeros, not only those on critical line

gets Weil expl. formula as trace formula

RH equivalent to

$$\mathrm{Tr}(\vartheta(h * h^{\#})) \geq 0 \qquad \forall h \in \mathcal{S}(C_{\mathbb{Q}})$$

where $(h_1 * h_2)(g) = \int h_1(k)\, h_2(k^{-1}g)\,(d^*k)$ ← multipl. Haar meas

and

$$h^{\#}(g) = |g|^{-1}\,\overline{h(g^{-1})}$$

A positivity statement for the trace of a correspondence

⇒ this looks somewhat like Weil proof for function fields...

$\mathbb{K} = \mathbb{F}_q(\mathbf{C})$ $\quad$ $\Sigma_{\mathbb{K}}$ = places = pts of $\mathbf{C}(\bar{\mathbb{F}}_q)$

degree n_v = # orbit of Fr on $\mathbf{C}(\bar{\mathbb{F}}_q) \to \Sigma_{\mathbb{K}}$

$$\zeta_{\mathbb{K}}(s) = \prod_{v \in \Sigma_{\mathbb{K}}} (1 - q^{-n_v s})^{-1} = \frac{P(q^{-s})}{(1-q^{-s})(1-q^{1-s})}$$

$P(T) = \prod (1 - \lambda_n T)$ eigenvalues of Frobenius acting on $H^1_{\acute{e}t}(\bar{C}, \mathbb{Q}_\ell)$

$RH \iff |\lambda_n| = q^{1/2}$

Correspondences: divisors (codim one subvarieties; lin. combinations of) $Z \subset C \times C$

degree, codegree, trace

$$d(Z) = Z \bullet (\{pt\} \times C)$$
$$d'(Z) = Z \bullet (C \times \{pt\})$$
$$Tr(Z) = d(Z) + d'(Z) - Z \bullet \Delta$$

diagonal

Weil: $RH \iff Tr(Z * Z') > 0$

applied to corresp: Frobenius action

$Z_1 * Z_2 = (\pi_{13})_* (\pi_{12}^* Z_1 \circ \pi_{23}^* Z_2)$ and Z' - transpose of Z

In NCG setting: correspondences :=> graph of scaling action

$g \in C_{\mathbb{K}}$ $\quad$ $Z_g = \{ (x, g^{-1}x) \} \subset \mathbb{A}_{\mathbb{K}}/\mathbb{K}^* \times \mathbb{A}_{\mathbb{K}}/\mathbb{K}^*$

$Z(f) = \int_{C_{\mathbb{K}}} f(g) Z_g \, dg$ $\quad$ $f \in \mathcal{S}(C_{\mathbb{K}})$

degree: $d(Z(f)) = \hat{f}(1) = \int f(u)|u| \, d^*u$

codegree: $d'(Z(f)) = \hat{f}(0) = \int f(u) \, d^*u$

In Weil proof first adjust deg & codeg.
by adding trivial correspondences $\{pt\} \times C$ and $C \times \{pt\}$

here can do same:

$d(Z(f)) = \hat{f}(1)$ add h with

$$h(u, \lambda) = \sum_{n \in \mathbb{Z} \setminus \{0\}} \eta(n\lambda) \qquad \lambda \in \mathbb{R}_+^*, \; u \in \hat{\mathbb{Z}}^*$$
$$C_{\mathbb{Q}} = \hat{\mathbb{Z}}^* \times \mathbb{R}_+^*$$

can find a function h as above
which has $\hat{h}(1) \neq 0$

because $\displaystyle\int_{\mathbb{R}} \sum_n \eta(n\lambda)\, d\lambda \neq \sum_n \int_{\mathbb{R}} \eta(n\lambda)\, d\lambda = 0$

(Fubini's thm not nec. hold)

$h \in V$

Next step in Weil's proof
Riemann-Roch's theorem applied to

$Z \cap P \times C$
divisor $Z(P)$ on C

$Z(P)$ of degree g $\Rightarrow$ linearly equivalent (mod principal divisors)
to an effective divisor (positivity)

$d(Z * Z') = d(Z) d(Z') = g d'(Z) = d'(Z * Z)$

$$Tr(Z * Z') = 2g\, d'(Z) + (2g-2) d'(Z) - Y \cdot \Delta$$
$$\geq (4g-2) d'(Z) - (4g-4) d'(Z) = 2d'(Z) \geq 0$$

where $Z * Z' = d'(Z)\Delta + Y$

Y divisor on C $\qquad \{Q \in C : Q_i(P) = Q_j(P) \; i \neq j\}$
where $Z(P) = Q_1 + \cdots + Q_g$
effective divisor

estimate $Y \cdot \Delta \leq (4g-4) d'(Z)$
$\Delta \cdot \Delta = 2 - 2g = \chi(C)$ Eul. char.

Need in NCG: principal divisors & Riemann-Roch