Tuesday May 25

$\hat{X} = \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$ $\quad \mathbb{P} = \Gamma / G \quad$ G fin. ind. subgroup of $\Gamma = PSL_2(\mathbb{Z})$

$PSL_2(\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ free product

$\sigma : x \longmapsto -\frac{1}{x}$ (rotations by π & $\frac{2\pi}{3}$ on circle)

$\tau : x \longmapsto -\frac{1}{x-1}$

$X_G = (\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}))/G$

$\tilde{I} = PSL_2(\mathbb{Z}) \cdot i$ orbit of $i \in \mathbb{H}$

$\tilde{R} = PSL_2(\mathbb{Z}) \cdot \rho$ orbit of $\rho = e^{\frac{\pi i}{3}}$

$\langle x, y \rangle =$ oriented geodesic segment in $\overline{\mathbb{H}}$ connecting x, y

$\overline{\mathbb{H}} = \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

Modular complex (Manin): cell decomposition of $X_G(\mathbb{C})$

0-cells : cusps $G \backslash \mathbb{P}^1(\mathbb{Q})$ and elliptic points $I = G \backslash \tilde{I}$

$R = G \backslash \tilde{R}$

1-cells : $G \backslash \{ \langle g(\infty), g(i) \rangle \; g \in SL_2(\mathbb{Z}) \}$

$G \backslash \{ \langle g(i), g(\rho) \rangle \; g \in SL_2(\mathbb{Z}) \}$

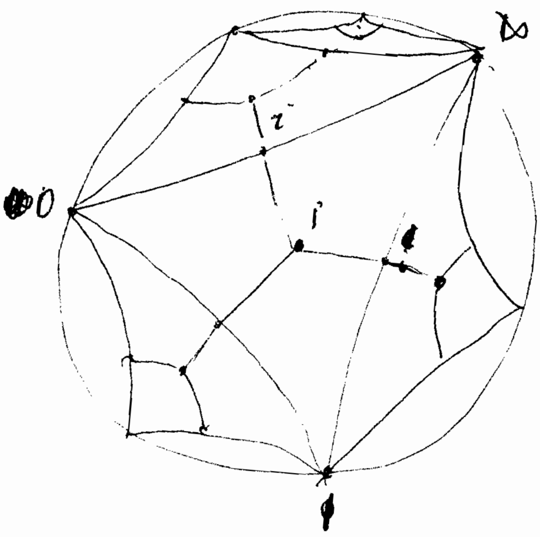
2-cells : $E = \{ i, \rho, 1+i, i\infty \}$ quadrangle

$G \backslash \{ PSL_2(\mathbb{Z}) \cdot E \}$

i.e. cells are image under quotient

$$\pi_G : \mathbb{H} \cup \mathbb{P}^1(\mathbb{Q}) \longrightarrow X_G(\mathbb{C})$$

of tessellation of $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$:



The arcs $\langle i, \rho \rangle$ and $\langle g(i), g(\rho) \rangle$ span a tree in $\mathbb{H}$

Cayley tree of $PSL_2(\mathbb{Z})$

The arcs $< g(0), g(i\infty) >$ $g \in PSL_2(\mathbb{Z})$ span the ideal triangulation of $\mathbb{H} \cup \mathbb{P}^1(\mathbb{Q})$

Boundary operators:

$$\partial : C_2 \to C_1$$

$$gE \longmapsto g\langle i, \rho \rangle + g\langle \rho, 1+i \rangle + g\langle 1+i, i\infty \rangle + g\langle i\infty, i \rangle$$

$$\partial : C_1 \to C_0$$

$$g\langle i\infty, i \rangle \mapsto g(i) - g(i\infty)$$

$$g\langle i, \rho \rangle \mapsto g(\rho) - g(i)$$

$$H_1(X_G) = \frac{\mathrm{Ker}(\partial : C_1 \to C_0)}{\mathrm{Im}(\partial : C_2 \to C_1)}$$

Case of relative homology: $\mathbb{Z}[\text{cusps}] = C_0 / \mathbb{Z}[R \cup I]$

$$0 \to C_2 \xrightarrow{\partial} C_1 \xrightarrow{\bar{\partial}} \mathbb{Z}[\text{cusps}] \to 0$$ quotient complex

