Thursday May 27

Homotopy quotients in NCG (a brief sketch)

$\mathcal{A}$ = C^*-algebra describing an NC space (e.g. a "bad quotient" X/G)

Y a commutative space (only unique up to htpy) s.t.

K-theory of C^*-algebra $\mathcal{A}$ (analytic) can be computed geometrically using the commut. space Y

More precise formulation for group actions: Baum-Connes conjecture

In specific case we're looking at

$$\mu : K^1(X_\varepsilon) \xrightarrow{\cong} K_0(C^*(S(\Lambda, V), \tilde{\sigma}))$$

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μ is "Kasparov map"

Conclusion: The 3-manifold X_ε is a homotopy quotient model of the NC space $C^*(S(\Lambda, V), \tilde{\sigma}) \cong \mathcal{A}_\theta \rtimes V$ $= \mathbb{T}_\theta / \mathrm{Aut}(\mathbb{T}_\theta)$

- Spectral geometry on $C^*(\Lambda, \sigma)$, $C^*(S(\Lambda, V), \tilde{\sigma})$ and on X_ε

- Relations between these and the Shimizu L-function

The Dirac operator on X_ε (Atiyah-Donnelly-Singer)

Rel. to Shimizu L-function

$$\partial\!\!\!/_{X_\varepsilon} = c(dt)\frac{\partial}{\partial t} + c(e^{t}dx)\frac{\partial}{\partial x} + c(e^{-t}dy)\frac{\partial}{\partial y}$$

because $\{dt, e^{t}dx, e^{-t}dy\}$ basis of tangent bundle of $S(\mathbb{R}^2, \mathbb{R}, \varepsilon) = \mathbb{R}^2 \rtimes_\varepsilon \mathbb{R}$

$c(\omega)$ = Clifford multiplication by ω

This gives

$$\partial\!\!\!/_{X_\varepsilon} = \frac{\partial}{\partial t}\sigma_0 + e^{t}\frac{\partial}{\partial x}\sigma_1 + e^{-t}\frac{\partial}{\partial y}\sigma_2$$

$\sigma_0, \sigma_1, \sigma_2$ = Pauli matrices

$$\partial\!\!\!/_{X_\varepsilon} = \begin{pmatrix} \frac{\partial}{\partial t} & e^{-t}\frac{\partial}{\partial y} - ie^{t}\frac{\partial}{\partial x} \\ e^{-t}\frac{\partial}{\partial y} + ie^{t}\frac{\partial}{\partial x} & -\frac{\partial}{\partial t} \end{pmatrix}$$

Connes-Landi isospectral deformations

$(C^\infty(X), L^2(X,S), \partial\!\!\!/_X)$ sp triple of a spin Riemannian mfld

$T^2 \subset \mathrm{Isom}(X)$

$$\pi(f) = \sum_{n,m\in\mathbb{Z}} \pi(f_{n,m}) \qquad f \in C^\infty(X)$$

Now relation to Shimizu L-function (as in Atiyah-Donnelly-Singer)

Restrict $D_{\theta,\theta'}$ to complement of zero modes $\lambda=0$

$$D_{\theta,\theta',0} = \sum_{\mu \in (\Lambda\setminus\{0\})/V} D^{\mu}_{\theta,\theta',0}$$

$$D^{\mu}_{\theta,\theta',0} \overset{\text{unitarity equiv.}}{\sim} D^{\mu}_{\theta} B_{\theta}$$

$$D^{\mu}_{\theta} \psi_\lambda = \operatorname{sign}(N(\mu)) \, |N(\mu)|^{1/2} \, \psi_\lambda$$

if $\lambda = A^{k}_{\varepsilon}(\mu)$ $\quad \mu \in \mathcal{F}_V$ fundam. domain & $k \in \mathbb{Z}$

$$B_\theta \psi_\lambda = (\varepsilon^{k}\sigma_1 + \varepsilon^{-k}\sigma_2)\, \psi_\lambda$$

$$\rightsquigarrow \sum_{\mu \in \frac{\Lambda\setminus\{0\}}{V}} \operatorname{sign}(N(\mu)) \, |N(\mu)|^{-s/2} = L(\Lambda, V, s)$$

$$= \operatorname{Tr}(F \, |D^{\mu}_{\theta}|^{-s/2})$$

Shimizu L-function from NC geometry of A_θ-torus

$$\not{D}_{\theta,\theta'} = \begin{pmatrix} 0 & \delta_{\theta'} - i\delta_\theta \\ \delta_{\theta'} + i\delta_\theta & 0 \end{pmatrix} = \lambda_1 \sigma_1 + \lambda_2 \sigma_2$$

where $\lambda = (\lambda_1, \lambda_2)$
$= (n + m\theta, n + m\theta')$

$\delta_\theta : \psi_{n,m} \longmapsto (n + m\theta)\, \psi_{n,m}$

$\delta_{\theta'} : \psi_{n,m} \longmapsto (n + m\theta')\, \psi_{n,m}$

$\psi_\lambda = \psi_{(n,m)}$

(Note: part of operator $\not{D}_{X_\Sigma} = \left(\frac{\partial}{\partial t}\sigma_0 + 2\pi i\, \lambda_1 \sigma_1 + 2\pi i\, \lambda_2 \sigma_2\right)$ in Fourier modes along the fiber tori)

