Lyapunov spectrum:

$$\lambda(\beta) := \lim_{n\to\infty} \frac{1}{n} \log |(T^n)'(\beta)|$$

$$= \lim_{n\to\infty} \frac{1}{n} \log \prod_{k=0}^{n-1} |T'(T^k\beta))|$$

Lyapunov exponent of the map T

For a dynamical system it measures how fast nearby orbits tend to diverge i.e. a measure of how chaotic the dynamics is.

$\lambda(\beta)$ is a T-invariant function (function of orbits of T-action)

in the case of $T\alpha = \frac{1}{\alpha} - [\frac{1}{\alpha}]$ shift of continued fraction expansion

(1) $\lambda(\beta) = 2 \lim_{n\to\infty} \frac{1}{n} \log q_n(\beta)$ (*)

(2) For almost all $\beta \in [0,1]$ (with Lebesgue measure)

$$\lambda(\beta) = \frac{\pi^2}{6\log 2} = \lambda_0$$

(3) There is an exceptional set in $[0,1]$ with Hausdorff $\dim_H = 1$ but Hausdorff measure $= 0$ where limit (*) does not exist

(4) Lyapunov spectrum: level sets of $\lambda(\beta)$

$$L_c = \{\alpha \in [0,1] : \lambda(\alpha) = c\}$$

$$[0,1] = \bigcup_{c\in\mathbb{R}} L_c \cup \text{Exceptional set}$$

Each L_c is an uncountable dense T-invariant subset with different Hausdorff dimensions

Limiting modular symbols and Lyapunov spectrum

Given a value $c \in \mathbb{R}$ and $L_c = \{ \alpha \in [0,1] : \lambda(\alpha) = c \}$ level set of Lyapunov exponent of T

$\forall \beta \in L_c$ the limiting modular symbol

$$\{\{*, \beta\}\}_G = \lim_{n \to \infty} \frac{1}{cn} \sum_{k=1}^{n} \varphi \circ T^k(t_0)$$

where $t_0 \in \mathbb{P} = \Gamma / G$ base pt. and

$$\varphi(s) = \{ g(0), g(i\infty) \}_G \qquad g \in PGL_2(\mathbb{Z}) = \Gamma \text{ s.t. representative of the coset } s \in \Gamma/G$$

In fact $\varphi(t_0) = \{0, i\infty\}_G$

$$\varphi(T^k(t_0)) = \{ g_k(\beta)^{-1}(0), g_k(\beta)^{-1}(i\infty) \}_G = \left\{ \frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)} \right\}_G$$

and by same argument as last time using geodesics between $\frac{p_{k-1}(\beta)}{q_{k-1}(\beta)}, \frac{p_k(\beta)}{q_k(\beta)}, \frac{p_{k+1}(\beta)}{q_{k+1}(\beta)}, \dots$ approaching β

Ruelle and Perron-Frobenius operators

$$(\mathcal{L}_{\sigma} f)(x,t) = \sum_{k=1}^{\infty} \frac{1}{(x+k)^{2\sigma}} f\left(\frac{1}{x+k}, \begin{pmatrix} 0 & 1 \\ 1 & k \end{pmatrix} t\right)$$

In general for a dynamical system T

- Ruelle transfer operator

$$(\mathcal{L}_\sigma f)(x,t) = \sum_{(y,s)\in T^{-1}(x,t)} \exp(h(y,s))\, f(y,s)$$

where here take

$$h(x,t) = -2\sigma \log |T'(x,t)|$$

a weighted sum over preimages: measures complexity of dynamical system

Perron-Frobenius operator

$$\int_{[0,1]\times\mathbb{P}} \bar{f} \cdot g\circ T \, d\mu_{\text{Lebesgue}} = \int_{[0,1]\times\mathbb{P}} \overline{\mathcal{P}(f)} \cdot g \, d\mu_{\text{Leb.}}$$

Adjoint of composition with T

transforms dynamical properties of T into functional analytic properties of $\mathcal{P}$

In good cases $\mathcal{L}_\sigma$ & $\mathcal{P}$ related: for our case

$$\mathcal{L}_\sigma|_{\sigma=1} = \mathcal{P}$$

Functional analytic properties of $\mathcal{P} = \mathcal{L}_1$.

- $\mathcal{L}_1$ compact, trace class operator
- $\mathcal{L}_1$ has top eigenvalue $\lambda = 1$, simple; eigenfunction $\frac{1}{1+x}$ up to normalization factor
- rest of spectrum of $\mathcal{L}_1$ in a ball $r < 1$
- complete set of eigenfunctions

(Under condition Red acts transitively on $\mathbb{P}$)

Lyapunov exponent

λ_σ = top eigenvalue of $\mathcal{L}_\sigma$

$\Rightarrow \quad \lambda(\beta) = \frac{d}{d\sigma} \lambda_\sigma \Big|_{\sigma=1}$ almost everywhere in ~~Hausdorff~~ Lebesgue meas.