Computes $H_1(X_G, R \cup I)$

$\mathbb{Z}[\mathbb{P}] = \text{span}\{ g\langle i,\rho\rangle \quad g = \text{representatives of } \mathbb{P} \}$

~~[illegible]~~

Subcomplex

$$0 \to \mathbb{Z}[P] \xrightarrow{\partial} \mathbb{Z}[R \cup I] \longrightarrow 0$$

computes $H_1(X_G \setminus \text{cusps})$

while $H_1(X_G \setminus \text{cusps}, R \cup I) = \mathbb{Z}[\mathbb{P}]$ gen. by the $g\langle i,\rho\rangle$ relative cycles

Notation: $H_A^B := H_1(X_G \setminus A, B; \mathbb{Z})$

pairing $H_A^B \times H_B^A \longrightarrow \mathbb{Z}$

in particular $H_{R\cup I}^{\text{cusps}}$, $H_{\text{cusps}}^{R\cup I}$, H^{cusps}, H_{cusps}

long exact sequence of relative cohomology:

$$0 \to H_{\text{cusps}} \longrightarrow H_{\text{cusps}}^{R\cup I} \xrightarrow{(\tilde{\beta}_R, \tilde{\beta}_I)} H_0(R) \oplus H_0(I) \to \mathbb{Z} \to 0$$

H_{cusps}, $H_{\text{cusps}}^{R\cup I}$ and with

$H_0(I) = \mathbb{Z}[\mathbb{P}_I]$ $\qquad \mathbb{P}_I = \langle\sigma\rangle\backslash\mathbb{P} = G\backslash\tilde{I}$

$H_0(R) = \mathbb{Z}[\mathbb{P}_R]$ $\qquad \mathbb{P}_R = \langle\tau\rangle\backslash\mathbb{P} = G\backslash\tilde{R}$

So get

$$0 \to H_{\text{cusps}} \to \mathbb{Z}[\mathbb{P}] \to \mathbb{Z}[\mathbb{P}_R] \oplus \mathbb{Z}[\mathbb{P}_I] \to \mathbb{Z} \to 0$$

or dually

$$0 \to H^{\text{cusps}} \longrightarrow \mathbb{Z}^{|\mathbb{P}|} \xrightarrow{(\beta_R, \beta_I)} \mathbb{Z}^{|\mathbb{P}_I|} \oplus \mathbb{Z}^{|\mathbb{P}_R|} \longrightarrow \mathbb{Z} \to 0$$

functions $\mathbb{P} \to \mathbb{Z}$ $\qquad$ with $H_{R\cup I}^{\text{cusps}} = \mathbb{Z}^{|\mathbb{P}|}$

In terms of generators of modular complex

Generators δ_s $s \in \mathbb{P}$ $\delta = \{g(0), g(i\infty)\}_G$

modular symbol

g repres. of $s \in \mathbb{P}$

relations given by 2-cells:

$$\circledast \begin{cases} \delta_s \oplus \delta_{\sigma s} & (\text{or } \delta_s \text{ if } s = \sigma s) \\ \delta_s \oplus \delta_{\tau s} \oplus \delta_{\tau^2 s} & (\text{or } \delta_s \text{ if } s = \tau s) \end{cases}$$

$\Rightarrow H^{\text{cusps}} =$ span of δ_s w/ relations $\circledast$

Now holographic image of the modular complex on $(\mathbb{P}^1(\mathbb{R}) \times \mathbb{P})$ Γ

Noncommutative space

$$C(\hat{X}) \rtimes \Gamma \qquad \hat{X} = \mathbb{P}^1(\mathbb{R}) \times \mathbb{P}$$

Since $\Gamma = \mathbb{Z}/2\mathbb{Z} * \mathbb{Z}/3\mathbb{Z}$ acting on a tree $\mathcal{T}$

$\Rightarrow$ Pimsner long exact sequence for K-theory of the crossed product algebra