This $\not{D}_{\theta,\theta'}$ defines a Dirac operator and a spectral triple for the NC torus $\mathcal{A}_\theta$ w/ Hilbert space $\{\psi_\lambda\} \cong \ell^2(\mathbb{Z}^2)$

Relation of this spectral triple to spectral geometry on NC space $C^*(S(\Lambda, V), \tilde\sigma)$:

As seen before $C^*(S(\Lambda,V), \tilde\sigma) \cong C^*(\Lambda, \sigma) \rtimes V$

$\Rightarrow \mathcal{A}_\theta \rtimes_{A_\Sigma} \mathbb{Z}$ crossed product by $\mathbb{Z}$ of an NC torus

$\rightsquigarrow$ extend a spectral triple on $\mathcal{A}_\theta$ to a spectral triple on $\mathcal{A} \rtimes_\alpha \mathbb{Z}$

(work in progress w/ Bellissard-Reihani)

Result in this case: Resulting Dirac op on crossed prod is again $\not{D}_{X_\Sigma}$

$$[D, \pi_{\xi_1,\xi_2}(f)] = \Big[\sum_{n,m} [D, \pi(f)]_{n,m}\, e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)} \Big]$$

$$= \sum_{n,m} [D, \pi(f)]_{n,m}\, e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)}$$

still a Dirac op. (bounded commutators)

$\Rightarrow$ Induced spectral triple from $\partial\!\!\!/_{X_\varepsilon}$ to NC torus with real multiplication

$$\mathcal{A}_\theta = C^*(\Lambda, \sigma)$$

On standard torus $\mathbb{R}^2/\mathbb{Z}^2$ two Kronecker foliations

$s_1 + s_2\theta$ and $s_1 + s_2\theta'$ where $\theta' \in \mathbb{K}$ is the Galois conjugate of θ

$\mathbb{Z} + \mathbb{Z}\theta$ $\quad \mathbb{Z} + \mathbb{Z}\theta'$ pseudolattices in $\mathbb{R}$

$\rightsquigarrow$ lattice $\Lambda \subset \mathbb{R}^2$

$$\Theta_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$ expanding along line $L_\theta = \{ s_1 + s_2\theta \}$
contracting along line $L_{\theta'} = \{ s_1 + s_2\theta' \}$

flows all other pts in $\mathbb{R}^2$ along hyperbola with asymptotes L_θ & $L_{\theta'}$

$e^t \frac{\partial}{\partial t}$ $\quad e^{-t}\frac{\partial}{\partial y}$ in $\partial\!\!\!/_{X_\varepsilon}$ are the leafwise derivatives along these two foliations

with e^t & e^{-t} factors

scaling of transverse measure under the flow

$$\alpha_\tau(\pi(f_{n,m})) = e^{2\pi i(n\tau_1 + m\tau_2)}\, \pi(f_{n,m})$$

$$\tau = (\tau_1, \tau_2) \in T^2$$

$$\alpha_\tau(T) = U(\tau)\, T\, U(\tau)^* \qquad \forall T \in \mathcal{B}(\mathcal{H}) \quad \forall \tau \in T^2$$

$U(\tau)$ unitary transformation T^2-action on $\mathcal{H}$

$U(\tau) = \exp(2\pi i \tau L)$ $\quad L = (L_1, L_2)$ infinitesimal generator of the action

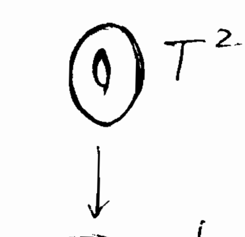
$$\rightsquigarrow \quad \pi_{\xi_1,\xi_2}(f) = \sum_{n,m} \pi(f_{n,m})\, e^{-2\pi i(\xi_1 n L_2 + \xi_2 m L_1)}$$

$$\pi_{\xi_1,\xi_2}(f_{n,m}) \cdot \pi_{\xi_1,\xi_2}(h_{k,r}) = \pi_{\xi_1,\xi_2}(f_{n,m} * h_{k,r})$$

$$f_{n,m} * h_{k,r} = \exp\left(-2\pi i(\xi_1 n r + \xi_2 m k)\right) f_{n,m} h_{k,r}$$

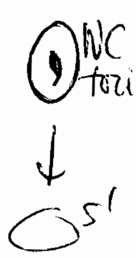
$\Rightarrow$ NC torus with $\sigma((n,m),(k,r)) = \exp\left(-2\pi i(\xi_1 n r + \xi_2 m k)\right)$

Geometrically: take X_Σ :



fibration of tori over S^1

and replace with



fibration of NC tori over S^1

Same Hilbert space $\mathcal{H}$ and same Dirac operator

$U(\tau) D U(\tau)^* = D$ $\quad$ since T^2 acts on X by isometries