Thursday ~~Tuesday~~ May 20 ①

$\ell_G(\beta) = \{\{*, \beta\}\}_G \in H_1(X_G, \mathbb{R})$

g = genus of $X_G(\mathbb{C})$

$\hat{f}_1, \dots, \hat{f}_g$ complex basis of (cohomology H^1)

~~[illegible]~~ $C_2(G)$ = cusp forms of weight 2

(1-forms, holom; vanishing at cusps)

$\langle \gamma, f \rangle = \int_\gamma f(z)\,dz$ pairing $H_1(X_G, \mathbb{R}) \times C_2(G) \longrightarrow \mathbb{R}$

$\gamma \in H_1(X_G, \mathbb{R})$

$\mathcal{F}_\alpha := \{ \beta \in \mathbb{R} : (\langle \ell_G(\beta), f_1 \rangle, \dots, \langle \ell_G(\beta), f_{2g} \rangle) = \alpha \in \mathbb{R}^{2g} \}$

$f_{2k-1} = \mathrm{Re}(\hat{f}_k) \quad f_{2k} = \mathrm{Im}(\hat{f}_k)$ level sets of the limiting mod. symbol

For arbitrary finite index subgroups $G \subset \Gamma = PSL_2(\mathbb{Z})$ with $g \geq 1$

$\exists$ convex differentiable function

$h_G : \mathbb{R}^{2g} \to \mathbb{R}$ s.t. for all $\alpha \in \nabla h_G(\mathbb{R}^{2g}) \subset \mathbb{R}^{2g}$

$$\dim_H(\mathcal{F}_\alpha) = \hat{h}_G(\alpha)$$ where

$$\hat{h}_G(x) := \inf_{t \in \mathbb{R}^{2g}} \left(h_G(t) - \langle x | t \rangle \right)$$ Legendre transform

and $h_G(0) = 1$ unique minimum for h_G and

$\ell_G(\mathcal{F}_\alpha) = \{\rho_\alpha\}$ $\rho_\alpha \in H_1(X_G, \mathbb{R})$ determined uniquely by

$$(\langle \rho_\alpha, f_1 \rangle, \dots, \langle \rho_\alpha, f_{2g} \rangle) = \alpha$$

Also $\overline{\nabla h_G(\mathbb{R}^{2g})} = \{\alpha \in \mathbb{R}^{2g} : \mathcal{F}_\alpha \neq \phi\}$

i.e. get all values in this way

Reference: M. Kesseböhmer, B.O. Stratmann
"Homology at infinity: fractal geometry of limiting symbols for modular subgroups"
Topology 46 (2007) 469–491

Pushing modular forms on the invisible boundary:

f a complex valued function defined on pairs of coprime integers (q, q') with $q \geq q' \geq 1$

with a decay condition $f(q, q') = O(q^{-\varepsilon})$ some $\varepsilon > 0$

For $\alpha \in (0,1]$ set

$$\ell(f, \alpha) = \sum_{n=1}^{\infty} f(q_n(\alpha), q_{n-1}(\alpha))$$

$q_n(\alpha)$ = successive denominators of cont. fr. expansion of α

A general analysis result (Lévy's lemma 1929)

$$\int_0^1 \ell(f, \alpha)\, d\alpha = \sum{}' \frac{f(q, q')}{q(q+q')}$$

Where here $\sum{}'$ means on the domain $q \geq q' \geq 1$ $(q, q') = 1$ $q, q' \in \mathbb{N}$

One can see why Lévy's lemma is true by noticing that, for all q, q' with $q \geq q' \geq 1$ and $(q,q')=1$ $\exists!$ $n \geq 0$ s.t. one can find an $\alpha \in (0,1]$ with

$q_n(\alpha) = q \quad q_{n-1}(\alpha) = q'$ and these α's form an interval of length $\dfrac{1}{q(q+q')}$

~~the intervalin~~ $p_{n-1}(\alpha) q_n(\alpha) - p_n(\alpha) q_n(\alpha) = (-$

$$\frac{p_{n-1}(\alpha) z + p_n(\alpha)}{q_{n-1}(\alpha) z + q_n(\alpha)} \qquad z \in (0,1]$$