(K-theory of crossed prod's by groups acting on trees)

$$\begin{tikzcd}
K_0(C(\hat{X})) \arrow[r, "\alpha"] & K_0(C(\hat{X}) \rtimes \Gamma_0) \oplus K_0(C(\hat{X}) \rtimes \Gamma_1) \arrow[r, "\tilde{\alpha}"] & K_0(C(\hat{X}) \rtimes \Gamma) \arrow[d, "\partial"] \\
K_1(C(\hat{X}) \rtimes \Gamma) \arrow[u, "\partial"] & K_1(C(\hat{X}) \rtimes \Gamma_0) \oplus K_1(C(\hat{X}) \rtimes \Gamma_1) \arrow[l, "\tilde{\beta}"'] & K_1(C(\hat{X})) \arrow[l, "\beta"']
\end{tikzcd}$$

Bott periodicity so only six terms of long exact sequence

Here $\Gamma_0 = \mathbb{Z}/2\mathbb{Z}$ and $\Gamma_1 = \mathbb{Z}/3\mathbb{Z}$

Vertices of tree $= \Gamma/\Gamma_0 \cup \Gamma/\Gamma_1$ So Γ_0, Γ_1 are the two possible stabilizers of vertices

Edges of tree $= \Gamma$

From this exact sequence extract piece:

$$\begin{tikzcd}
0 \arrow[r] & \ker(\beta) \arrow[r, hook] & K_1(C(\hat{X})) \arrow[r, "\beta"] & K_1(C(\hat{X})\rtimes\Gamma_0)\oplus K_1(C(\hat{X})\rtimes\Gamma_1) \arrow[d] \\
 & & & \mathrm{Im}(\tilde{\beta}) \arrow[d] \\
 & & & 0
\end{tikzcd}$$

<u>Result</u>: there is a natural isomorphism between this long exact sequence and

$$0 \to H^{cusps} \longrightarrow H^{cusps}_{R\cup I} \longrightarrow \mathbb{Z}^{|\mathbb{P}_I|} \oplus \mathbb{Z}^{|\mathbb{P}_R|} \longrightarrow \mathbb{Z} \longrightarrow 0$$

which computes homology of mod curves via mod complex & mod symbols

<u>Pf</u>: $K_0(C(\mathbb{P}^1(\mathbb{R}))) = \mathbb{Z} = K_1(C(\mathbb{P}^1(\mathbb{R})))$

$K_i(C(\hat{X})) = \mathbb{Z}^{|\mathbb{P}|}$ $K_i(C(\hat{X})\rtimes\mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}^{|\mathbb{P}_I|}$

$K_i(C(\hat{X})\rtimes\mathbb{Z}/3\mathbb{Z}) = \mathbb{Z}^{|\mathbb{P}_R|}$

thus exact sequence for K-theory becomes

$$\begin{tikzcd}
\mathbb{Z}^{|\mathbb{P}|} \arrow[r, "\alpha"] & \mathbb{Z}^{|\mathbb{P}_I|}\oplus\mathbb{Z}^{|\mathbb{P}_R|} \arrow[r] & K_0(C(\hat{X})\rtimes\Gamma) \arrow[d] \\
K_1(C(\hat{X})\rtimes\Gamma) \arrow[u] & \mathbb{Z}^{|\mathbb{P}_I|}\oplus\mathbb{Z}^{|\mathbb{P}_R|} \arrow[l, "\tilde{\beta}"'] & \mathbb{Z}^{|\mathbb{P}|} \arrow[l, "\beta"']
\end{tikzcd}$$

Split this into two parts

$$0 \to \mathrm{Ker}(\alpha) \hookrightarrow \mathbb{Z}^{|\mathbb{P}|} \xrightarrow{\alpha} \mathbb{Z}^{|\mathbb{P}_I|} \oplus \mathbb{Z}^{|\mathbb{P}_R|} \xrightarrow{\tilde{\alpha}} \mathrm{Im}(\tilde{\alpha}) \to 0$$

$$0 \to \mathrm{Ker}(\beta) \hookrightarrow \mathbb{Z}^{|\mathbb{P}|} \xrightarrow{\beta} \mathbb{Z}^{|\mathbb{P}_I|} \oplus \mathbb{Z}^{|\mathbb{P}_R|} \xrightarrow[\tilde{\beta}]{} \mathrm{Im}(\tilde{\beta}) \to 0$$

morphisms in these sequences are induced by

$$\beta_y : C(\hat{X}) \longrightarrow C(\hat{X}) \rtimes \Gamma_{t(y)}$$

$$\beta_y(a) = \gamma_{y^t}^{-1}(a)$$

$$\beta_{\bar{y}} : C(\hat{X}) \to C(\hat{X}) \rtimes \Gamma_{o(y)}$$

$$\beta_{\bar{y}}(a) = \gamma_{y^o}^{-1}(a)$$

y edge $\langle i, \rho \rangle$ $\quad o(y) = i \quad t(y) = \rho$

$\Gamma_{o(y)} = \mathbb{Z}/2\mathbb{Z} \quad \Gamma_{t(y)} = \mathbb{Z}/3\mathbb{Z}$ stabilizers of these pts.