In general can also define $\mathcal{L}_{\sigma, E}$ for any E T-invariant subset of $[0,1] \times \mathbb{P}$ (by counting only preimages $(y,s) \in E$)

$\Rightarrow \quad \lambda_{\sigma, E}$ top eigenvalue of $\mathcal{L}_{\sigma, E}$

$$\lambda(\beta) = \frac{d}{d\sigma} \lambda_\sigma \Big|_{\sigma = \dim_H(E)}$$ almost everywhere in the Hausdorff measure of dimension $\dim_H E$

Case of limiting mod symbols on quadratic irrationalities

$$\{\{*, \beta\}\}_G = \frac{1}{\lambda(\beta)\,\ell} \sum_{k=1}^{\ell} \{ g_k(\beta)^{-1}(0), g_k(\beta)^{-1}(i\infty) \}_G$$

$\beta = [\overline{k_1, \dots, k_\ell}]$ periodic continued fraction of period ℓ

Vanishing on average result

$\{\{*, \beta\}\}_G = 0$ almost everywhere in Lebesgue measure

(or $\mu_H(E)$-a.e. if on a T-inv. subset E of $\dim_H(E)$)

In fact computing averages of iterates

$$\cancel{\frac{1}{\lambda_0 n}\sum_{k=1}^{n}\varphi\circ T^k}$$

$$R_\tau(\beta,s) := \frac{1}{\tau}\{x, y(t)\}_G$$ want to show this converges weakly to zero i.e.

$$\lim_{\tau\to\infty}\int_E R_\tau(\beta,s)\, f(\beta,s)\, d\mu_H(\beta,s) = 0$$

(then improve weak convergence to strong convergence almost everywhere)

to show weak convergence use Perron-Frobenius operator

replace $R_\tau(\beta,S)$ with

$$\frac{1}{\lambda_0 n}\sum_{k=1}^{n} \varphi\circ T^k$$ (limit computes $\{x,\beta Y\}_G$)

then $$\int_E \frac{1}{\lambda_0 n}\sum_{k=1}^{n}\varphi\circ T^k\cdot f\, d\mu$$

$$= \frac{1}{\lambda_0 n}\sum_{k=1}^{n}\int_E \varphi\, \mathcal{L}_1^k(f)\, d\mu$$

Because of properties of $\mathcal{L}_1$

$f = \sum a_m f_m$ $\quad f_m$ eigenfunctions

$a_0 f_0 \qquad \mathcal{L}_1(f) = a_0 f_0 + \lambda_1 a_1 f_1 + \lambda_2 a_2 f_2 + \cdots$

$\lambda_0 = 1$ top eigenvalue $\qquad \lambda_i < r < 1$

$\Rightarrow \mathcal{L}_1^k(f) \longrightarrow a_0 f_0 \qquad f_0$ is the T-invariant density and $a_0 = \int f \, d\mu$

$$\Rightarrow \quad \mathcal{L}_1^k(f) \longrightarrow \left(\int f \, d\mu\right) f_0$$

$$\Rightarrow \quad \int_{[0,1]\times \mathbb{P}} \{\{*, \beta\}\}_G \, f(\beta, t) \, d\mu(\beta, t)$$

$$= \left(\int_{[0,1)\times\mathbb{P}} f(\beta, t) \, d\mu(\beta, t)\right)\left(\int_{[0,1]\times\mathbb{P}} \varphi \, f_0 \, d\mu\right)$$

φ function only of $s \in \mathbb{P}$

$$\Rightarrow \left(\int f \, d\mu\right) \cdot \frac{1}{\# \mathbb{P}} \sum_{s \in \mathbb{P}} \varphi(s)$$

but $\displaystyle\sum_{s \in \mathbb{P}} \varphi(s) = \sum_{s \in \mathbb{P}} \{ g_s(0), g_s(i\infty) \}_G = 0$

because each term ~~appears twice with opposite~~ changes sign under action of inversion $\sigma \in GL_2(\mathbb{Z})$ $\sigma^2 = 1$ and sum globally invariant under σ

<u>However</u>: vanishing almost everywhere on any T-inv. subset of positive μ_H-measure (of some $\dim_H$) does not mean $\{\{*, \beta\}\}_G$ trivial

Besides the real quadratic case there are many non-trivial values of $\{\{*, \beta\}\}_G$ but they all happen on sets of measure zero