Note: Can extend summation by removing $(q,q')=1$ condition by extending f

choose $K : \mathbb{N} \to \mathbb{Z}$ arbitr. function & t number

$$F(q,q') := K(d)\, d^{-t} f(q,q') \qquad \text{with } d = \gcd(q,q')$$

then
$$\sum{}' \frac{f(q,q')}{q(q+q')} = \zeta(K,t)^{-1} \sum_{q \geq q' \geq 1} \frac{F(q,q')}{q(q+q')}$$

$$\zeta(K,t) = \sum_{d \geq 1} K(d)\, d^{-t}$$

An identity about modular symbols proved by Manin ~72

$$\sum_{d \mid m} \sum_{b=1}^{d} \int_{\{0, \frac{b}{d}\}} \omega = (\sigma(m) - c_m) \int_0^{i\infty} \pi_G^*(\omega)$$

where $G = \Gamma_0(N)$, $(m,N)=1$, $\pi^*(\omega)$ = cusp form for $\Gamma_0(N)$

eigenfunction of Hecke operators T_m

$$T_m\, \pi^*(\omega) = \underset{\substack{\uparrow \\ \text{eigenvalue}}}{c_m}\, \pi^*(\omega)$$

$$\sigma(m) = \sum_{d \mid m} d$$

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Multiply by m^{-2-t}; sum over m prime to N:

$$\sum_{m:(m,N)=1} \frac{1}{m^{2+t}} \sum_{d|m} \sum_{b=1}^{d} \int_{\{0,\frac{b}{d}\}_G} \omega =$$

$$= \left[\sum_{m:(m,N)=1} \frac{\sigma(m)}{m^{2+t}} - L_\omega^{(N)}(2+t) \right] \int_0^{i\infty} \pi^*(\omega)$$

L-function of the Hecke eigenform ω with omitted Euler N-factor

if N not prime: omitted all m not with $(m,N)=1$ in sum

Note then: any

$\{0,\frac{q'}{q}\}_G$ symbol, where $(q,q')=1$,

occurs as $\{0,\frac{b}{d}\}_G$ for some $d|m$ and some b

iff $q|m$ and in this case it occurs $\tau(m/q)$ times

where $\tau(n) = \sum_{d|n} 1 = \#\{d: d|n\}$

$$\Rightarrow \sum_{m:(m,N)=1} \sum_{q|m} \frac{\tau(\frac{m}{q})}{m^{2+t}} \sum_{\substack{q'\le q\\(q,q')=1}} \{0,\frac{q'}{q}\}_G$$

$$= \sum_{n:(n,N)=1} \frac{\tau(n)}{n^{2+t}} \sum_{q:(q,N)=1} \frac{\sum_{q'\le q\,(q,q')=1} \{0,\frac{q'}{q}\}_G}{q^{2+t}} =$$

$$= \zeta^{(N)}(2+t)^2 \left[\sum_{q:(q,N)=1} \frac{1}{q^{2+t}} \sum_{\substack{q' \le q \\ (q,q')=1}} \left\{0, \frac{q'}{q}\right\}_G \right]$$

Note also that $\zeta^{(N)}(1+t)\,\zeta^{(N)}(2+t) = \sum_{m:(m,N)=1} \frac{\sigma(m)}{m^{2+t}}$

$$\Longrightarrow \sum_{q:(q,N)=1} \frac{1}{q^{2+t}} \sum_{\substack{q' \le q \\ (q,q')=1}} \int_{\{0, \frac{q'}{q}\}_G} \omega =$$

$$\left(\frac{\zeta^{(N)}(1+t)}{\zeta^{(N)}(2+t)} - \frac{L_\omega^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2} \right) \int_0^{i\infty} \pi^*(\omega)$$

Taking $f(q,q') = \frac{q+q'}{q^{1+t}} \left\{0, \frac{q'}{q}\right\}$

for all $(q,q')=1$

for N prime: $\left\{0, \frac{q'}{q}\right\} = \{0, i\infty\}$ for N/q get

write $\int_0^1 \ell(f,\alpha)\,d\alpha = \sum' \frac{f(q,q')}{q(q+q')}$

for this $f(q,q')$

gives

$$\int_0^1 d\alpha \sum_{n=0}^{\infty} \frac{q_{n+1}(\alpha)+q_n(\alpha)}{q_{n+1}(\alpha)^{1+t}} \int_{\{0,\frac{q_n(\alpha)}{q_{n+1}(\alpha)}\}_G} \omega$$

$$= \left[\frac{\zeta^{(N)}(1+t)}{\zeta^{(N)}(2+t)} - \frac{L_\omega^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2} \right] \int_0^{i\infty} \pi^*(\omega)$$