y^o, y^t = edges in $\mathcal{T}$ tree with $t(y^t), o(y^o) \in \{i, \rho\}$

$\gamma_{y^t} \in \Gamma$ s.t. $\gamma_{y^t} \, y^t = y$; $\gamma_{y^o} \in \Gamma$ s.t. $\gamma_{y^o} \, y^o = y$

acts as τ $\qquad$ acts as σ

$$\Rightarrow \quad \beta : K_1(C(\hat{X})) \to K_1(C(\hat{X}) \rtimes \mathbb{Z}/2\mathbb{Z}) \oplus K_1(C(\hat{X}) \rtimes \mathbb{Z}/3\mathbb{Z})$$

$$\delta_s \longmapsto \delta_{[s]_I} \oplus \delta_{[s]_R}$$

$$\Rightarrow \quad \mathrm{Ker}(\beta) = H^{\text{cusps}}$$

same generators and relations

$\Sigma a_s \delta_s \in \mathrm{Ker}(\beta) \iff a_s + a_{\sigma s} = 0$ (or $a_s = 0$ if $\sigma s = s$) $\qquad a_s + a_{\tau s} + a_{\tau^2 s} = 0$ (or $a_s = 0$ if $\tau s = s$)

Slight difference between α_1, β maps
i.e. between $K_1(C(\hat{X})\rtimes\Gamma)$ and $K_0(C(\hat{X})\rtimes\Gamma)$

K_0 has an additional <u>torsion</u> term which keeps into account order of the stabilizers

$$K_0(C(\hat{X})\rtimes\Gamma) = H^{cusps} \oplus \mathbb{Z} \oplus \mathbb{I}$$

$$\mathbb{Z}\oplus\mathbb{I} = \mathbb{Z}^2/\mathbb{Z}(l,1) \oplus T(n_1,\dots,n_k)$$

if G has signature $(\underset{\nwarrow \text{genus}}{g}; \underbrace{n_1,\dots,n_k}_{\text{depend on orders \& \# of ell. pts}}, \overset{\text{\# cusps}}{q})$

$l = \text{l.c.m.}(n_1,\dots,n_k)$

while $K_1(C(\hat{X})\rtimes\Gamma) = H^{cusps} \oplus \mathbb{Z}$

Noncommutative tori with real multiplication and L-functions of real quadratic fields (and Lorentzian spectral triples)

$\mathbb{K} = \mathbb{Q}(\sqrt{d})$ real quadratic field

two embeddings (Galois conjugate) of $\mathbb{K} \overset{\alpha_i}{\hookrightarrow} \mathbb{R}$

$L \subset \mathbb{K}$ rk 2-subgroup $L \simeq \mathbb{Z}^2$ abstr. isom. (lattice or pseudolattice)

each embedding $\alpha_i(L) \subset \mathbb{R}$ not a lattice but a pseudolattice (dense image $\mathbb{Z}+\mathbb{Z}\theta$ some θ irrat. quadr. $\theta\in\mathbb{K}$)

U_L^+ = group of totally positive units in $\mathbb{K}$ preserving L

$$\{u \in \mathcal{O}_{\mathbb{K}}^* : uL \subset L \,;\ \alpha_i(u) \in \mathbb{R}_+^*\}$$

$U_L^+ \cong \mathbb{Z}$ generator ϵ_L $\quad U_L^+ = \epsilon_L^{\mathbb{Z}}$

if $L = \mathcal{O}_{\mathbb{K}}$ ring of integers ϵ_L is a fundam. unit

Obtain from L a lattice in $\mathbb{R}^2$ by setting

$$\Lambda = \{ (\alpha_1(\ell), \alpha_2(\ell)) ; \ell \in L \}$$

U_L^+ gives action on Λ by

$$\lambda = (\alpha_1(\ell), \alpha_2(\ell)) \longmapsto A_\varepsilon(\lambda) = (\varepsilon\, \alpha_1(\ell), \varepsilon' \alpha_2(\ell))$$

where $\varepsilon' = \varepsilon^{-1}$ = Galois conjugate of ε

Can form the semidirect product $\quad V = U_L^+$ or a fin. ind subgroup

$$S(\Lambda, V) = \Lambda \rtimes_\varepsilon V$$

discrete group

subgroup of a (solvable) Lie group

$$S(\mathbb{R}^2, \mathbb{R}) = \mathbb{R}^2 \rtimes \mathbb{R}$$ where action of $\mathbb{R}$ on $\mathbb{R}^2$

is by 1-param. subgroup of $SL_2(\mathbb{R})$

$$\Theta_t(x,y) = (e^t x, e^{-t} y)$$

Say $\varepsilon >$ [illegible]; $0 < \varepsilon' < 1$

$$A_\varepsilon = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon' \end{pmatrix} \in SL_2(\mathbb{R})$$

Can form a 3-manifold as quotient

$$X_\varepsilon = S(\Lambda, V) \backslash S(\mathbb{R}^2, \mathbb{R}) \qquad \pi_1(X_\varepsilon) = S(\Lambda, V)$$

since $S(\mathbb{R}^2, \mathbb{R})$ contractible

Topology of X_ε : cohomology

$$H^{\text{even}}(X_\varepsilon, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z} \oplus \operatorname{coker}(1 - A_\varepsilon) \qquad H^{\text{odd}}(X_\varepsilon, \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}$$

(from computing $H_1(X_\varepsilon, \mathbb{Z}) = \pi_1(X_\varepsilon) / [\pi_1(X_\varepsilon), \pi_1(X_\varepsilon)]$ $= \pi_1(X_\varepsilon)^{ab}$ and then using dualities)

More explicit description: $\alpha_1(L) \subset \mathbb{R}$ basis $\{1, \theta\}$

$\overset{\|}{\mathbb{Z} + \mathbb{Z}\theta}$

then action of $V = \varepsilon^{\mathbb{Z}}$ given by a matrix (generator ε)

$$\varphi_\varepsilon = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \quad \text{with}$$

$$\varepsilon = a + b\theta \qquad \varepsilon' = c + d\theta$$

$\frac{1}{\theta}, \frac{1}{\theta'}$ are the fixed pts of φ_ε acting by fract. lin. transf

(since $\theta^{-1} = \frac{\varepsilon}{\varepsilon\theta} = \frac{a\theta^{-1} + b}{c\theta^{-1} + d}$)

$$A_\varepsilon^k : (n + m\theta, n + m\theta') \longmapsto (\varepsilon^k(n + m\theta), \varepsilon'^k(n + m\theta'))$$

is then equivalent to

$$(n, m) \longmapsto (n, m)\varphi_\varepsilon^k$$

Hence write equivalently $S(\Lambda, V) = \mathbb{Z}^2 \rtimes_{\varphi_\varepsilon} \mathbb{Z}$

$$(\lambda, \varepsilon^k) \leftrightsquigarrow (n, m, k) \quad \text{if } \alpha(\lambda) = n + m\theta$$

Noncommutative tori $\qquad UV = e^{2\pi i\theta} VU$

$$\mathcal{A}_\theta = C(S^1) \rtimes_\theta \mathbb{Z} \qquad \text{action by irrational rotation } \exp(2\pi i \theta)$$

An equivalent description of the C^*algebra of the noncommutative torus is as twisted group algebra (as in the integer quantum Hall effect)

$$C^*(\mathbb{Z}^2, \sigma)$$

$C^*_r(\Gamma,\sigma)$: $\sigma : \Gamma \times \Gamma \to U(1)$ multiplier

$$\sigma(\gamma_1,\gamma_2)\,\sigma(\gamma_1\gamma_2,\gamma_3) = \sigma(\gamma_1,\gamma_2\gamma_3)\,\sigma(\gamma_2,\gamma_3)$$