ϕ = Euler function

$$+ \underbrace{\sum_{d=1}^{\infty} \frac{\phi(Nd)}{(Nd)^{2+t}} \int_0^{i\infty} \pi^*(\omega)}_{\displaystyle \left[\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{\zeta^{(N)}(1+t)}{\zeta^{(N)}(2+t)} \right] \int_0^{i\infty} \pi^*(\omega)}$$

$$= \left[\frac{\zeta(1+t)}{\zeta(2+t)} - \frac{L_\omega^{(N)}(2+t)}{\zeta^{(N)}(2+t)^2} \right] \int_0^{i\infty} \pi^*(\omega)$$

Riem. zeta funct
& L-function of ω

period of Hecke eigenforms
~~(Mellin transf)~~

A combination of information on the boundary $\mathbb{P}^1(\mathbb{R})$

"holography" type phenomenon

Another relation between $\mathcal{L}_\beta$ ~~Ra~~ Ruelle transfer op. on $[0,1]\times\mathbb{P}$ and geometry of mod. curve X_G:

Selberg zeta function:

$g \in GL_2(\mathbb{Z})$ set

$$D(g) = Tr(g)^2 - 4\det(g)$$

$$N(g) = \left(\frac{Tr(g) + D(g)^{1/2}}{2}\right)^2$$

- g hyperbolic if $Tr(g)$ & $D(g)$ are both positive
- g primitive if <u>not</u> a power $g = h^n$ of some $h \in GL_2(\mathbb{Z})$

$$\chi_s(g) = \frac{N(g)^{-s}}{1-\det(g)N(g)^{-1}} \quad \text{for } g \text{ hyperbolic}$$

$\mathbb{P} = GL_2(\mathbb{Z})/G$ coset space $\quad \rho_{\mathbb{P}} : GL_2(\mathbb{Z}) \to \mathbb{Z}[\mathbb{P}]$ representation (from action on $\mathbb{P}$)

$$Z_G(s) := \prod_{g \in Prim} \prod_{m=0}^{\infty} \det\left(1 - \det(g)^m N(g)^{-s-m} \rho_{\mathbb{P}}(g)\right)$$

Selberg zeta function

$g \in Prim$: $GL_2(\mathbb{Z})$-conjugacy classes of primitive hyperbolic elements of $GL_2(\mathbb{Z})$

then:

$$Z_G(s) = \det(1 - \mathcal{L}_s)$$

as a compact trace class op on $B_{\mathbb{C}}$

to see this:

$$-\log\det(1-\mathcal{L}_s) = \sum_{l=1}^{\infty} \frac{\mathrm{Tr}(\mathcal{L}_s^l)}{l}$$

$$= \mathrm{Tr}\left(\sum_{l=1}^{\infty} \frac{1}{l} \left(\sum_{n=1}^{\infty} \pi_{s,n} \right)^l \right)$$

where

$\pi_{s,n}$ = n-th summand of $\mathcal{L}_s$

$$\pi_{s,n}(f)(x,t) = \frac{1}{(x+n)^{2s}} f\left(\frac{1}{x+n}, \begin{pmatrix} 0 & 1 \\ 1 & n \end{pmatrix} t\right)$$

set $\pi_s(g) = \prod_i \pi_{s,k_i}$ where $g = \begin{pmatrix} 0 & 1 \\ 1 & k_1 \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ 1 & k_{l(g)} \end{pmatrix}$

for $g \in \mathrm{Red}$

Hyp = conj. classes of all hyperbolic matrices

$k(g)$ max integer s.t. $g = h^{k(g)}$ for some h

$$\tau_g = \mathrm{Tr}(\rho_{\mathbb{P}}(g)) = \#\{t \in \mathbb{P} : g(t) = t\}$$

then

$$\sum_{g \in \mathrm{Hyp}} \frac{1}{k(g)} \pi_s(g) \tau_g = \sum_{g \in \mathrm{Prim}} \sum_{k=1}^{\infty} \frac{1}{k} \frac{N(g)^{-ks} \tau_{g^k}}{1 - \det(g)^k N(g)^{-k}}$$

But also have

$$-\log Z_G(s) = \sum_{g \in \text{Prim}} \sum_{m=0}^{\infty} \mathrm{Tr} \sum_{k=1}^{\infty} \frac{1}{k} \det(g)^{mk} N(g)^{-(s+m)k} \rho_P(g^k)$$

$$= \sum_{g \in \text{Prim}} \sum_{k=1}^{\infty} \frac{1}{k} \frac{N(g)^{-ks} \tau_{g^k}}{1 - \det(g)^k N(g)^{-k}}$$