(phase associated to potential of a magnetic field)

$$\sigma((n,m),(n',m')) = \exp\left(-2\pi i\left(\xi_1\, nm' + \xi_2\, mn'\right)\right)$$

$U = R^\sigma_{(0,1)} \qquad V = R^\sigma_{(1,0)}$ acting on $\ell^2(\mathbb{Z}^2)$ as

$$(Uf)(n,m) = e^{-2\pi i \xi_2 n} f(n, m+1)$$

$$(Vf)(n,m) = e^{-2\pi i \xi_1 m} f(n+1, m)$$

satisfy $UV = e^{2\pi i\theta} VU$ for $\theta = \xi_2 - \xi_1$

leaves some freedom in the choice of ξ_2, ξ_1 with fixed $\theta = \xi_2 - \xi_1$
use this freedom to adjust cocycle σ so that:

$$\sigma((n,m),(n',m')) = \sigma((n,m)\varphi, (n',m')\varphi) \qquad \forall \varphi = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$$

This condition gives

$$\exp\Big(-2\pi i \big((\xi_1+\xi_2)(ab\,nn' + cd\,mm') + (\xi_1 cb + \xi_2 ad)\,mn' + (\xi_1 ad + \xi_2 cb)\,nm' \big)$$

So if $\xi_2 = -\xi_1$ obtain invariance under $SL_2(\mathbb{Z})$

So choose $\xi_2 = \frac{\theta}{2} = -\xi_1$

Then if σ satisfies this invariance condition then it defines a multiplier $\tilde{\sigma} : S(\Lambda,V) \times S(\Lambda,V) \longrightarrow U(1)$

$$\tilde{\sigma}((n,m,k),(n',m',k')) := \sigma((n,m),(n',m')\varphi_\varepsilon^{k})$$

here identify as before $S(\Lambda,V) = \mathbb{Z}^2 \rtimes_{\varphi_\varepsilon} \mathbb{Z}$

check the cocycle condition:

$$\sigma((n_1,m_1),(n_2,m_2)\varphi_\varepsilon^{k_1})\,\sigma((n_1,m_1)+(n_2,m_2)\varphi_\varepsilon^{k_1},(n_3,m_3)\varphi_\varepsilon^{k_1+k_2}))$$
$$= \sigma((n_1,m_1),(n_2,m_2)\varphi_\varepsilon^{k_1}+(n_3,m_3)\varphi_\varepsilon^{k_1+k_2})\,\sigma((n_2,m_2)\varphi_\varepsilon^{k_1},(n_3,m_3)\varphi_\varepsilon^{k_1+k_2}))$$

Then can describe the crossed product

$$A_\theta \rtimes V \cong C^*(\mathbb{Z}^2,\sigma) \rtimes_{\varphi_\varepsilon} \mathbb{Z} \cong C^*(S(\Lambda,V),\tilde{\sigma})$$

when θ from $\alpha_1(L) = \mathbb{Z} + \mathbb{Z}\theta$

<u>Notice</u>: what is the NC space

$$\mathcal{A}_\theta \rtimes V \quad ?$$

A quotient $\underset{\text{NC torus}\nearrow}{\mathbb{T}_\theta} / \mathrm{Aut}(\mathbb{T}_\theta)$

Similar object to quotient of an elliptic curve by its automorphisms $\quad E/\mathrm{Aut}(E)$

In case of imaginary quadratic fields coords of torsion pts of this space generate abelian extensions of $\mathbb{K}$ if E is a CM elliptic curve

Here we have a real quadratic field and RM NC torus $\mathbb{T}_\theta$
Can use the NC space $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$ to obtain some numbers possibly related to abelian extensions of $\mathbb{K}$?

Conjectural generators of abelian extensions of $\mathbb{K}$ (Stark numbers) obtained from a family of L-functions of real quadr. field $\mathbb{K}$

$\Rightarrow$ Show that these L-function come naturally from NC geometry of $\mathbb{T}_\theta/\mathrm{Aut}(\mathbb{T}_\theta)$

Concentrate on one of these L-functions (Shimizu L-function)

$$L(\Lambda, V, s) = \sum_{\mu \in (\Lambda \setminus \{0\})/V} \mathrm{sign}(N(\mu))\, |N(\mu)|^{-s}$$